JOHNSON’S PROJECTION, KALTON’S PROPERTY \((M^*)\), AND \(M\)-IDEALS OF COMPACT OPERATORS

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Abstract. Let \(X\) and \(Y\) be Banach spaces. We give a “non-separable” proof of the Kalton-Werner-Lima-Oja theorem that the subspace \(K(X, X)\) of compact operators forms an \(M\)-ideal in the space \(L(X, X)\) of all continuous linear operators from \(X\) to \(X\) if and only if \(X\) has Kalton’s property \((M^*)\) and the metric compact approximation property. Our proof is a quick consequence of two main results. First, we describe how Johnson’s projection \(P\) on \(L(X, Y)^*\) applies to \(f \in L(X, Y)^*\) when \(f\) is represented via a Borel (with respect to the relative weak* topology) measure on \(B_{X^*} \otimes B_{Y^*}^{w^*} \subseteq L(X, Y)^*\): If \(Y^*\) has the Radon-Nikodým property, then \(P\) “passes under the integral sign”. Our basic theorem en route to this description — a structure theorem for Borel probability measures on \(B_{X^*} \otimes B_{Y^*}^{w^*}\) — also yields a description of \(K(X, Y)^*\) due to Feder and Saphar. Second, we show that property \((M^*)\) for \(X\) is equivalent to every functional in \(B_{X^*} \otimes B_{X^*}^{w^*}\) behaving as if \(K(X, X)\) were an \(M\)-ideal in \(L(X, X)\).

1. Introduction

Throughout this paper, \(X\) and \(Y\) will be Banach spaces over the same scalar field \(K\) where \(K = \mathbb{R}\) or \(K = \mathbb{C}\). The closed unit ball and the unit sphere of \(X\) will be denoted, respectively, by \(B_X\) and \(S_X\), and \(\overline{B}(x, r)\) is the closed ball in \(X\) with center \(x\) and radius \(r\). For a set \(A \subseteq X\), we denote its convex hull by \(\text{co } A\), and its linear span by \(\text{span } A\). The symbol \(L(X, Y)\) will stand for the space of continuous linear operators from \(X\) to \(Y\), and \(K(X, Y)\) for its subspace of compact operators. We shall write \(L(X)\) and \(K(X)\) instead of \(L(X, X)\) and \(K(X, X)\), respectively. The identity operator on \(X\) will be denoted by \(I_X\) or simply by \(I\).

According to the terminology in [GKS], a closed subspace \(Z\) of \(X\) is said to be an ideal in \(X\) if there exists a continuous linear projection \(P\) on \(X^*\) with \(\ker P = Z^\perp = \{x^* \in X^*: x^*|_Z = 0\}\) and \(\|P\| = 1\). It is straightforward to verify that if \(Z\) is an ideal in \(X\), then, for every \(x^* \in X^*\), the functional

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$Px^* \in X^*$ is a norm-preserving extension of the restriction $x^*|Z \in Z^*$. If the ideal projection $P$ satisfies $\|x^*\| = \|Px^*\| + \|x^* - Px^*\|$ for all $x^* \in X^*$, then $Z$ is said to be an $M$-ideal in $X$ (for $M$-ideals, see the monograph [HWW]).

The space $X$ is said to have the metric compact approximation property (shortly, MCAP) if there is a net $(K_\alpha)$ in $B_{K(X)}$ such that $\lim_{\alpha} K_\alpha x = x$ for all $x \in X$. The net $(K_\alpha)$ is called a metric compact approximation of the identity (shortly, MCAI). If also $\lim_{\alpha} K_\alpha^* x^* = x^*$ for all $x^* \in X^*$, then $(K_\alpha)$ is called a shrinking MCAI, and $X$ is said to have the shrinking MCAP.

Note that (see [J], proof of Lemma 1) if $(K_\alpha)$ is any weak$^*$ convergent (in $K(X)^{\ast \ast}$) MCAI of $Y$, then $K(X,Y)$ is an ideal in $L(X,Y)$ with respect to the Johnson projection $P$ on $L(X,Y)^*$ defined by

$$ (1.1) \quad Pf(T) = \lim_{\alpha} f(K_\alpha T), \quad T \in L(X,Y), \ f \in L(X,Y)^*. $$

The space $X$ is said to have property $(M^*)$ (see [HWW], page 296), if whenever $x^*, u^* \in X^*$, $\|u^*\| \leq \|x^*\|$, and $(x^*_\alpha) \subset X^*$ is a bounded net such that $x^*_\alpha \rightharpoonup 0$, one has

$$ \limsup_{\alpha} \\|u^* + x^*_\alpha\| \leq \limsup_{\alpha} \\|x^* + x^*_\alpha\|. $$

The following Kalton-Werner-Lima-Oja theorem characterizes $M$-ideals of compact operators on $X$.

**Theorem 1.1.** The following assertions are equivalent.

(i) $K(X)$ is an $M$-ideal in $L(X)$.

(ii) $X$ has property $(M^*)$ and the MCAP.

Property $(M^*)$ (in its sequential form) was introduced in [K] where it was proven that, for separable $X$, $K(X)$ is an $M$-ideal in $L(X)$ if and only if $X$ has property $(M^*)$ and a very strong form of the MCAP; this result was extended to the nonseparable case in [O1]. In [KW], Theorem 1.1 was proven for separable $X$, a simpler proof was given in [L]. Finally, in [O2], it was shown that $K(X)$ is an $M$-ideal in $L(X)$ if and only if $K(Z)$ is an $M$-ideal in $L(Z)$ for all separable closed subspaces $Z$ of $X$ having the MCAP (a somewhat simpler proof can be modeled after [P]), thus proving Theorem 1.1 also in the general case (note that if $X$ has property $(M^*)$, then also every closed subspace of $X$ has property $(M^*)$; moreover, $X$ has property $(M^*)$ if and only if every separable closed subspace of $X$ has property $(M^*)$ (see [O3])). The shortest known proof of Theorem 1.1 is given in [O3].
The aim of this paper is to give a direct “non-separable” proof of Theorem 1.1. We develop ideas from [L] and [O3].

Let us fix some more notation, point out some observations, and agree in some conventions.

