Mean-Square Filtering for Polynomial States with Poisson Noises

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This paper presents the optimal finite-dimensional filter for incompletely measured polynomial system states, confused with white Poisson noises, over linear observations.

The problem is treated proceeding from the general expression for the stochastic Ito differential of the optimal estimate and the error variance [1].

As a result, the Ito differentials for the optimal estimate and error variance corresponding to the stated filtering problem are first derived.
Introduction II

The procedure for obtaining a closed system of the filtering equations for any polynomial state with white Poisson noises over linear observations is then established, which yields the explicit closed form of the filtering equations in the particular case of a third-order state equation.

In the example, performance of the designed optimal filter is verified against the conventional mean-square polynomial filter designed for systems with white Gaussian noises.
Let \((\Omega, F, P)\) be a complete probability space with an increasing right-continuous family of \(\sigma\)-algebras \(F_t, t \geq t_0\), and let \((N_1(t), F_t, t \geq t_0)\) and \((N_2(t), F_t, t \geq t_0)\) be independent Poisson processes. Consider the following nonlinear differential equation:

\[
dx(t) = f(x, t)dt + b(t)dN_1(t), \quad x(t_0) = x_0, \tag{1}
\]

and a linear differential equation for the observation process

\[
dy(t) = (A_0(t) + A(t)x(t))dt + B(t)dN_2(t). \tag{2}
\]

Here \(x(t) \in \mathbb{R}^n\), \(y(t) \in \mathbb{R}^m\), \(m \leq n\); \(x_0 \in \mathbb{R}^n\) is a Poisson vector such that \(x_0\), \(N_1(t) \in \mathbb{R}^p\), and \(N_2(t) \in \mathbb{R}^q\) are independent. It is assumed that \(B(t)B^T(t)\) is a positive definite matrix, therefore, \(m \leq q\). All coefficients in (1)–(2) are deterministic functions of appropriate dimensions.
Filtering Problem for Incompletely Measured Polynomial States over Linear Observations II

The nonlinear function \( f(x, t) \) is considered polynomial of \( n \) variables.

In accordance with [2], a \( p \)-degree polynomial of a vector \( x(t) \in R^n \) is regarded as a \( p \)-linear form of \( n \) components of \( x(t) \)

\[
f(x, t) = a_0(t) + a_1(t)x + a_2(t)xx^T + \ldots + a_p(t)x \ldots p \text{ times} \ldots x,
\]

\( a_0(t) \) is a vector of dimension \( n \),
\( a_1(t) \) is a matrix of dimension \( n \times n \),
\( a_2(t) \) is a 3D tensor of dimension \( n \times n \times n \),
\( a_p(t) \) is an \((p + 1)\)D tensor of dimension \( n \times \ldots (p+1) \text{ times} \ldots \times n \),
\( x \times \ldots p \text{ times} \ldots \times x \) is a \( p \)D tensor of dimension \( n \times \ldots p \text{ times} \ldots \times n \).
Filtering Problem for Incompletely Measured Polynomial States over Linear Observations III

The estimation problem is to find the optimal estimate \( \hat{x}(t) \) of the system state \( x(t) \), that minimizes the Euclidean 2-norm

\[
J = E[(x(t) - \hat{x}(t))^T(x(t) - \hat{x}(t)) \mid F_t^Y]
\]

at every time moment \( t \).

As known [1], this optimal estimate is given by the conditional expectation

\[
\hat{x}(t) = m(t) = E(x(t) \mid F_t^Y)
\]

As usual, the matrix function

\[
P(t) = E[(x(t) - m(t))(x(t) - m(t))^T \mid F_t^Y]
\]

is the estimation error variance.
The optimal filtering equations could be obtained using the formula for the Ito differential of the conditional expectation $m(t) = E(x(t))$ and $P(t) = E((x(t) − m(t))(x(t) − m(t))^T)$:

$$dm(t) = E(f(x, t) \mid F_t^Y) dt + E(x(t)[A(t)(x(t) − m(t))]^T \mid F_t^Y) \times$$

$$(B(t)B^T(t))^{-1}(dy(t) − (A_0(t) + A(t)m(t)))$$

$$m(t_0) = E(x(t_0)),$$

$$dP(t) = (E((x(t) − m(t))(f(x, t))^T \mid F_t^Y) +$$

$$E(f(x, t)(x(t) − m(t))^T) \mid F_t^Y) +$$

$$b(t)b^T(t) − P(t)A^T(t)(B(t)B^T(t))^{-1}A(t)P(t))dt,$$

$$P(t_0) = E[(x(t_0) − m(t_0))(x(t_0) − m(t_0))^T].$$
The filtering equations (4) and (5) are reduced to the form:

