Appendix A. MODEL DESCRIPTION

This appendix presents the comprehensive expressions derived to establish the model of the crane's mechanical systems. The following sections are arranged in a stepwise manner starting by establishing the kinematics, next the Lagrangian and it's derivatives and finally the generalized forces. These are the three main steps to go through in order to obtain the equations of motion presented in (**??**). The components of this equation system are given section A.4.
The appendix also contains expression for the disturbance model and tool point kinematics. The parameters for crane model are given in Table A.1.

Table A.1. Model parameters

Par.	Value	Description
y_A	$15.84m$	Global distance
x_D	$1.75m$	Global distance
y_D	$13.07m$	Global distance
l_1	$10.89m$	Local distance, point A - CoG 1
l_{AB}	$23.5m$	Local distance
l_{AE}	$6m$	Local distance
l_{AF}	$17.15m$	Local distance
l_2	$7.6m$	Local distance, point B - CoG 2
l_{BC}	$18.5m$	Local distance
l_{BG}	$2.3m$	Local distance
m_1	$17475kg$	Mass of inner jib
m_1	$10195kg$	Mass of outer jib
m_{load}	$10000kg$	Mass of payload
β_e	$15000bar$	Effective oil stiffness
Q_{max}	$900l/min$	Max. flow pr. control valve
A_{p_1}	$160850mm^2$	Piston area of cylinder
CR_1	0.53	Cylinder area ratio
$l_{1_{min}}$	$5.075m$	Minimum length of cylinder
h_1	$3.975m$	Stroke of cylinder
A_{p_2}	$141370mm^2$	Piston area of cylinder
CR_2	0.56	Cylinder area ratio
$l_{2_{min}}$	$4.67m$	Minimum length of cylinder
h_2	$3.57m$	Stroke of cylinder

A.1 Kinematics

For the kinetic energy:

$$\mathbf{r}_1 = \begin{bmatrix} l_1 \cdot \cos\phi_1 \\ y_A + l_1 \cdot \sin\phi_1 \end{bmatrix} \tag{A.1}$$

$$\dot{\mathbf{r}}_1 = \begin{bmatrix} -\dot{\phi}_1 \cdot l_1 \cdot \sin\phi_1 \\ \dot{\phi}_1 \cdot l_1 \cdot \cos\phi_1 \end{bmatrix} \tag{A.2}$$

$$\mathbf{r}_2 = \begin{bmatrix} l_{AB} \cdot \cos\phi_1 + l_2 \cdot \cos\phi_2 \\ y_A + l_{AB} \cdot \sin\phi_1 + l_2 \cdot \sin\phi_2 \end{bmatrix} \tag{A.3}$$

$$\dot{\mathbf{r}}_2 = \begin{bmatrix} -\dot{\phi}_1 \cdot l_{AB} \cdot \sin\phi_1 - \dot{\phi}_2 \cdot l_2 \cdot \sin\phi_2 \\ \dot{\phi}_1 \cdot l_{AB} \cdot \cos\phi_1 + \dot{\phi}_2 \cdot l_2 \cdot \cos\phi_2 \end{bmatrix} \tag{A.4}$$

$$\mathbf{r}_{load} = \begin{bmatrix} l_{AB} \cdot \cos\phi_1 + l_{BC} \cdot \cos\phi_2 \\ y_A + l_{AB} \cdot \sin\phi_1 + l_{BC} \cdot \sin\phi_2 \end{bmatrix} \tag{A.5}$$

$$\dot{\mathbf{r}}_{load} = \begin{bmatrix} -\dot{\phi}_1 \cdot l_{AB} \cdot \sin\phi_1 - \dot{\phi}_2 \cdot l_{BC} \cdot \sin\phi_2 \\ \dot{\phi}_1 \cdot l_{AB} \cdot \cos\phi_1 + \dot{\phi}_2 \cdot l_{BC} \cdot \cos\phi_2 \end{bmatrix} \tag{A.6}$$

For the generalized forces:

$$\mathbf{r}_E = \begin{bmatrix} l_{AE} \cdot \cos\phi_1 \\ y_A + l_{AE} \cdot \sin\phi_1 \end{bmatrix} \tag{A.7}$$

$$\mathbf{r}_F = \begin{bmatrix} l_{AF} \cdot \cos\phi_1 \\ y_A + l_{AF} \cdot \sin\phi_1 \end{bmatrix} \tag{A.8}$$

$$\mathbf{r}_G = \begin{bmatrix} l_{AB} \cdot \cos\phi_1 + l_{BG} \cdot \cos\phi_2 \\ y_A + l_{AB} \cdot \sin\phi_1 + l_{BG} \cdot \sin\phi_2 \end{bmatrix} \tag{A.9}$$