Recall that, for $x^{**} \in X^{**}$ and $y^* \in Y^*$, the functional $x^{**} \otimes y^* \in \mathcal{L}(X,Y)^*$ is defined by $(x^{**} \otimes y^*)(T) = x^{**}(T^*y^*)$, $T \in \mathcal{L}(X,Y)$. Denote

$$B_{X^{**}} \otimes B_{Y^*} = \{x^{**} \otimes y^* : x^{**} \in B_{X^{**}}, y^* \in B_{Y^*}\} \subset \mathcal{L}(X,Y)^*.$$ 

Let $\phi \in \overline{B_{X^{**}} \otimes B_{Y^*}}^{w^*}$. Observe that $\phi_{|K(X,Y)} = x^{**} \otimes y^*_{|K(X,Y)}$ for some $x^{**} \otimes y^* \in B_{X^{**}} \otimes B_{Y^*}$. Moreover, if $\phi_{|K(X,Y)} \neq 0$, and $\tilde{x}^{**} \in X^{**}$ and $\tilde{y}^* \in Y^*$ are such that $\phi_{|K(X,Y)} = \tilde{x}^{**} \otimes \tilde{y}^*\big|_{K(X,Y)}$, then $\tilde{x}^{**} = \alpha x^{**}$ and $\tilde{y}^* = \frac{1}{\alpha} y^*$ for some $\alpha \in \mathbb{K}$. Thus the functional $g_{\phi} := x^{**} \otimes y^* \in \mathcal{L}(X,Y)^*$ is well-defined.

Let us make the convention that, unless explicitly stated otherwise, whenever considering topological properties (such as compactness, openness, Borelness) of subsets of the sets $\overline{B_{X^{**}} \otimes B_{Y^*}}^{w^*} \subset B_{\mathcal{L}(X,Y)^*}$, $B_{X^{**}}$, and $B_{Y^*}$, the topology we have in mind is the relative weak* topology of the respective set.

Since, for every $T \in \mathcal{L}(X,Y)$, there is some $\phi \in C := \overline{B_{X^{**}} \otimes B_{Y^*}}^{w^*} \subset B_{\mathcal{L}(X,Y)^*}$ such that $\text{Re} \langle \phi, T \rangle = \|T\|$, by the Hahn-Banach separation theorem, it quickly follows that $\overline{C}^{w^*} = B_{\mathcal{L}(X,Y)^*}$. Thus, for every $f \in \mathcal{L}(X,Y)^*$, as a consequence of the Riesz representation theorem, there is a regular Borel probability measure $\mu$ on $C$ such that $f(T) = \int_C \phi(T) \, d\mu(\phi)$, $T \in \mathcal{L}(X,Y)$. In Section 2, we prove the following characterization of Johnson’s projection.

**Theorem 1.2.** Let $Y^*$ have the Radon-Nikodým property, let $Y$ have the shrinking MCAP with $(K_\alpha) \subset B_{K(Y)}$, being a weak* convergent (in $K(Y)^{**}$) shrinking MCAI, and let $\mu$ be a regular Borel (with respect to the relative weak* topology) probability measure on $C := \overline{B_{X^{**}} \otimes B_{Y^*}}^{w^*} \subset B_{\mathcal{L}(X,Y)^*}$.

Then there is a Borel set $C' \subset C$ such that

(a) $\int_{C \setminus C'} |\phi(S)| \, d\mu(\phi) = 0$ for all $S \in K(X,Y)$;

(b) for every $T \in \mathcal{L}(X,Y)$, the function $C \ni \phi \mapsto g_{\phi}(T) \chi_{C'}(\phi) \in \mathbb{K}$ is measurable;

(c) letting $P$ be the Johnson projection defined by (1.1), and defining $f \in \mathcal{L}(X,Y)^*$ by $f(T) = \int_{C'} \phi(T) \, d\mu(\phi)$, $T \in \mathcal{L}(X,Y)$, one has

$$Pf(T) = \int_{C'} g_{\phi}(T) \, d\mu(\phi) = \int_{C'} P\phi(T) \, d\mu(\phi), \quad T \in \mathcal{L}(X,Y).$$

If $K(X,Y)$ were an $M$-ideal in $\mathcal{L}(X,Y)$, then, for any $\phi \in \overline{B_{X^{**}} \otimes B_{Y^*}}^{w^*}$, one would have $\|g_{\phi}\| + \|\phi - g_{\phi}\| \leq 1$. In Section 3, we prove the following theorem revealing the essence of property $(M^*)$: Every $\phi \in \overline{B_{X^{**}} \otimes B_{X^{**}}}$
behaves, in a sense, like it would do if \( \mathcal{K}(X) \) were an \( M \)-ideal in \( \mathcal{L}(X) \). We denote \( \mathcal{L} := \text{span}(\mathcal{K}(X) \cup \{I\}) \subset \mathcal{L}(X) \) and, for \( f \in \mathcal{L}(X)^* \), \( \|f\|_\mathcal{L} := \|f|_\mathcal{L} \).

**Theorem 1.3.** The following assertions are equivalent.

(i) \( X \) has property \((M^*)\).

(ii) For every \( \phi \in B_{X^*} \otimes B_{X^*}^w \), one has \( \|g_\phi\| + \|\phi - g_\phi\| \leq 1 \).

(iii) For every \( \phi \in B_{X^*} \otimes B_{X^*}^w \), one has \( \|g_\phi\| + \|\phi - g_\phi\|_\mathcal{L} \leq 1 \).

Theorems 1.2 and 1.3 put together easily yield (the implication (ii)\( \Rightarrow \) (i)) of Theorem 1.1. We also use Theorem 1.2 to indicate a large class of pairs of Banach spaces \( X \) and \( Y \) for which \( \mathcal{K}(X,Y) \) has Phelps’ property \( U \) in \( \mathcal{L}(X,Y) \) (i.e., every functional \( f \in \mathcal{K}(X,Y)^* \) has a unique norm-preserving extension to \( \mathcal{L}(X,Y) \)).