\[
\begin{align*}
    dm(t) &= E(f(x, t) | F_t^Y)dt + P(t)A^T(t)(B(t)B^T(t))^{-1} \times \\
    & \quad (dy(t) - (A_0(t) + A(t)m(t))dt), \\
    m(t_0) &= E(x(t_0) | F_{t_0}^Y), \\
    dP(t) &= (E((x(t) - m(t))(f(x, t))^T | F_t^Y) + \\
    & \quad E(f(x, t)(x(t) - m(t))^T) | F_t^Y) + \\
    & \quad b(t)b^T(t) - P(t)A^T(t)(B(t)B^T(t))^{-1}A(t)P(t))dt, \\
    P(t_0) &= E[(x(t_0) - m(t_0)(x(t_0) - m(t_0)^T].
\end{align*}
\]
Filtering Problem for Incompletely Measured Polynomial States over Linear Observations VIII

\[ E(f(x, t) \mid F_t^Y) \] and \[ E((x(t) - m(t))f^T(x, t)) \mid F_t^Y \] can be represented as functions of \( m(t) \) and \( P(t) \) using the following property of a Poisson random variable \( x(t) - m(t) \): all its conditional moments can be represented as functions of the variance \( P(t) \), namely,

\[
\begin{align*}
m_1 &= E[(x(t) - m(t)) \mid Y(t)] = 0, \\
m_2 &= E[(x(t) - m(t))^2 \mid Y(t)] = P, \\
m_3 &= E[(x(t) - m(t))^3 \mid Y(t)] = P, \\
m_4 &= E[(x(t) - m(t))^4 \mid Y(t)] = 3P^2 + P, \ldots \text{ etc.}
\end{align*}
\]
Filtering Problem for Incompletely Measured Polynomial States over Linear Observations VIII

\[ E(f(x, t) \mid F_t^Y) \text{ and } E((x(t) - m(t))f^T(x, t)) \mid F_t^Y) \]
can be represented as functions of \( m(t) \) and \( P(t) \) using the following property of a Gaussian random variable \( x(t) - m(t) \): all its conditional moments can be represented as functions of the variance \( P(t) \), namely,

\[
\begin{align*}
m_1 &= E[(x(t) - m(t)) \mid Y(t)] = 0, \\
m_2 &= E[(x(t) - m(t))^2 \mid Y(t)] = P, \\
m_3 &= E[(x(t) - m(t))^3 \mid Y(t)] = 0, \\
m_4 &= E[(x(t) - m(t))^4 \mid Y(t)] = 3P^2, ..., \\
m_{2k-1} &= E[(x(t) - m(t))^{2k-1} \mid Y(t)] = 0, \\
m_{2k} &= E[(x(t) - m(t))^{2k} \mid Y(t)] = (2k - 1)!!P^{2k} = 1 \times 3 \times \ldots \times (2k - 1)P^{2k}.
\end{align*}
\]
Optimal Filter for Third-Order Polynomial State I

Let the function

\[ f(x, t) = a_0(t) + a_1(t)x + a_2(t)xx^T + a_3(t)xxx^T \]  \hspace{1cm} (12)

be a third-order polynomial, where \( x \) is an \( n \)-dimensional vector, \( a_0(t) \) is an \( n \)-dimensional vector, \( a_1(t) \) is a \( n \times n \)-dimensional matrix, \( a_2(t) \) is a 3D tensor of dimension \( n \times n \times n \), \( a_3(t) \) is a 4D tensor of dimension \( n \times n \times n \times n \).

In this case, the representations for \( E(f(x, t) \mid F_t^Y) \) and \( E((x(t) - m(t))(f(x, t))^T \mid F_t^Y) \) as functions of \( m(t) \) and \( P(t) \) are derived as follows:
Optimal Filter for Third-Order Polynomial State II

\[
E(f(x, t)) = a_0(t) + a_1(t)m(t) + a_2(t)m(t)m^T(t) + a_2(t)P(t) + 3a_3(t)m(t)P(t) + a_3(t)m(t)m^T(t) + a_3(t)P(t) \ast 1, \quad (13)
\]

\[
E(f(x, t)(x(t) - m(t))^T)) + E((x(t) - m(t))(f(x, t))^T) = (14)
\]

\[
a_1(t)P(t) + P(t)a_1^T(t) + 2a_2(t)m(t)P(t) + a_2(t)P(t) \ast 1 + (a_2(t)2m(t)P(t) + P(t)\ast 1))^T + a_3(t)[(P(t)\ast (1\ast 1^T)) + 3P(t)P(t) + 3m(t)m^T(t)P(t) + 3(m(t)P(t)) \ast 1^T] + (a_3(t)[(P(t) \ast (1 \ast 1^T)) + 3P(t)P(t) + 3m(t)m^T(t)P(t) + 3(m(t)P(t)) \ast 1^T])^T.
\]
Optimal Filter for Third-Order Polynomial State III

