CORRECTION TO: "ON THE COMPACT APPROXIMATION PROPERTY"

VEGARD LIMA, ÅSVALD LIMA, AND OLAV NYGAARD

Abstract. The proof of our paper "On the compact approximation property" contains a proof with a small gap.

The gap is in the proof of the fact that if a Banach spaces X has the compact approximation property, then for every Banach space Y and every weakly compact operator $T : Y \to X$, the space

$$\mathfrak{E} = \{S \circ T : S \text{ compact operator on } X\}$$

is an ideal in $\mathfrak{F} = \operatorname{span}(\mathfrak{E}, \{T\})$.

The aim of this note is to fill this gap.

1. A correction

We will correct the proof of Theorem 2.2 (i) $\Rightarrow$ (ii) in [2]. The proof there works for reflexive spaces, but the general case contains typos and after correcting those it is not obvious that the proof works.

Let X and Y denote Banach spaces and let $\mathcal{F}(Y, X)$, $\mathcal{K}(Y, X)$ and $\mathcal{W}(Y, X)$ denote the spaces of finite rank, compact and weakly compact operators from Y to X. We will need the following version of Theorem 2.3 in [3].

Theorem 1.1. *Let X and Y be Banach spaces. Let G be a finite dimensional subspace of $\mathcal{W}(Y, X)$. There there exists a reflexive space Z, a norm one operator $J : Z \to X$, and a linear isometry $\Phi : G \to \mathcal{W}(Y, Z)$ such that $T = J \circ \Phi(T)$ for all $T \in G$. In particular,*

$$K = \overline{\operatorname{conv}}\{Ty : T \in B_G, y \in B_Y\}$$

is a weakly compact subset of X and if $H = G \cap \mathcal{K}(Y, X)$, then

$$K_H = \overline{\operatorname{conv}}\{Ty : T \in B_H, y \in B_Y\}$$

is a compact subset of X.

Proof. The only part of the statement that is not proved in Theorem 2.3 in [3] is that K_H is compact. We will use the compactness criterion from Exercise I.4 in [1]. Given $\varepsilon > 0$, let $\{T_1, \dots, T_n\}$ be an ε-net of B_H. Let K_ε be the closed convex hull of the compact set $\overline{T_1(B_Y)} \cup \dots \cup \overline{T_n(B_Y)}$. By Mazur's Compactness Theorem, K_ε is compact. Since $K_H \subset K_\varepsilon + \varepsilon B_X$ and $\varepsilon > 0$ arbitrary we get that K_H is compact. $\square$

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Theorem 1.2. *If X has the compact approximation property, then for every Banach space Y and every weakly compact operator $T : Y \to X$, the space*

$$\mathfrak{E} = \{S \circ T : S \in \mathcal{K}(X, X)\}$$

is an ideal in $\mathfrak{F} = \operatorname{span}(\mathfrak{E}, \{T\})$.

Proof. The proof of Theorem 2.2 (i) $\Rightarrow$ (ii) in [2] proves this result for every *reflexive* space Y.

Let Y be a Banach space and let $T \in \mathcal{W}(Y, X)$. Define $\mathfrak{E}_T = \{S \circ T : S \in \mathcal{K}(X, X)\}$ and $\mathfrak{F}_T = \operatorname{span}(\mathfrak{E}_T, \{T\})$. Without loss of generality $\|T\| = 1$.

Let $\varepsilon > 0$. Let G be a finite dimensional subspace of $\mathfrak{F}_T \subset \mathcal{W}(Y, X)$. Let $H = G \cap \mathfrak{E}_T$.

By Theorem 2.3 in [3], see Theorem 1.1 above, there is a reflexive space Z, a norm-one operator $J : Z \to X$, and a linear isometry $\Phi : G \to \mathcal{W}(Y, Z)$ such that $U = J \circ \Phi(U)$ for all $U \in G$.

We have proved our result for reflexive spaces so $\mathfrak{E}_J$ is an ideal in $\mathfrak{F}_J$ and by Lemma 1.4 in [3] there exists a net $(S_\gamma) \subseteq \mathcal{K}(X, X)$ such that $\sup_\gamma \|S_\gamma J\| \leq \|J\|$ and $S_\gamma J \to J$ in the strong operator topology. We get for $U \in G$

$$\sup_\gamma \|S_\gamma U\| = \sup_\gamma \|S_\gamma J \Phi(U)\| \leq \|J\| \|\Phi(U)\| \leq \|U\|.$$

We saw in Theorem 1.1 that K_H is compact in X. By the original Davis-Figiel-Johnson-Pełczyński construction (see Lemma 2.1 in [3]) K_H is compact in Z. Hence there exists an index γ_0 such that $\sup_{x \in K_H} \|S_{\gamma_0} Jx - Jx\| \leq \varepsilon$. We get that for $ST \in B_H$

$$\|S_{\gamma_0} ST - ST\| = \sup_{y \in B_Y} \|S_{\gamma_0} STy - STy\| \leq \sup_{x \in K_H} \|S_{\gamma_0} Jx - Jx\| \leq \varepsilon.$$

Define $v : G \to \mathfrak{E}_T$ by $v(U) = S_{\gamma_0} \circ U$. Then $\|v(U)\| = \|S_{\gamma_0} U\| \leq \|U\|$ and by the above v almost fixes U if $U = S \circ T \in \mathfrak{E}_T$. In summary, there exists a linear operator $v : G \to \mathfrak{E}_T$ such that

(a) $\|v\| \leq 1$ and
(b) $\|v(U) - U\| \leq \varepsilon \|U\|$ for all $U \in H = G \cap \mathfrak{E}_T$.

By the local formulation of ideals $\mathfrak{E}_T$ is an ideal in $\mathfrak{F}_T$. □

References

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Department of Mathematics, Agder University College, Serviceboks 422, 4604 Kristiansand, Norway

E-mail address: Asvald.Lima@hia.no

E-mail address: Vegard.Lima@gmail.com

E-mail address: Olav.Nygaard@hia.no