2. Proof of Theorem 1.2

Theorem 1.2 follows from

**Theorem 2.1.** Let \( Y^* \) (respectively, \( X^{**} \)) have the Radon-Nikodým property, and let \( \mu \) be a regular Borel (with respect to the relative weak* topology) probability measure on \( C := \overline{B_{X^*}} \otimes B_{Y^*}^w \subset B_{\mathcal{L}(X,Y)^*} \). Denote by \( \mathcal{C} \) the collection of compact subsets \( A \) of \( C \) with the following property:

- there is a norm compact set \( Y^*_A \subset S_{Y^*} \) (respectively, \( X^{**}_A \subset S_{X^{**}} \)) such that, for every \( \phi \in A \), there are \( y^* \in Y^*_A \) and \( x^{**} \in B_{X^{**}} \) (respectively, \( y^* \in B_{Y^*} \) and \( x^{**} \in X^{**} \)) with \( g_\phi = x^{**} \otimes y^* \).

Then there are pairwise disjoint Borel sets \( C_j \subset C \), \( j \in \{0\} \cup \mathbb{N} \), such that \( C = \bigcup_{j=0}^\infty C_j \), where \( \int_{C_0} |\phi(S)| \, d\mu(\phi) = 0 \) for all \( S \in \mathcal{K}(X,Y) \), and \( C_j \subset \mathcal{C} \), \( j \in \mathbb{N} \).

**Proof.** Let \( D \subset C \) be a Borel subset such that \( \int_D |\phi(S)| \, d\mu(\phi) > 0 \) for some \( S \in \mathcal{K}(X,Y) \). By a standard exhaustion argument, it suffices to show that there is a subset \( A \subset D \) with \( A \subset \mathcal{C} \) and \( \mu(A) > 0 \). Without loss of generality, we may assume that \( |\phi(S)| = |g_\phi(S)| \geq 2\delta \) for some \( \delta > 0 \) and all \( \phi \in D \), and that \( D \) is (weak*) compact. We consider only the case when \( Y^* \) has the Radon-Nikodým property. (The proof is symmetric if \( X^{**} \) has the Radon-Nikodým property.) Let \( \mathcal{U} \subset B_Y \) be a finite \( \delta \)-net for \( S^{**}[B_{X^*}] \). For each \( y \in \mathcal{U} \), denote \( L_y := B_{X^{**}} \cap (S^{**})^{-1}[\overline{B}(y,\delta)] \), then \( L_y \) is (weak*) compact, and thus the set \( D_y := \{ \phi \in D : g_\phi = x^{**} \otimes z^* \text{ for some } x^{**} \in L_y \text{ and } z^* \in B_{Y^*} \} \) is (weak*) compact too. Moreover, for some \( y \in \mathcal{U} \), one must have \( \mu(D_y) > 0 \). For simplicity, relabel \( L_y \) and \( D_y \), respectively, by \( L \) and \( D \).
Denote by $\mathcal{K}$ the collection of compact (in the relative weak* topology) subsets of $B_{Y^*}$, and let $\mathcal{K}_\delta := \{ y^* \in B_{Y^*} : y^*(y) = \delta \} \in \mathcal{K}$. For each $K \in \mathcal{K}$ and each compact subset $H \subset D$, denote, respectively, 

$$C_K := \{ \phi \in D : g_\phi = x^{**} \otimes ty^* \}
$$

for some $x^{**} \in L$, $y^* \in K \cap \mathcal{K}_\delta$, and $t \in \mathbb{K}$ with $ty^* \in B_{Y^*}$.

and

$$K_H := \{ y^* \in \mathcal{K}_\delta : g_\phi = x^{**} \otimes ty^* \}
$$

for some $\phi \in H$, $x^{**} \in L$, and $t \in \mathbb{K}$ with $ty^* \in B_{Y^*}$.

Observe that $C_K$ is a compact (and thus Borel) (with respect to the relative weak* topology) subset of $C$, and $K_H \in \mathcal{K}$. Indeed, let $\phi \in D$, $x^{**} \in L$, $y^* \in \mathcal{K}_\delta$, and $t \in \mathbb{K}$ with $ty^* \in B_{Y^*}$ be such that $g_\phi = x^{**} \otimes ty^*$. One has $\delta \leq ||y^*|| \leq 1$ and, since

$$1 \geq |t| |y^*(y)| \geq |\phi(S)| - |\phi(S) - ty^*(y)| \geq 2\delta - ||ty^*|| ||S^* x^{**} - y|| \geq \delta,$$

$$1 \leq |t| \leq \frac{1}{2}. \text{ The (weak*) compactness of both } C_K \text{ and } K_H \text{ now quickly follows. Notice also that } H \subset C_{K_H}.$$

Notice that $\rho : \mathcal{K} \ni K \mapsto \int_{C_K} |\phi(S)| d\mu(\phi) \in [0, 1]$ is a regular content. To see that $\rho$ is regular, let $K \in \mathcal{K}$ and $\varepsilon > 0$. We have to find a $K' \in \mathcal{K}$ with $K' \supset K$ such that $\rho(K') < \rho(K) + \varepsilon$. To this end, choose a compact set $H \subset D \setminus C_K$ such that $\int_H |\phi(S)| d\mu(\phi) > \int_{D \setminus C_K} |\phi(S)| d\mu(\phi) - \varepsilon$. Since $K_H \cap K = \emptyset$, there are disjoint (in the relative weak* topology) open sets $U, V \subset B_{Y^*}$ such that $K \subset U$ and $K_H \subset V$. Letting $K' := B_{Y^*} \setminus V \in \mathcal{K}$ one has $K' \supset U \supset K$, and

$$\rho(K') = \int_{C_{K'}} |\phi(S)| d\mu(\phi) \leq \int_{D \setminus C_{K_H}} |\phi(S)| d\mu(\phi) \leq \int_{D \setminus H} |\phi(S)| d\mu(\phi) =
$$

$$= \int_{C_K} |\phi(S)| d\mu(\phi) + \int_{D \setminus C_K} |\phi(S)| d\mu(\phi) - \int_H |\phi(S)| d\mu(\phi) < \rho(K) + \varepsilon,$$

as desired.