For the pressure gradients:

$$\dot{\mathbf{r}}_E = \begin{bmatrix} -\dot{\phi}_1 \cdot l_{AE} \cdot \sin\phi_1 \\ \dot{\phi}_1 \cdot l_{AE} \cdot \cos\phi_1 \end{bmatrix} \tag{A.10}$$

$$\dot{\mathbf{r}}_F = \begin{bmatrix} -\dot{\phi}_1 \cdot l_{AF} \cdot \sin\phi_1 \\ \dot{\phi}_1 \cdot l_{AF} \cdot \sin\phi_1 \end{bmatrix} \tag{A.11}$$

$$\dot{\mathbf{r}}_G = \begin{bmatrix} -\dot{\phi}_1 \cdot l_{AB} \cdot \sin\phi_1 - \dot{\phi}_2 \cdot l_{BG} \cdot \sin\phi_2 \\ -\dot{\phi}_1 \cdot l_{AB} \cdot \cos\phi_1 - \dot{\phi}_2 \cdot l_{BG} \cdot \cos\phi_2 \end{bmatrix} \tag{A.12}$$

$$\mathbf{r}_{DE} = \mathbf{r}_E - \mathbf{r}_D \tag{A.13}$$

$$\dot{\mathbf{r}}_{DE} = \dot{\mathbf{r}}_E \tag{A.14}$$

$$v_1 = \frac{1}{|\mathbf{r}_{DE}|} \cdot \mathbf{r}_{DE}^T \cdot \dot{\mathbf{r}}_{DE} \tag{A.15}$$

$$\mathbf{r}_{FG} = \mathbf{r}_G - \mathbf{r}_F \tag{A.16}$$

$$\dot{\mathbf{r}}_{FG} = \dot{\mathbf{r}}_G - \dot{\mathbf{r}}_F \tag{A.17}$$

$$v_2 = \frac{1}{|\mathbf{r}_{FG}|} \cdot \mathbf{r}_{FG}^T \cdot \dot{\mathbf{r}}_{FG} \tag{A.18}$$

A.2 The Lagrangian

Equations (A.2), (A.4) and (A.6) are used to obtain the kinetic energy for the Lagrangian:

$$\begin{aligned} L &= T_1 + T_2 + T_{load} - V_1 - V_2 + V_{load} \\ &= \frac{1}{2} \cdot m_1 \cdot (\dot{\mathbf{r}}_1^T \cdot \dot{\mathbf{r}}_1) + \frac{1}{2} \cdot m_2 \cdot (\dot{\mathbf{r}}_2^T \cdot \dot{\mathbf{r}}_2) + \\ &\quad \frac{1}{2} \cdot m_{load} \cdot (\dot{\mathbf{r}}_{load}^T \cdot \dot{\mathbf{r}}_{load}) - \\ &\quad m_1 \cdot g \cdot (y_A + l_1 \cdot \sin\phi_1) - \\ &\quad m_2 \cdot g \cdot (y_A + l_{AB} \cdot \sin\phi_1 + l_2 \cdot \sin\phi_2) - \\ &\quad m_{load} \cdot g \cdot (y_A + l_{AB} \cdot \sin\phi_1 + l_{BC} \cdot \sin\phi_2) \end{aligned} \tag{A.19}$$

The partial derivatives of the Lagrangian:

$$\frac{\partial L}{\partial \dot{\phi}_1} = \dot{\phi}_1 \cdot l_1^2 \cdot m_1 + \dot{\phi}_1 \cdot l_{AB}^2 \cdot m_2 + \dot{\phi}_1 \cdot l_{AB}^2 \cdot m_{load} + \dot{\phi}_2 \cdot l_{AB} \cdot l_2 \cdot m_2 \cdot \cos(\phi_1 - \phi_2) + \dot{\phi}_2 \cdot l_{AB} \cdot l_{BC} \cdot m_{load} \cdot \cos(\phi_1 - \phi_2) \tag{A.20}$$

$$\frac{\partial L}{\partial \phi_1} = -g \cdot l_1 \cdot m_1 \cdot \cos\phi_1 - g \cdot l_{AB} \cdot m_2 \cdot \cos\phi_1 - g \cdot l_{AB} \cdot m_{load} \cdot \cos\phi_1 - \dot{\phi}_1 \cdot \dot{\phi}_2 \cdot l_{AB} \cdot l_2 \cdot m_2 \cdot \sin(\phi_1 - \phi_2) - \dot{\phi}_1 \cdot \dot{\phi}_2 \cdot l_{AB} \cdot l_{BG} \cdot m_{load} \cdot \sin(\phi_1 - \phi_2) \tag{A.21}$$