Where, the vector $1$ is an $n$-dimensional vector with all its components equal to 1, and

$$
(a_3(t)P(t) \ast 1)_i = \sum_{j,k,l} a_{3 \, ijk\ell}(t)P_{jk}(t)1_{l}, \quad i = 1, \ldots, n,
$$

$$
(a_3(t)P(t) \ast 1 \ast 1^T)_{ij} = \sum_{h,k,l} a_{3 \, ihk\ell}(t)P_{hk}(t)1_{l}1_{j}, \quad i, j = 1, \ldots, n,
$$

$$
(a_3(t)m(t)P(t) \ast 1^T)_{ij} = \sum_{h,k,l} a_{3 \, ihk\ell}(t)m_{h}(t)P_{kl}(t)1_{j}, \quad i, j = 1, \ldots, n.
$$

Substituting the expression (13) in (10) and the expression (14) in (11), the filtering equations for the optimal estimate $m(t)$ and the error variance $P(t)$ are obtained:
Optimal Filter for Third-Order Polynomial State IV

\[
dm(t) = (a_0(t) + a_1(t)m(t) + a_2(t)m(t)m^T(t) + a_2(t)P(t) +
3a_3(t)m(t)P(t) + a_3(t)m(t)m^T(t) + a_3(t)P(t) \ast 1 +
P(t)A^T(t)(B(t)B^T(t))^{-1}[dy(t) - (A_0(t) + A(t)m(t))]dt, \quad (15)
\]
\[
m(t_0) = E(x(t_0) \mid F^Y_t),
\]
\[
dP(t) = (a_1(t)P(t) + P(t)a_1^T(t) + 2a_2(t)m(t)P(t) + a_2(t)P(t) \ast 1 +
(a_2(t)(2m(t)P(t) + P(t) \ast 1))^T + a_3(t)[(P(t) \ast (1 \ast 1^T)) + 3P(t)P(t) +
3m(t)m^T(t)P(t) + 3(m(t)P(t)) \ast 1^T] + (a_3(t)[(P(t) \ast (1 \ast 1^T)) +
3P(t)P(t) + 3m(t)m^T(t)P(t) + 3(m(t)P(t)) \ast 1^T])]^T + b(t)b^T(t))dt -
P(t)A^T(t)(B(t)B^T(t))^{-1}A(t)P(t)dt. \quad (16)
\]
\[
P(t_0) = E((x(t_0) - m(t_0))(x(t_0) - m(t_0))^T \mid F^Y_t)).
\]
Optimal Filter for Third-Order Polynomial State II

\[ E(f(x, t)) = a_0(t) + a_1(t)m(t) + a_2(t)m(t)m^T(t) + a_2(t)P(t) + \]
\[ 3a_3(t)m(t)P(t) + a_3(t)m(t)m(t)m^T(t), \quad (13a) \]

\[ E((x(t) - m(t))(f(x, t))^T)) + E((x(t) - m(t))(f(x, t))^T) = (14a) \]
\[ a_1(t)P(t) + P(t)a_1^T(t) + 2a_2(t)m(t)P(t) + \]
\[ (a_2(t)(2m(t)P(t))^T + a_3(t)[3P(t)P(t) + 3m(t)m^T(t)P(t))] + \]
\[ (a_3(t)[3P(t)P(t) + 3m(t)m^T(t)P(t)])^T. \]
\[ dm(t) = (a_0(t) + a_1(t)m(t) + a_2(t)m(t)m^T(t) + a_2(t)P(t) + \\
3a_3(t)m(t)P(t) + a_3(t)m(t)m^T(t) + \\
P(t)A^T(t)(B(t)B^T(t))^{-1}[dy(t) - (A_0(t) + A(t)m(t))dt], \quad (15a) \]
\[
m(t_0) = E(x(t_0) | F^Y_t),
\]
\[
dP(t) = (a_1(t)P(t) + P(t)a_1^T(t) + 2a_2(t)m(t)P(t) + \\
(a_2(t)(2m(t)P(t))^T + a_3(t)[3P(t)P(t) + 3m(t)m^T(t)P(t)] + \\
(a_3(t)[3P(t)P(t) + 3m(t)m^T(t)P(t)])^T + b(t)b^T(t))dt - \\
P(t)A^T(t)(B(t)B^T(t))^{-1}A(t)P(t)dt. \quad (16a) \]
\[
P(t_0) = E((x(t_0) - m(t_0))(x(t_0) - m(t_0))^T | F^Y_t).\]
Example 1