Let $\nu$ be the regular Borel (with respect to the relative weak* topology) measure on $B_{Y^*}$ induced by the regular content $\rho$, i.e., for a Borel set $E \subset B_{Y^*}$, $\nu(E) = \inf \{ \lambda(U) : E \subset U \in \mathcal{U} \}$ where $\mathcal{U}$ is the collection of open subsets of $B_{Y^*}$ and $\lambda(U) = \sup \{ \rho(K) : U \supset K \in \mathcal{K} \}$, $U \in \mathcal{U}$, is the inner content induced by $\rho$. Since $C_{K_D} = D$, one has $\nu(K_D) \geq \rho(K_D) = \int_D |\phi(S)| d\mu(\phi) > 0$. Since $Y^*$ has the Radon-Nikodým property, by [B], Theorem 4.3.11,(a)$\Rightarrow$(b), and Lemmas 4.3.6 and 4.3.10, there is a norm compact set $K_0 \subset K_D$ such that $\nu(K_0) > 0$. Now we can take $C_{K_0}$ to be the desired $A$, because $C_{K_0} \subset D$, $C_{K_0} \in \mathcal{C}$ (one
can take \( Y_{C_{K_0}}^* = \left\{ \frac{1}{n} y^*: y^* \in K_0 \right\} \), and, since, by the regularity of \( \rho \),
\[
\int_{C_{K_0}} |\phi(S)| \, d\mu(\phi) = \rho(K_0) = \nu(K_0) > 0,
\]
also \( \mu(C_{K_0}) > 0 \). \( \square \)

**Proof of Theorem 1.2.** Let the sets \( C_j \), \( j \in \{0\} \cup \mathbb{N} \), be as in Theorem 2.1. Put \( C' = \bigcup_{j=1}^\infty C_j \). Let \( T \in S_{\mathcal{L}(X,Y)} \). Choose an increasing sequence of indices \( (j_n)_{n=1}^\infty \subset \mathbb{N} \) so that \( \mu \left( \bigcup_{j=j_n+1}^\infty C_j \right) < \frac{1}{n} \), \( n \in \mathbb{N} \). For each \( n \in \mathbb{N} \), let \( A_n \subset S_{Y^*} \) be a finite \( \frac{1}{n} \)-net for \( \bigcup_{j=1}^{j_n} Y_{C_j}^* \) where the sets \( Y_{C_j}^* \) are as in Theorem 2.1. Choose an increasing sequence of indices \( (\alpha_n)_{n=1}^\infty \) so that, whenever \( n \in \mathbb{N} \), for each \( \alpha \geq \alpha_n \), one has \( \|K_\alpha y^* - y^*\| < \frac{1}{n} \) for all \( y^* \in A_n \).

Now let \( n \in \mathbb{N} \) be fixed and let \( \alpha \geq \alpha_n \). Suppose that \( \phi \in \bigcup_{j=1}^{j_n} C_j \), and let \( x^{**} \in B_{X^{**}} \) and \( y^* \in Y_{C_j}^* \) (\( j \in \{1, \ldots, j_n\} \)) be such that \( g_\phi = x^{**} \otimes y^* \). For some \( y^*_\phi \in A_n \), one has \( \|y^* - y^*_\phi\| < \frac{1}{n} \). Thus
\[
|g_\phi(T) - \phi(K_\alpha T)| = |g_\phi(T) - g_\phi(K_\alpha T)| \leq \|T^{**} x^{**}\| \|y^* - K_\alpha y^*\|
\leq \|y^* - y^*_\phi\| + \|y^*_\phi - K_\alpha y^*_\phi\| + \|K_\alpha\| \|y^*_\phi - y^*\| < \frac{3}{n}.
\]
It follows that \( \phi(K_\alpha T) \to g_\phi(T) \) for each \( \phi \in C' \); thus the function \( C' \ni \phi \mapsto g_\phi(T) \chi_{C'}(\phi) \in \mathbb{K} \) is measurable.

Letting, again, \( n \in \mathbb{N} \) be fixed and \( \alpha \geq \alpha_n \), one has
\[
\left| \int_{C'} g_\phi(T) \, d\mu(\phi) - f(K_\alpha T) \right| \leq \int_{C'} |g_\phi(T) - \phi(K_\alpha T)| \, d\mu(\phi)
= \int_{\bigcup_{j=1}^{j_n} C_j} |g_\phi(T) - \phi(K_\alpha T)| \, d\mu(\phi) + \int_{\bigcup_{j=j_n+1}^{\infty} C_j} |g_\phi(T) - \phi(K_\alpha T)| \, d\mu(\phi)
< \frac{3}{n} + \frac{2}{n} = \frac{5}{n},
\]
and it follows that \( Pf(T) = \lim_{\alpha} f(K_\alpha T) = \int_{C'} g_\phi(T) \, d\mu(\phi) \). \( \square \)

**Remark 2.1.** The assumption in Theorem 1.2 for \( (K_\alpha) \) to be weak* convergent (in \( \mathcal{K}(Y)^{**} \)) is, in fact, superfluous: A description of \( \mathcal{K}(X,Y)^* \) due to Feder and Saphar (see [FS], Theorem 1 or Corollary 2.2 below) yields that if \( Y^* \) has the Radon-Nikodým property, then every shrinking MCAI of \( Y \) is weak* convergent (in \( \mathcal{K}(Y)^{**} \)).

**Remark 2.2.** Suppose that, in Theorem 1.2, \( Y \) is separable. Then \( Y \) has a shrinking MCAI which is a sequence, label it \( (K_n)_{n=1}^\infty \). By [FS], Theorem 1 (or Corollary 2.2 below), one has \( Pg(T) = \lim_{n \to \infty} Pg(K_n T) = \lim_{n \to \infty} g(K_n T) \) for every \( g \in \mathcal{L}(X,Y)^* \) and every \( T \in \mathcal{L}(X,Y) \) (for details, see [P], Lemma 1.2). Thus, one has, for any \( T \in \mathcal{L}(X,Y) \), by Lebesgue's
bounded convergence theorem,

\[ Pf(T) = \lim_{n \to \infty} f(K_nT) = \lim_{n \to \infty} \int_C \phi(K_nT) \, d\mu(\phi) = \lim_{n \to \infty} \int_C g_\phi(K_nT) \, d\mu(\phi) \]

\[ = \int_C \lim_{n \to \infty} g_\phi(K_nT) \, d\mu(\phi) = \int_C g_\phi(T) \, d\mu(\phi) = \int_C P\phi(T) \, d\mu(\phi). \]

Notice that the Feder-Saphar description of \( K(X,Y)^* \) which was used in Remarks 2.1 and 2.2 is, in fact, a consequence of Theorem 2.1.