$$\frac{\partial L}{\partial \dot{\phi}_2} = \dot{\phi}_2 \cdot l_2^2 \cdot m_2 + \dot{\phi}_2 \cdot l_{BC}^2 \cdot m_{load} + \dot{\phi}_1 \cdot l_{AB} \cdot l_2 \cdot m_2 \cdot \cos(\phi_1 - \phi_2) + \dot{\phi}_1 \cdot l_{AB} \cdot l_{BC} \cdot m_{load} \cdot \cos(\phi_1 - \phi_2) \tag{A.22}$$

$$\frac{\partial L}{\partial \phi_2} = -g \cdot l_2 \cdot m_2 \cdot \cos\phi_2 - g \cdot l_{BC} \cdot \cos\phi_2 + \dot{\phi}_1 \cdot \dot{\phi}_2 \cdot l_{AB} \cdot l_2 \cdot m_2 \cdot \sin(\phi_1 - \phi_2) + \dot{\phi}_1 \cdot \dot{\phi}_2 \cdot l_{AB} \cdot l_{BC} \cdot m_{load} \cdot \sin(\phi_1 - \phi_2) \tag{A.23}$$

The time derivatives of the partial derivatives:

$$dt\left\{\frac{\partial L}{\partial \dot{\phi}_1}\right\} = \ddot{\phi}_1 \cdot l_1^2 \cdot m_1 + \ddot{\phi}_1 \cdot l_{AB}^2 \cdot m_2 + \ddot{\phi}_1 \cdot l_{AB}^2 \cdot m_{load} + \ddot{\phi}_2 \cdot l_{AB} \cdot l_2 \cdot m_2 \cdot \cos(\phi_1 - \phi_2) - (\dot{\phi}_1 - \dot{\phi}_2) \cdot \dot{\phi}_2 \cdot l_{AB} \cdot l_2 \cdot m_2 \cdot \sin(\phi_1 - \phi_2) + \ddot{\phi}_2 \cdot l_{AB} \cdot l_{BC} \cdot m_{load} \cdot \cos(\phi_1 - \phi_2) \; (\dot{\phi}_1 - \dot{\phi}_2) \cdot \dot{\phi}_2 \cdot l_{AB} \cdot l_{BC} \cdot m_2 \cdot \sin(\phi_1 - \phi_2) \tag{A.24}$$

$$dt\left\{\frac{\partial L}{\partial \dot{\phi}_2}\right\} = \ddot{\phi}_2 \cdot l_2^2 \cdot m_2 + \ddot{\phi}_2 \cdot l_{BC}^2 \cdot m_{load} + \ddot{\phi}_1 \cdot l_{AB} \cdot l_2 \cdot m_2 \cdot \cos(\phi_1 - \phi_2) - (\dot{\phi}_1 - \dot{\phi}_2) \cdot \dot{\phi}_1 \cdot l_{AB} \cdot l_2 \cdot m_2 \cdot \sin(\phi_1 - \phi_2) + \ddot{\phi}_1 \cdot l_{AB} \cdot l_{BC} \cdot m_{load} \cdot \cos(\phi_1 - \phi_2) - (\dot{\phi}_1 - \dot{\phi}_2) \cdot \dot{\phi}_1 \cdot l_{AB} \cdot l_{BC} \cdot m_{load} \cdot \sin(\phi_1 - \phi_2) \tag{A.25}$$

A.3 Generalized Forces

The force vectors are expressed by means of θ_1 and θ_2:

$$\mathbf{F}_E = \begin{bmatrix} F_1 \cdot \cos\theta_1 \\ F_1 \cdot \sin\theta_1 \end{bmatrix} \tag{A.26}$$

$$\mathbf{F}_F = \begin{bmatrix} -F_2 \cdot \cos\theta_2 \\ -F_2 \cdot \sin\theta_2 \end{bmatrix} \tag{A.27}$$

$$\mathbf{F}_G = \begin{bmatrix} F_2 \cdot \cos\theta_2 \\ F_2 \cdot \sin\theta_2 \end{bmatrix} \tag{A.28}$$

where

$$F_1 = A_{p_1} \cdot (p_1 - CR_1 \cdot p_{r_1}) \tag{A.29}$$

$$F_2 = A_{p_2} \cdot (p_2 - CR_2 \cdot p_{r_2}) \tag{A.30}$$

are the forces produced by the cylinders.
The partial derivatives of the three position vectors from (A.7)-(A.9):