Let the bi-dimensional real state $x(t)$ satisfy the third-order system

$$
\dot{x}_1(t) = x_2(t), \quad x_1(0) = x_{10},
$$

(17)

$$
\dot{x}_2(t) = 0,1x_2^3(t) + \psi_1(t), \quad x_2(0) = x_{20},
$$

and the scalar observation process be given by

$$
y(t) = x_1(t) + \psi_2(t),
$$

(18)

Here $x(t) \in R^2$, $y(t) \in R$, $\psi_1(t)$ and $\psi_2(t)$ are white Poisson noises. $A = [1 0] \in R^{(1 \times 2)}$ is non-square and non-invertible, $a_1 = [0 1 \mid 0 0]$, the 3D tensor coefficient $a_2$ consists of zeros only, and the 4D tensor coefficient $a_3$ has only one non-zero entry, $a_3 \ 2222 = 0,1$, whereas its other entries are zeros.
The filtering equations (15),(16) take the following particular form:

\[
\dot{m}_1(t) = m_2(t) + P_{11}(t)[y(t) - m_1(t)],
\]

\[
\dot{m}_2(t) = 0.1m_2^3(t) + 0.3P_{22}(t)m_2(t) + 0.1P_{22}(t) + P_{12}(t)[y(t) - m_1(t)],
\]

with \( m(0) = E(x(0) \mid y(0)) = m_0 \),

\[
\dot{P}_{11}(t) = 2P_{12}(t) - P_{11}^2(t),
\]

\[
\dot{P}_{12}(t) = 1.1P_{22}(t) + 0.3m_2^2(t)P_{12}(t) + 0.3m_2(t)P_{22}(t) + 0.3P_{22}(t)P_{12}(t) - P_{11}(t)P_{12}(t),
\]

\[
\dot{P}_{22}(t) = 1 + 0.2P_{22}(t) + 0.6m_2^2(t)P_{22}(t) + 0.6m_2(t)P_{22}(t) + 0.6P_{22}^2(t) - P_{12}^2(t),
\]

with \( P(0) = E((x(0) - m(0))(x(0) - m(0))^T \mid y(0)) = P_0 \).
Example III

The estimates obtained upon solving the equations (19)–(20) are compared to the estimates satisfying the conventional mean-square polynomial filter equations for the third-order state (17) over the incomplete linear observations (18) (see [2]):

\[ \dot{m}_{k1}(t) = m_{k2}(t) + P_{k11}(t)[y(t) - m_{k1}(t)], \quad (21) \]
\[ \dot{m}_{k2}(t) = 0.1m_{k2}^3(t) + 0.3P_{k22}(t)m_{k2}(t) + P_{k12}(t)[y(t) - m_{k1}(t)], \]
\[ \dot{P}_{k11}(t) = 2P_{k12}(t) - P_{k11}^2(t), \quad (22) \]
\[ \dot{P}_{k12}(t) = P_{k22}(t) + 0.3m_{k2}^2(t)P_{k12}(t) + \]
\[ 0.3P_{k22}(t)P_{k12}(t) - P_{k11}(t)P_{k12}(t), \]
\[ \dot{P}_{k22}(t) = 1 + 0.6m_{k2}^2(t)P_{k22}(t) + 0.6P_{k22}^2(t) - P_{k12}^2(t). \]
Example III

For each of the two filters (19)–(20) and (21)–(22), and the reference system (17)–(18), involved in simulation, the following initial values are assigned:

\[ x_{10} = -2.5, \quad x_{20} = -0.35, \quad m_{10} = -14.6, \quad m_{20} = -1.38, \quad P_{110} = 20, \quad P_{120} = 0.9, \quad P_{220} = 0.06. \]

Realizations of white Poisson noises \( \psi_1(t) \) and \( \psi_2(t) \) in (21) are generated using the Simulink chart suggested in [3].
**Figura:** Graph of the error between the real state $x_1(t)$, satisfying (17), and the optimal filter estimate $m_1(t)$, satisfying (19), in the entire simulation interval $[0, 2]$. 
Figura: Graph of the error between the real state $x_2(t)$, satisfying (17), and the optimal filter estimate $m_2(t)$, satisfying (19), in the entire simulation interval $[0, 2]$. 
**Figura:** Graph of the error between the real state $x_1(t)$, satisfying (17), and the estimate $m_{k1}(t)$, satisfying (21), in the simulation interval $[0, 1.7842]$. 
Figura: Graph of the error between the real state $x_2(t)$, satisfying (17), and the estimate $m_{k2}(t)$, satisfying (21), in the simulation interval $[0, 1,7842]$. 
* The estimates calculated by using the obtained optimal filter for a third-order polynomial state with white Poisson noise over incomplete linear observations have definitely better convergence properties than the estimates given by the conventional mean-square polynomial filter.