**Corollary 2.2** (see [FS], Theorem 1). Suppose that \( X^{**} \) or \( Y^* \) has the Radon-Nikodým property, and let \( g \in K(X,Y)^* \) and \( \varepsilon > 0 \). Then there are \( x_j^{**} \in X^{**} \) and \( y_j^* \in Y^* \), \( j \in \mathbb{N} \), such that \( g = \sum_{j=1}^\infty x_j^{**} \otimes y_j^* \) and \( \sum_{j=1}^\infty \|x_j^{**}\| \|y_j^*\| < \|g\| + \varepsilon \).

**Proof.** It suffices to show that there are \( n \in \mathbb{N} \), \( x_1^{**}, \ldots, x_n^{**} \in X^{**} \), and \( y_1^*, \ldots, y_n^* \in Y^* \) such that \( \|g - \sum_{j=1}^n x_j^{**} \otimes y_j^*\| < \varepsilon \) and \( \sum_{j=1}^n \|x_j^{**}\| \|y_j^*\| \leq \|g\| \). One may clearly assume that \( \|g\| = 1 \).

Let \( f \in S_{L(X,Y)^*} \) be some extension of \( g \). As explained in Introduction, there is a regular Borel (with respect to the relative weak* topology) probability measure \( \mu \) on \( C := \overline{B_{X^{**}}} \otimes B_{Y^*} \subset B_{L(X,Y)^*} \) such that \( f(T) = \int_C \phi(T) \, d\mu(\phi), \ T \in \mathcal{L}(X,Y) \). Now, in Theorem 2.1, one has \( \mu(C_0) = 0 \), and one may also assume that \( \hat{\mathcal{C}} := \mathcal{C} \setminus C_0 \subset \mathcal{C} \).

We consider only the case when \( Y^* \) has the Radon-Nikodým property. (The proof is symmetric if \( X^{**} \) has the Radon-Nikodým property.) Let \( \{y_1^*, \ldots, y_n^*\} \subset S_{Y^*} \ (n \in \mathbb{N}) \) be an \( \frac{\varepsilon}{n} \)-net for the set \( Y_{\mathcal{C}}^* \) from Theorem 2.1. Choose \( y_j \in S_{Y^*} \) such that \( |y_j^*(y_j) - 1| < \frac{\varepsilon}{n}, \ j \in \{1, \ldots, n\} \). For each \( j \in \{1, \ldots, n\} \), the set

\[ B_j := \{ \phi \in \hat{\mathcal{C}} : g_\phi = x_{\phi}^{**} \otimes y_{\phi}^* \}

for some \( x_{\phi}^{**} \in B_{X^{**}} \) and \( y_{\phi}^* \in Y_{\mathcal{C}}^* \) with \( \|y_{\phi}^* - y_j^*\| \leq \frac{\varepsilon}{n} \)

is (weak*) compact; thus the set \( E_j := B_j \setminus \bigcup_{i=1}^{j-1} B_i \) is Borel, and we may define \( x_j^{**} \in X^{**} \) by \( x_j^{**}(x^*) = \int_{E_j} \phi(x^* \otimes y_j) \, d\mu(\phi) = \int_{E_j} g_\phi(x^* \otimes y_j) \, d\mu(\phi), \ x^* \in X^* \). Now, whenever \( j \in \{1, \ldots, n\} \), one has \( \|x_j^{**}\| \|y_j^*\| \leq \mu(E_j) \), and, since, for all \( \phi \in E_j \),

\[ \|y_{\phi}^* - y_j^*(y_j)\| y_j^* \]

\[ \leq |1 - y_j^*(y_j)| \|y_{\phi}^*\| + |y_j^*(y_j) - y_{\phi}^*(y_j)| \|y_{\phi}^*\| + |y_{\phi}^*(y_j)| \|y_{\phi}^* - y_j^*\| < \varepsilon, \]
one has, for every $S \in B_{K(X,Y)}$,
\[
\left| \int_{E_J} \phi(S) \, d\mu(\phi) - (x_J^{**} \otimes y_J^*)(S) \right| = \left| \int_{E_J} g_\phi(S) \, d\mu(\phi) - x_J^{**}(S^*y_J^*) \right| \\
= \left| \int_{E_J} g_\phi(S) \, d\mu(\phi) - \int_{E_J} g_\phi(S^*y_J^* \otimes y_J) \, d\mu(\phi) \right| \\
= \left| \int_{E_J} x_\phi^{**}(S^*y_J^*) \, d\mu(\phi) - \int_{E_J} x_\phi^{**}(S^*y_J^*) \, y_\phi(y_J) \, d\mu(\phi) \right| \\
\leq \int_{E_J} \|S^*x_\phi^{**}\| \|y_J^* - y_\phi(y_J)\| \, d\mu(\phi) < \epsilon. 
\]
It follows that $\sum_{j=1}^n \|x_J^{**}\| \|y_J^*\| \leq \|g\|$, and, for every $S \in B_{K(X,Y)}$,
\[
\left| g(S) - \sum_{j=1}^n (x_J^{**} \otimes y_J^*)(S) \right| = \left| \sum_{j=1}^n \int_{E_J} \phi(S) \, d\mu(\phi) - \sum_{j=1}^n (x_J^{**} \otimes y_J^*)(S) \right| \\
\leq \sum_{j=1}^n \left| \int_{E_J} \phi(S) \, d\mu(\phi) - (x_J^{**} \otimes y_J^*)(S) \right| < \epsilon, 
\]
as desired.

\[
\square
\]

3. Proofs of Theorems 1.3 and 1.1

The implication (ii) of Theorem 1.3 is contained in

**Proposition 3.1.** Let both $X$ and $Y$ have property $(M^*)$. Then, for any $\phi \in B_{X^{**}} \otimes B_{Y^*}$, one has $\|g_\phi\| + \|\phi - g_\phi\| \leq 1$.