$$\frac{\partial \mathbf{r}_E}{\partial \phi_1} = \begin{bmatrix} -l_{AE} \cdot \sin\phi_1 \\ l_{AE} \cdot \cos\phi_1 \end{bmatrix} \tag{A.31}$$

$$\frac{\partial \mathbf{r}_E}{\partial \phi_2} = 0 \tag{A.32}$$

$$\frac{\partial \mathbf{r}_F}{\partial \phi_1} = \begin{bmatrix} -l_{AF} \cdot \sin\phi_1 \\ l_{AF} \cdot \cos\phi_1 \end{bmatrix} \tag{A.33}$$

$$\frac{\partial \mathbf{r}_F}{\partial \phi_2} = 0 \tag{A.34}$$

$$\frac{\partial \mathbf{r}_G}{\partial \phi_1} = \begin{bmatrix} -l_{AB} \cdot \sin\phi_1 \\ l_{AB} \cdot \cos\phi_1 \end{bmatrix} \tag{A.35}$$

$$\frac{\partial \mathbf{r}_G}{\partial \phi_2} = \begin{bmatrix} -l_{BG} \cdot \sin\phi_2 \\ l_{BG} \cdot \cos\phi_2 \end{bmatrix} \tag{A.36}$$

The generalized forces:

$$Q_1 = \mathbf{F}_E^T \cdot \frac{\partial \mathbf{r}_E}{\partial \phi_1} + \mathbf{F}_F^T \cdot \frac{\partial \mathbf{r}_F}{\partial \phi_1} + \mathbf{F}_G^T \cdot \frac{\partial \mathbf{r}_G}{\partial \phi_1} \tag{A.37}$$

$$Q_2 = \mathbf{F}_E^T \cdot \frac{\partial \mathbf{r}_E}{\partial \phi_2} + \mathbf{F}_F^T \cdot \frac{\partial \mathbf{r}_F}{\partial \phi_2} + \mathbf{F}_G^T \cdot \frac{\partial \mathbf{r}_G}{\partial \phi_2} \tag{A.38}$$

A.4 Equations of Motion

The components of the equations of motion:

$$a_{11} = l_1^2 \cdot m_1 + l_{AB}^2 \cdot m_2 + l_{AB}^2 \cdot m_{load} \tag{A.39}$$

$$a_{12} = l_{AB} \cdot l_2 \cdot m_2 \cdot \cos(\phi_1 - \phi_2) + l_{AB} \cdot l_{BC} \cdot m_{load} \cdot \cos(\phi_1 - \phi_2) \tag{A.40}$$

$$a_{21} = l_{AB} \cdot l_2 \cdot m_2 \cdot \cos(\phi_1 - \phi_2) + l_{AB} \cdot l_{BC} \cdot m_{load} \cdot \cos(\phi_1 - \phi_2) \tag{A.41}$$

$$a_{22} = l_2^2 \cdot m_2 + l_{BC}^2 \cdot m_{load} \tag{A.42}$$

$$\begin{aligned} b_1 = &-\dot{\phi}_2^2 \cdot l_{AB} \cdot l_2 \cdot m_2 \cdot \sin(\phi_1 - \phi_2) - \dot{\phi}_2^2 \cdot l_{AB} \cdot l_{BC} \cdot m_{load} \cdot \sin(\phi_1 - \phi_2) - g \cdot l_1 \cdot m_1 \cdot \cos\phi_1 - g \cdot l_{AB} \cdot m_2 \cdot \cos\phi_1 - g \cdot l_{AB} \cdot m_{load} \cdot \cos\phi_1 - \\ &A_{p_1} \cdot (p_1 - CR_1 \cdot p_{r_1}) \cdot \cos\theta_1 \cdot l_{AE} \cdot \sin\phi_1 + A_{p_1} \cdot (p_1 - CR_1 \cdot p_{r_1}) \cdot \sin\theta_1 \cdot l_{AE} \cdot \cos\phi_1 + \\ &A_{p_2} \cdot (p_2 - CR_2 \cdot p_{r_2}) \cdot \cos\theta_2 \cdot l_{AF} \cdot \sin\phi_1 - A_{p_2} \cdot (p_2 - CR_2 \cdot p_{r_2}) \cdot \sin\theta_2 \cdot l_{AF} \cdot \cos\phi_1 - \\ &A_{p_2} \cdot (p_2 - CR_2 \cdot p_{r_2}) \cdot \cos\theta_2 \cdot l_{AB} \cdot \sin\phi_1 + A_{p_2} \cdot (p_2 - CR_2 \cdot p_{r_2}) \cdot \sin\theta_2 \cdot l_{AB} \cdot \cos\phi_1 \end{aligned} \tag{A.43}$$