* On the contrary, the estimation error given by the conventional mean-square polynomial filter behaves unstably and diverges to infinity before the asymptotic time of the reference state.
References I


Optimal State Filtering and Parameter Identification for Linear Systems with Poisson Noises

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1. Introduction

2. Filtering Problem Statement

3. Optimal Filter and Identifier for Linear Systems with Unknown Parameters

4. Example
This paper presents the optimal filter and parameter identifier for linear stochastic systems with unknown multiplicative and additive parameters over linear observations, where unknown parameters are considered Poisson processes.

The filtering problem is formalized considering the unknown parameters as additional system states satisfying linear stochastic differential equations with zero drift and unit diffusion.
The original problem is reduced to the filtering problem for incompletely measured polynomial (bilinear) Poisson system states over linear observations, whose solution is obtained in the companion paper.

This presents the optimal algorithm for simultaneous state estimation and parameter identification in linear Poisson systems with unknown multiplicative and additive parameters over linear observations. The obtained optimal filter for the extended state vector also serves as the optimal identifier for the unknown parameters.
Since the original identification problem is reduced to the filtering problem for the extended system state including both state and parameters, the identifiability condition for the original system coincides with the observability condition for the extended system.

Performance of the designed optimal state filter and parameter identifier is verified for both, stable and unstable, linear uncertain systems and compared against the conventional mean-square filter designed for polynomial systems with white Gaussian noises.
Filtering Problem for Linear Systems with Unknown Parameters I

Let $(\Omega, F, P)$ be a complete probability space with an increasing right-continuous family of $\sigma$-algebras $F_t, t \geq t_0$, and let $(N_1(t), F_t, t \geq t_0)$ and $(N_2(t), F_t, t \geq t_0)$ be independent Poisson processes. The $F_t$-measurable random process $(x(t), y(t))$ is described by a linear differential equation with an unknown vector parameter $\theta(t)$ for the system state

$$dx(t) = (a_0(\theta, t) + a(\theta, t)x(t))dt + b(t)dN_1(t), \quad x(t_0) = x_0, \quad (1)$$

and a linear differential equation for the observation process

$$dy(t) = (A_0(t) + A(t)x(t))dt + B(t)dN_2(t). \quad (2)$$
Filtering Problem for Linear Systems with Unknown Parameters II

Here, $x(t) \in R^n$ is the state vector, $y(t) \in R^m$ is the observation process, $m \leq n$, and $\theta(t) \in R^p$, $p \leq n \times n + n$, is the vector of unknown entries of matrix $a(\theta, t)$ and unknown components of vector $a_0(\theta, t)$. The latter means that both structures contain unknown components $a_{0i}(t) = \theta_k(t), k = 1, \ldots, p_1 \leq n$ and $a_{ij}(t) = \theta_k(t), k = p_1 + 1, \ldots, p \leq n \times n + n$, as well as known components $a_{0i}(t)$ and $a_{ij}(t)$, whose values are known functions of time. The initial condition $x_0 \in R^n$ is a Poisson vector such that $x_0, N_1(t)$, and $N_2(t)$ are independent. It is assumed that $B(t)B^T(t)$ is a positive definite matrix. All coefficients in (1)–(2) are deterministic functions of time of appropriate dimensions.
Filtering Problem for Linear Systems with Unknown Parameters III

It is considered that there is no useful information on values of the unknown parameters $\theta_k(t), k = 1, \ldots, p$, and this uncertainty even grows as time tends to infinity. In other words, the unknown parameters can be modeled as $F_t$-measurable Poisson processes

$$d\theta(t) = dN_3(t),$$

with unknown initial conditions $\theta(t_0) = \theta_0 \in \mathbb{R}^p$, where $(N_3(t), F_t, t \geq t_0)$ is a Poisson process independent of $x_0$, $N_1(t)$, and $N_2(t)$. 
Filtering Problem for Linear Systems with Unknown Parameters IV

The estimation problem is to find the optimal estimate
\[ \hat{z}(t) = [\hat{x}(t), \hat{\theta}(t)] \]
of the combined vector of the system states and unknown parameters \( z(t) = [x(t), \theta(t)] \), based on the observation process \( Y(t) = \{y(s), 0 \leq s \leq t\} \). As known, this optimal estimate is given by the conditional expectation

\[ \hat{z}(t) = m(t) = E(z(t) \mid F^Y_t) \]

of the system state \( z(t) = [x(t), \theta(t)] \) with respect to the \( \sigma \)-algebra \( F^Y_t \) generated by the observation process \( Y(t) \) in the interval \([t_0, t]\). As usual, the matrix function

\[ P(t) = E[(z(t) - m(t))(z(t) - m(t))^T \mid F^Y_t] \]

is the estimation error variance.
Optimal Filter and Identifier for Linear Systems with Unknown Parameters

To apply the optimal filtering equations for the state vector $z(t) = [x(t), \theta(t)]$, governed by the equations (1) and (3), over the linear observations (2) (see the companion paper), the state equation (1) should be written in the polynomial form. For this purpose, a matrix $a_1(t) \in \mathbb{R}^{(n+p)\times(n+p)}$, a cubic tensor $a_2(t) \in \mathbb{R}^{(n+p)\times(n+p)\times(n+p)}$, and a vector $c_0(t) \in \mathbb{R}^{(n+p)}$ are introduced as follows.