**Proof.** Let $\phi \in B_{X^{**}} \otimes B_{Y^*}$-w* and let $\phi_\alpha = x_\alpha^{**} \otimes y_\alpha^* \in B_{X^{**}} \otimes B_{Y^*}$ be such that $w^*\text{-lim}_\alpha \phi_\alpha = \phi$ in $L(X,Y)^*$. We may assume that $w^*\text{-lim}_\alpha x_\alpha^{**} = x^{**}$ in $X^{**}$ and $w^*\text{-lim}_\alpha y_\alpha^* = y^*$ in $Y^*$ for some $x^{**} \in B_{X^{**}}$ and $y^* \in B_{Y^*}$. Denote $g = g_\phi = x^{**} \otimes y^*$ and $h = \phi - g$. We must show that $\|g\| + \|h\| \leq 1$. The case $y^* = 0$ is trivial, so let us assume that $y^* \neq 0$. Fix arbitrary $S \in S_{K(X,Y)}$ with $S^*y^* \neq 0$ and $T \in S_{L(X,Y)}$. It suffices to show that $|g(S) + h(T)| \leq 1$. To this end, pick $y_n \in S_Y$, $n \in \mathbb{N}$, such that $y^*(y_n) \to \|y^*\|$ and denote $K_n = \frac{y^*}{\|y^*\|} \otimes y_n \in B_{K(Y)}$, $n \in \mathbb{N}$. Then $K_n^*y^* = y^*(y_n) \frac{y^*}{\|y^*\|} \to y^*$, thus
\[
g(K_nT) = x^{**}(T^*K_n^*y^*) = T^{**}x^{**}(K_n^*y^*) \to T^{**}x^{**}(y^*) = g(T). 
\]
Fix an arbitrary $\varepsilon > 0$ and choose $n \in \mathbb{N}$ such that $\|K_n^*y^* - y^*\| < \varepsilon$ and $|g(K_nT) - g(T)| < \varepsilon$. Find $v^* \in B_{Y^*}$ with $\|v^*\| \leq \|y^*\|$ such that $\|T^*v^*\| > \frac{\|S^*y^*\|}{1+\varepsilon}$ and $x \in B_X$ such that $(T^*v^*)(x) = \frac{S^*y^*}{1+\varepsilon}$, and put $U = \frac{S^*y^*}{\|S^*y^*\|} \otimes x \in B_{K(X)}$. Then $U^*T^*v^* = T^*v^*(x) \frac{S^*y^*}{\|S^*y^*\|} = \frac{1+\varepsilon}{1+\varepsilon}S^*y^*$, thus $S^*y^* =
(1 + \varepsilon) \| \alpha \| U^* T^* v^* \). Now, since \( X^* \) and \( Y^* \) have property \((M^*)\), one has
\[
|g(S) + h(T)| = |\phi(S) + h(T) + g(T) - g(K_n T) - g(K_n T) - g(K_n T)| \\
\leq |\phi(S + T - K_n T)| + |g(K_n T) - g(T)| \\
< \lim_{\alpha} [x_{\alpha}^*(S^* y_{\alpha}^* + T^* y_{\alpha}^* - T^* K_n^* y_{\alpha}^*)] + \varepsilon \\
\leq \limsup_{\alpha} \| S^* y_{\alpha}^* + T^* y_{\alpha}^* - T^* K_n^* y_{\alpha}^* \| + \varepsilon \\
\leq \limsup_{\alpha} \| U^* T^* v^* + T^* y_{\alpha}^* - y^* \| + \varepsilon \| U^* T^* v^* \| + \| T^* y^* - T^* K_n y^* \| + \varepsilon \\
\leq \limsup_{\alpha} \| T^* v^* + T^* (y_{\alpha}^* - y^*) \| + \| T^* \| \| y^* - K_n y^* \| + 2\varepsilon \\
\leq \limsup_{\alpha} \| v^* + y_{\alpha}^* - y^* \| + 3\varepsilon \leq \limsup_{\alpha} \| y^* + y_{\alpha}^* - y^* \| + 3\varepsilon \leq 1 + 3\varepsilon.
\]

Letting \( \varepsilon \to 0 \) yields \( |g(S) + h(T)| \leq 1 \), as desired. \( \square \)

Observe that, if \( X \) is infinite-dimensional, then whenever \( S \in K(X) \) and \( \lambda \in \mathbb{K} \) are such that \( \| S + \lambda I \| < 1 \), one has \( |\lambda| < 1 \) (because otherwise \( \| \frac{1}{2} S + I \| < 1 \) and thus \( \frac{1}{2} S \) would be invertible). Hence, for all \( h \in K(X)^1 \subset \mathcal{L}(X)^* \), one has \( \| h \|_\mathcal{L} = |h(I)| \) because
\[
\| h \|_\mathcal{L} = \sup \{ |\phi(S + \lambda I) : S \in K(X), \lambda \in \mathbb{K}, \| S + \lambda I \| < 1 \} \\
= \sup \{ |\lambda| |h(I) : S \in K(X), \lambda \in \mathbb{K}, \| S + \lambda I \| < 1 \} \leq |h(I)| \leq \| h \|_\mathcal{L}.
\]

**Proof of Theorem 1.3.** (i)\( \Rightarrow \) (ii) is obvious from Proposition 3.1.

(ii)\( \Rightarrow \) (iii) is more than obvious.

(iii)\( \Rightarrow \) (i). Let (iii) hold, let \( x^*, u^* \in X^* \) be such that \( \| u^* \| \leq \| x^* \| \), and let \( (x_{\alpha}^*) \subset X^* \) be a bounded weak* null net. We must show that
\[
\limsup_{\alpha} \| u^* + x_{\alpha}^* \| \leq \limsup_{\alpha} \| x^* + x_{\alpha}^* \|.
\]

We may assume that \( \| u^* \| < \| x^* \| \) and that \( \limsup_{\alpha} \| u^* + x_{\alpha}^* \| = \lim_{\alpha} \| u^* + x_{\alpha}^* \| \). In this case \( M := \limsup_{\alpha} \| x^* + x_{\alpha}^* \| > 0 \) (because otherwise we would have \( x_{\alpha}^* \to -x^* \) in norm, hence also \( x_{\alpha}^* \to -x^* \) weak* and thus \( x^* = 0 \) implying that \( \| u^* \| < 0 \)); thus we may assume that \( M_\alpha := \| x^* + x_{\alpha}^* \| > 0 \) for all \( \alpha \) and also that \( M_\alpha \to M \). Pick \( S \in B_{K(X)} \) such that \( S^* x^* = u^* \) (note that such a rank one \( S \) exists). By passing to product index, we may assume that there is a net \( (x_\alpha) \subset S_X \) such that
\[
\lim_{\alpha} \| S^* x^* + x_{\alpha}^* \| = \lim_{\alpha} \| S^* x^*(x_\alpha) + x_{\alpha}^*(x_\alpha) \|.
\]