$$\begin{aligned} b_2 = &\dot{\phi}_1^2 \cdot l_{AB} \cdot l_2 \cdot m_2 \cdot \sin(\phi_1 - \phi_2) + \dot{\phi}_1^2 \cdot l_{AB} \cdot l_{BC} \cdot m_{load} \cdot \sin(\phi_1 - \phi_2) - g \cdot l_2 \cdot m_2 \cdot \cos\phi_2 - g \cdot l_{BC} \cdot m_{load} \cdot \cos\phi_2 - \\ &A_{p_2} \cdot (p_2 - CR_2 \cdot p_{r_2}) \cdot \cos\theta_2 \cdot l_{BG} \cdot \sin\phi_2 + A_{p_2} \cdot (p_2 - CR_2 \cdot p_{r_2}) \cdot \sin\theta_2 \cdot l_{BG} \cdot \cos\phi_2 \end{aligned} \tag{A.44}$$

A.5 Generalized Disturbance Forces

The disturbance forces:

$$\delta\mathbf{F}_1 = \begin{bmatrix} 0 \\ \ddot{y}_h(t) \cdot m_1 \end{bmatrix} \tag{A.45}$$

$$\delta\mathbf{F}_2 = \begin{bmatrix} 0 \\ \ddot{y}_h(t) \cdot m_2) \end{bmatrix} \tag{A.46}$$

$$\delta\mathbf{F}_{load} = \begin{bmatrix} 0 \\ \ddot{y}_h(t) \cdot m_{load} \end{bmatrix} \tag{A.47}$$

The partial derivatives of the three position vectors:

$$\frac{\partial \mathbf{r}_1}{\partial \phi_1} = \begin{bmatrix} -l_1 \cdot \sin\phi_1 \\ l_1 \cdot \cos\phi_1 \end{bmatrix} \tag{A.48}$$

$$\frac{\partial \mathbf{r}_1}{\partial \phi_2} = 0 \tag{A.49}$$

$$\frac{\partial \mathbf{r}_2}{\partial \phi_1} = \begin{bmatrix} -l_{AB} \cdot \sin\phi_1 \\ l_{AB} \cdot \cos\phi_1 \end{bmatrix} \tag{A.50}$$

$$\frac{\partial \mathbf{r}_2}{\partial \phi_2} = \begin{bmatrix} -l_2 \cdot \sin\phi_2 \\ l_2 \cdot \cos\phi_2 \end{bmatrix} \tag{A.51}$$

$$\frac{\partial \mathbf{r}_{load}}{\partial \phi_1} = \begin{bmatrix} -l_{AB} \cdot \sin\phi_1 \\ l_{AB} \cdot \cos\phi_1 \end{bmatrix} \tag{A.52}$$

$$\frac{\partial \mathbf{r}_{load}}{\partial \phi_2} = \begin{bmatrix} -l_{BC} \cdot \sin\phi_2 \\ l_{BC} \cdot \cos\phi_2 \end{bmatrix} \tag{A.53}$$

The disturbance forces:

$$\delta b_1 = \delta\mathbf{F}_1^T \cdot \frac{\partial \mathbf{r}_1}{\partial \phi_1} + \delta\mathbf{F}_2^T \cdot \frac{\partial \mathbf{r}_2}{\partial \phi_1} + \delta\mathbf{F}_{load} \cdot \frac{\partial \mathbf{r}_{load}}{\partial \phi_1} \tag{A.54}$$

$$\delta b_2 = \delta\mathbf{F}_1^T \cdot \frac{\partial \mathbf{r}_1}{\partial \phi_2} + \delta\mathbf{F}_2^T \cdot \frac{\partial \mathbf{r}_2}{\partial \phi_2} + \delta\mathbf{F}_{load} \cdot \frac{\partial \mathbf{r}_{load}}{\partial \phi_2} \tag{A.55}$$

A.6 Tool Point Position

$$X_{tp} = l_{AB} \cdot \cos\phi_1 + l_{BC} \cdot \cos\phi_2 \tag{A.56}$$

$$Y_{tp} = y_A + l_{AB} \cdot \sin\phi_1 + l_{BC} \cdot \sin\phi_2 \tag{A.57}$$