The equation for the $i$-th component of the state vector is given by

$$dx_i(t) = (a_{0i}(t)+\sum_{j=1}^{n} a_{ij}(t)x_j(t))dt+\sum_{j=1}^{n} b_{ij}(t)dN_{1j}(t), \quad x_i(t_0) = x_{0i}.$$
Then:

1. If the variable $a_{0i}(t)$ is a known function, then the $i$-th component of the vector $c_0(t)$ is set to this function, $c_{0i}(t) = a_{0i}(t)$. Otherwise, if the variable $a_{0i}(t)$ is an unknown function, then the $(i, n+i)$-th entry of the matrix $a_1(t)$ is set to 1.

As noted, the number of unknown components of the vector $a_0$ is equal to $p_1 \leq n$.

2. If the variable $a_{ij}(t)$ is a known function, then the $(i,j)$-th component of the matrix $a_1(t)$ is set to this function, $a_{1ij}(t) = a_{ij}(t)$. Otherwise, if the variable $a_{ij}(t)$ is an unknown function, then the $(i, n+p_1+k, j)$-th entry of the cubic tensor $a_2(t)$ is set to 1, where $k$ is the number of this current unknown.
entry in the matrix $a_{ij}(t)$, counting the unknown entries consequently by rows from the first to $n$-th entry in each row. The number of unknown entries in the matrix $a_1$ is equal to $p - p_1 \leq n^2$, where $p_1 \leq n$ is the number of unknown components of the vector $a_0$, and $p \leq n^2 + n$ is the total number of unknown variables.

3. All other unassigned entries of the matrix $a_1(t)$, cubic tensor $a_2(t)$, and vector $c_0(t)$ are set to 0.
Using the introduced notation, the state equations (1),(3) for the vector \( z(t) = [x(t), \theta(t)] \in \mathbb{R}^{n+p} \) can be rewritten as

\[
dz(t) = (c_0(t) + a_1(t)z(t) + a_2(t)z(t)z^T(t))dt + \text{diag}[b(t), I_{p \times p}]d[N_1^T(t), N_3^T(t)]^T,
\]

\[z(t_0) = [x_0, \theta_0],\]

where the matrix \( a_1(t) \), cubic tensor \( a_2(t) \), and vector \( c_0(t) \) have already been defined, and \( I_{p \times p} \) is the \( p \times p \) - dimensional identity matrix. The equation (4) is bilinear with respect to the extended state vector \( z(t) = [x(t), \theta(t)] \).
Thus, the estimation problem is now reformulated as to find the optimal estimate $\hat{z}(t) = m(t) = [\hat{x}(t), \hat{\theta}(t)]$ for the state vector $z(t) = [x(t), \theta(t)]$, governed by the bilinear equation (4), based on the observation process $Y(t) = \{y(s), 0 \leq s \leq t\}$, satisfying the equation (2). The solution of this problem is obtained using the optimal filtering equations for bilinear-linear states with partially measured linear part over linear observations (see the companion paper) and given by

$$
dm(t) = (c_0(t) + a_1(t)m(t) + a_2(t)m(t)m^T(t) + a_2(t)P(t))dt + 
$$

$$
P(t)\begin{bmatrix} \mathbf{A}(t) \\ 0_{m \times p} \end{bmatrix}^T(B(t)B^T(t))^{-1}[dy(t) - (A_0(t) + A(t)m(t))]dt,
$$
Optimal Filter and Identifier for Linear Systems with Unknown Parameters VI

\[ m(t_0) = [E(x(t_0) \mid F_t^Y), E(\theta(t_0) \mid F_t^Y)], \]

\[ dP(t) = (a_1(t)P(t) + P(t)a_1^T(t) + 2a_2(t)m(t)P(t) + a_2(t)P(t)*1 + (a_2(t)(2m(t)P(t)+P(t)*1)))^T + (\text{diag}[b(t), I_p])(\text{diag}[b(t), I_p]^T))dt - P(t)[A(t), 0_{m\times p}]^T(B(t)B^T(t))^{-1}[A(t), 0_{m\times p}]P(t)dt, \]