Considering \( \phi_\alpha := x_\alpha \otimes \frac{1}{M_\alpha} (x^* + x_{\alpha}^*) \in B_{X^{**}} \otimes B_{X^*} \), we may assume that \( w^*\text{-lim}_\alpha \phi_\alpha = \phi \in \mathcal{L}(X)^* \) for some \( \phi \in \overline{B_{X^{**}}} \otimes B_{X^*} \), and that \( w^*\text{-lim}_\alpha x_\alpha = x^{**} \in X^{**} \) for some \( x^{**} \in B_{X^{**}} \). Then \( g_\phi = \frac{1}{M} x^{**} \otimes x^* \)
and \( \phi - g_\phi = \frac{1}{M} w^* - \lim_\alpha x_\alpha \otimes x_\alpha^* \). By (iii), one has
\[
\limsup_\alpha \|u^* + x_\alpha^*\| = \lim_\alpha |S^* x_\alpha^* + x_\alpha^*(x_\alpha)| = |M g_\phi(S) + M(\phi - g_\phi)(I)| \\
\leq M \left( \|g_\phi\| + \|\phi - g_\phi\|_\mathcal{L} \right) \\
= M = \limsup_\alpha \|x^* + x_\alpha^*\|.
\]
\[\square\]

Remark 3.1. In [L], Theorem 2.2 Å. Lima proved, combining knowledge on weak* strongly exposed points of \( B_\mathcal{X} \) with a clever slice-cutting technique, that if \( K(X) \) is a semi \( M \)-ideal in \( \text{span}(K(X) \cup \{I\}) \), then \( X \) has property \((M^*)\). This result is an immediate consequence of our implication (iii) \( \Rightarrow \) (i) whose proof was more or less elementary.

The following Corollary is well known. Our Theorem 1.3 yields a very simple proof for it.

Corollary 3.2 (see [HWW], page 297). Let \( X \) have property \((M^*)\). Then \( X \) is an \( M \)-ideal in \( X^{**} \).

Proof. Let \( x^{**} = x^* + x^\perp \in S_{X^{**}} \) (\( x^* \in X^* \), \( x^\perp \in X^\perp \)), and let \( \varepsilon > 0 \). It suffices to show that \( \|x^*\| + \|x^\perp\| < 1 + \varepsilon \). To this end, pick \( x^{**} \in S_{X^{**}} \) satisfying \( |x^+(x^{**})| \geq \|x^\perp\| - \varepsilon \), and observe that the functional
\[
\phi = x^{**} \otimes x^{***} : \mathcal{L}(X) \ni T \mapsto x^{***}(T^* x^{**})
\]
is in \( \overline{B_\mathcal{X}^* \otimes B_\mathcal{X}^*} \) (because whenever a net \( (x_\alpha^*) \subset B_\mathcal{X}^* \) is such that \( x_\alpha^* \to x^{***} \) weak* in \( X^{***} \), then \( x^{**} \otimes x_\alpha^* \to \phi \) weak* in \( \mathcal{L}(X)^* \)). Clearly, \( g_\phi = x^{**} \otimes x^* \) and thus, since
\[
\|\phi - g_\phi\| \geq |(\phi - g_\phi)(I)| = |x^+(x^{**})| \geq \|x^\perp\| - \varepsilon,
\]
by Theorem 1.3,
\[
\|x^*\| + \|x^\perp\| = \|g_\phi\| + \|x^\perp\| \leq \|g_\phi\| + \|\phi - g_\phi\| + \varepsilon \leq 1 + \varepsilon.
\]
\[\square\]

Now we are in a position to prove Theorem 1.1. The implication (ii) \( \Rightarrow \) (i) is the particular case with \( Y = X \) of the known

Proposition 3.3 (cf. Theorem 1.1 combined with [O1], Theorem 8). Let both \( X \) and \( Y \) have property \((M^*)\), and let \( Y \) have the MCAP. Then \( K(X,Y) \) is an \( M \)-ideal in \( \mathcal{L}(X,Y) \).

Proof. Let \( (K_\alpha) \subset B_{K(Y)} \) be an MCAI. By Corollary 3.2, \( Y \) is an \( M \)-ideal in its bidual; hence \( B_Y^* \) is the norm closed convex hull of its weak* strongly exposed points (see [HWW], page 127, Corollary 3.2). It easily follows that
Let $f \in S_{\mathcal{L}(X,Y)^*}$, and let $T_1, T_2 \in B_{\mathcal{L}(X,Y)}$ be arbitrary. It suffices to show that $|Pf(T_1)| + |(I - P)f(T_2)| \leq 1$. As explained in Introduction, there is a regular Borel probability measure $\mu$ on $C := \overline{B_{\mathcal{L}(X,Y)^*}} \subset B_{\mathcal{L}(X,Y)^*}$, such that $\int_C \phi(T) \, d\mu(\phi)$, $T \in \mathcal{L}(X,Y)$. By Theorem 1.3, one has $|g_\phi(T_1)| + |(\phi - g_\phi)(T_2)| \leq \|g_\phi\| + \|\phi - g_\phi\| \leq 1$ for all $\phi \in C$. Thus, letting the set $C'$ be as in Theorem 1.2 (notice that, since $Y$ is an $M$-ideal in its bidual, the dual $Y^*$ enjoys the Radon-Nikodým property (see [HWW], page 126, Theorem 3.1)), one has

$$
\begin{align*}
|Pf(T_1)| + |(I - P)f(T_2)| &= \left| \int_{C'} g_\phi(T_1) \, d\mu(\phi) \right| + \left| \int_{C'} (\phi - g_\phi)(T_2) \, d\mu(\phi) \right| + \left| \int_{C \setminus C'} \phi(T_2) \, d\mu(\phi) \right| \\
&\leq \int_{C'} (|g_\phi(T_1)| + |(\phi - g_\phi)(T_2)|) \, d\mu(\phi) + \mu(C \setminus C') \\
&\leq \mu(C') + \mu(C \setminus C') = 1.
\end{align*}
$$

\[\blacksquare\]

**Proof of Theorem 1.1.** (ii)$\Rightarrow$(i) is immediate from Proposition 3.3.

(i)$\Rightarrow$(ii). Let $\mathcal{K}(X)$ be an $M$-ideal in $\mathcal{L}(X)$ with $P \in \mathcal{L}(X)^*$ being the ideal projection. Property $(M^*)$ for $X$ follows immediately from the implication (ii)$\Rightarrow$(i) of Theorem 1.3. The argument to obtain the MCAP for $X$ is well known: By Goldstine’s theorem (or by the bipolar theorem), $B_{\mathcal{K}(X)}$ is dense in $B_{\mathcal{L}(X)}$ in the weak topology $\sigma(\mathcal{L}(X), \text{ran} P)$. Thus there is a net $(K_\alpha) \subset B_{\mathcal{K}(X)}$ such that $Pf(K_\alpha) \to Pf(I_X)$ for all $f \in \mathcal{L}(X)^*$. In particular, $x^*(K_\alpha x) \to x^*(I_X x)$ for all $x \in X$ and all $x^* \in X^*$ (because $P(x \otimes x^*) = x \otimes x^*$), i.e., $K_\alpha \to I_X$ in the weak operator topology of $\mathcal{L}(X)$. Since the weak and strong operator topologies yield the same dual space (see, e.g., [DSch], Theorem VI.1.4), after passing to convex combinations, we may assume that $K_\alpha x \to x$ for all $x \in X$, and thus $X$ has the MCAP. \[\blacksquare\]

We conclude by showing how Theorem 1.2 yields a result which produces multiple examples of pairs of Banach spaces $X$ and $Y$ for which $\mathcal{K}(X,Y)$ has Phelps’ property $U$ in $\mathcal{L}(X,Y)$. Recall that a closed subspace $Z$ of $X$ is said to have (Phelps’) property $U$ in $X$ if every $z^* \in Z^*$ admits a unique norm-preserving extension $x^* \in X^*$.

**Theorem 3.4.** Let $Y^*$ have the Radon-Nikodým property, let $Y$ have the shrinking MCAP, and suppose that, for every $x^{**} \in S_{X^{**}}$ and every $y^* \in S_{Y^*}$, the functional $x^{**} \otimes y^* \in \mathcal{L}(X,Y)^*$ itself is the only norm-preserving
extension of its restriction to $\mathcal{K}(X,Y)$. Then $\mathcal{K}(X,Y)$ has property $U$ in $\mathcal{L}(X,Y)$.

**Remark 3.2.** By a result of A. Lima (see [L], Lemma 3.4; see also [OP] for a recent easier proof), $x^{**} \otimes y^* \in \mathcal{L}(X,Y)^*$ itself is the only norm-preserving extension of its restriction to $\mathcal{K}(X,Y)$ whenever $x^{**} \in B_X \subset B_{X^{**}}$ is a denting point of $B_X$ or $y^* \in B_Y^*$ is a weak* denting point of $B_Y^*$. It is known (see [LLT1] and [LLT2]) that a point $x \in B_X$ is a denting point of $B_X$ if and only if it is both an extreme point and a point of weak-to-norm continuity of $B_X$, as well as a point $y^* \in B_Y^*$ is a weak*-denting point of $B_Y^*$ if and only if it is both an extreme point and a point of weak*-to-norm continuity of $B_Y^*$.

**Proof of Theorem 3.4.** Let $(K_0) \subset B_Y$ be a shrinking MCAI of $Y$, and let $P$ be the Johnson projection on $\mathcal{L}(X,Y)^*$ defined by (1.1) (notice that $(K_0)$ is weak* convergent (in $\mathcal{K}(Y)^*$) by Remark 2.1). Then $P\phi = g_\phi$ for all $\phi \in B_{X^{**}} \otimes B_{Y^{**}} =: C$.

Let $f \in S_{\mathcal{L}(X,Y)^*}$ be such that $\|Pf\| = \|f\| = 1$. It suffices to show that $Pf = f$. As explained in Introduction, there is a regular Borel (with respect to the relative weak* topology) probability measure $\mu$ on $C$ representing $f$, i.e., $f(T) = \int_C \phi(T) d\mu(\phi)$ for all $T \in \mathcal{L}(X,Y)$. Since $\|Pf\| = \|f|_\mathcal{K(X,Y)^*}\|$, one has, in Theorem 1.2, $\mu(C \setminus C') = 0$. Denote $C_1 := \{\phi \in C' : \|g_\phi\| = 1\}$. Then $\mu(C' \setminus C_1) = 0$ (the function $C \ni \phi \mapsto \|g_\phi\|$ is measurable since it is lower semicontinuous) because otherwise $\int_{C' \setminus C_1} \|g_\phi\| d\mu(\phi) < \mu(C' \setminus C_1)$ and thus

\[
\|Pf\| = \sup_{T \in B_{\mathcal{L}(X,Y)}} |Pf(T)| = \sup_{T \in B_{\mathcal{L}(X,Y)}} \left| \int_{C'} g_\phi(T) d\mu(\phi) \right| \\
\leq \sup_{T \in B_{\mathcal{L}(X,Y)}} \int_{C'} |g_\phi(T)| d\mu(\phi) \leq \int_{C'} \|g_\phi\| d\mu(\phi) \\
= \int_{C_1} \|g_\phi\| d\mu(\phi) + \int_{C' \setminus C_1} \|g_\phi\| d\mu(\phi) < \mu(C_1) + \mu(C' \setminus C_1) = 1.
\]

By our assumption, for any $\phi \in C_1$, one has $g_\phi = \phi$. From Theorem 1.2 it now follows that, for any $T \in \mathcal{L}(X,Y)$,

\[
Pf(T) = \int_{C_1} g_\phi(T) d\mu(\phi) = \int_{C_1} \phi(T) d\mu(\phi) = f(T).
\]

$$\Box$$

**References**

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