\[ P(t_0) = E((z(t_0) - m(t_0))(z(t_0) - m(t_0))^T \mid F_t^Y), \]

where the vector 1 is an \( n \)-dimensional vector with all its components equal to 1; \( 0_{m\times p} \) is the \( m \times p \) - dimensional zero matrix; \( P(t) \) is the conditional variance of the estimation error.
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\[ z(t) - m(t) \text{ with respect to the observations } Y(t); \text{ and the expression } a_2(t)P(t)\ast 1 \text{ is defined as follows:} \]

\[
(a_2(t)P(t)\ast 1)_{ih} = \sum_{j,k} a_{2\ ijk}(t)P_{kj}(t)1_h
\]
Theorem 1. The optimal finite-dimensional filter for the extended state vector $z(t) = [x(t), \theta(t)]$, governed by the equation (4), over the linear observations (2) is given by the equation (5) for the optimal estimate $\hat{z}(t) = m(t) = [\hat{x}(t), \hat{\theta}(t)] = E([x(t), \theta(t)] | F_t^Y)$ and the equation (6) for the estimation error variance $P(t) = E[(z(t) - m(t))(z(t) - m(t))^T | F_t^Y]$. This filter, applied to the subvector $\theta(t)$, also serves as the optimal identifier for the vector of unknown parameters $\theta(t)$ in the equation (1), yielding the estimate subvector $\hat{\theta}(t)$ as the optimal parameter estimate.
Example I

Let the scalar system state $x(t) \in R$ satisfy the linear equation with an scalar unknown parameter $\theta \in R$

$$\dot{x}(t) = \theta x(t) + \psi_1(t), \quad x(0) = x_0,$$  \hspace{1cm} (7)

and the scalar observation process $y(t) \in R$ be given by the linear equation

$$y(t) = x(t) + \psi_2(t),$$  \hspace{1cm} (8)

where $\psi_1(t)$ and $\psi_2(t)$ are white Poisson noises, which are the weak mean square derivatives of standard Poisson processes. The equations (7),(8) present the conventional form for the equations
Example II

(1)–(2), which is actually used in practice. The parameter $\theta$ is modeled as a standard Poisson process, i.e., satisfies the equation

$$d\theta(t) = dN_3(t), \quad \theta(0) = \theta_0,$$

which can also be written as

$$\dot{\theta}(t) = \psi_3(t), \quad \theta(0) = \theta_0, \quad (9)$$

where $\psi_3(t)$ is a white Poisson noise. $\psi_1(t)$ and $\psi_3(t)$ in the state and parameter equations are assumed to be independent white Poisson noises.
Example III

The filtering problem is to find the optimal estimate for the bilinear-linear state (7),(9), using linear observations (8) confused with independent randomly driven isolated disturbances modeled as Poisson white noises. The simulation time is set to $T = 7.5$. The filtering equations (5),(6) take the following particular form for the system (7)–(9)

$$
\dot{m}_1(t) = m_1(t)m_2(t) + P_{12}(t) + P_{11}(t)[y(t) - m_1(t)], \quad (10)
$$

$$
\dot{m}_2(t) = P_{12}(t)[y(t) - m_1(t)],
$$

with the initial conditions $m_1(0) = E(x_0 \mid y(0)) = m_{10}$ and $m_2(0) = E(\theta_0 \mid y(0)) = m_{20}$,

$$
\dot{P}_{11}(t) = 1 + 2P_{12}(t) + 4P_{11}(t)m_2(t) - P_{11}^2(t), \quad (11)
$$
Example IV

\[ \dot{P}_{12}(t) = P_{12}(t) + 2P_{12}(t)m_2(t) - P_{11}(t)P_{12}(t), \]

\[ \dot{P}_{22}(t) = 1 - P_{12}^2(t), \]

with the initial condition

\[ P(0) = E(([x_0, \theta_0] - m(0))([x_0, \theta_0] - m(0))^T | y(0)) = P_0. \]
Example V

The estimates obtained upon solving the equations (10),(11) are also compared to the estimates satisfying the conventional mean-square-Gaussian polynomial filter equations for the states (7),(9) over the incomplete linear observations (8):

\[
\dot{m}_{G1}(t) = m_{G1}(t)m_{G2}(t) + P_{G12}(t) + P_{G11}(t)[y(t) - m_{G1}(t)], \quad (12)
\]

\[
\dot{m}_{G2}(t) = P_{G12}(t)[y(t) - m_{G1}(t)],
\]

with the initial conditions \(m_{G1}(0) = E(x_0 \mid y(0)) = m_{G10}\) and \(m_{G2}(0) = E(\theta_0 \mid y(0)) = m_{G20}\),

\[
\dot{P}_{G11}(t) = 1 + 4P_{G11}(t)m_{G2}(t) - P_{G11}^2(t), \quad (13)
\]

\[
\dot{P}_{G12}(t) = 2P_{G12}(t)m_{G2}(t) - P_{G11}(t)P_{G12}(t),
\]

\[
\dot{P}_{G22}(t) = 1 - P_{G12}^2(t),
\]
Example VI

Numerical simulation results are obtained solving the system of filtering equations (10),(11) and (12),(13). The obtained values of $m_1(t)$, $m_2(t)$, $m_{G1}(t)$, and $m_{G2}(t)$ satisfying the equations (10), and (12), respectively, are compared to the real values of the state variable $x(t)$ and parameter $\theta(t)$ in (7),(9).

For each of the two filters (10),(11) and (12),(13), and the reference system (7)–(9) involved in simulation, the following initial values are assigned: $x_0 = 5000$, $m_{10} = 0.1$, $m_{20} = 0$, $P_{110} = P_{220} = 100$, $P_{120} = 10$. The unknown parameter $\theta$ is assigned as $\theta = 0.1$ in the first simulation and as $\theta = -0.1$ in the second one, thus considering the system (7) unstable and stable, respectively.
Example VII

The following graphs are obtained: graphs of the reference state variable \( x(t) \), and the optimal state estimate \( m_1(t) \) and parameter estimate \( m_2(t) \), satisfying the equation (10), in the unstable (\( \theta = 0.1 \)) and stable (\( \theta = -0.1 \)) cases, respectively, are shown in Figs. 1 and 2; graphs of the reference state variable \( x(t) \), and the conventional mean-square-Gaussian filter estimates \( m_{G1}(t) \) and \( m_{G2}(t) \), satisfying the equation (12), in the unstable (\( \theta = 0.1 \)) and stable (\( \theta = -0.1 \)) cases, respectively, are shown in Figs. 3 and 4. The graphs of all those variables are shown in the entire simulation interval from \( t_0 = 0 \) to \( T = 7.5 \).
Example VIII

![Graph showing real state value and optimal state estimate over time.](image-url)
Example IX

Figure: Graphs of the reference state variable $x(t)$ (thick line above), optimal state estimate $m_1(t)$ (thin line above) and optimal parameter estimate $m_2(t)$ (thick line below), satisfying (10), for the unstable system (7) in the simulation interval $[0, 7.5]$. 
Example X

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Figure: Graphs of the reference state variable $x(t)$ (thick line above), optimal state estimate $m_1(t)$ (thin line above) and optimal parameter estimate $m_2(t)$ (thick line below), satisfying (10), for the stable system (7) in the simulation interval $[0, 7.5]$. 
Example XII

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Figure: Graphs of the reference state variable $x(t)$ (thick line above), optimal state estimate $m_{G1}(t)$ (thin line above) and optimal parameter estimate $m_{G2}(t)$ (thick line below), satisfying (12) for the unstable system (7) in the simulation interval $[0, 7.5]$. 
Example XIV

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Example XV

**Figure:** Graphs of the reference state variable $x(t)$ (thick line above), optimal state estimate $m_{G1}(t)$ (thin line above) and optimal parameter estimate $m_{G2}(t)$ (thick line below), satisfying (12), for the stable system (7) in the simulation interval $[0, 7.5]$. 
Example XVI

It can be observed that, in both cases, the state estimate \( m_1(t) \) converges to the real state \( x(t) \) and the parameter estimate \( m_2(t) \) converges to the real value (0.1 or -0.1) of the unknown parameter \( \theta(t) \) rapidly, in less than 7.5 time units. This behavior can be classified as very reliable, especially taking into account large deviations in the initial values for the real state and its estimate and large values of the initial error variances.
Another advantage to be mentioned is that the designed filter and parameter identifier work equally well for stable and unstable systems, which correspond to operation of linear systems in nominal conditions and under persistent external disturbances, respectively. On the contrary, it can be observed that the state estimate $m_{G1}(t)$ does not even approximate the real state $x(t)$ at the time $T = 7.5$ in both cases, for the stable and unstable system (7), and the parameter estimate $m_{G2}(t)$ does not converge to the real value (0.1 or −0.1) of the unknown parameter $\theta(t)$. 
Conclusions I

This paper presents the optimal filtering and parameter identification problem for linear stochastic systems with unknown multiplicative and additive parameters over linear observations with an invertible observation matrix, where unknown parameters are considered Poisson processes.

The simulation results show very reliable behavior of the designed state filter and parameter identifier. On the contrary, the estimates given by the conventional state filter and parameter identifier for systems with Gaussian noises diverge from the real state and parameter values.