METRIC APPROXIMATION PROPERTIES AND TRACE MAPPINGS

ÅSVALD LIMA AND EVE OJA

Abstract. We establish necessary and sufficient conditions involving trace mappings and Hahn-Banach extension operators for a Banach space to have metric or metric compact approximation properties. We also study metric approximation properties for dual spaces. As an application, alternative (hopefully enlightening) proofs are given for the well-known result that the dual space has the metric approximation property whenever it has the approximation property and the Radon-Nikodým property.

1. Introduction

Let X and Y be Banach spaces. We denote by $\mathcal{L}(X,Y)$ the Banach space of bounded linear operators from X to Y, and by $\mathcal{F}(X,Y)$, $\mathcal{K}(X,Y)$, and $\mathcal{W}(X,Y)$ its subspaces of finite rank operators, compact operators, and weakly compact operators.

Recall that a Banach space X is said to have the *metric approximation property* (MAP) if for every compact set K in X and every $\varepsilon > 0$, there is an operator $T \in \mathcal{F}(X,X)$ with $\|T\| \le 1$ such that $\|Tx - x\| \le \varepsilon$ for all $x \in K$. Recall that X is said to have the *metric compact approximation property* (MCAP) if for every compact set K in X and every $\varepsilon > 0$, there is an operator $T \in \mathcal{K}(X,X)$ with $\|T\| \le 1$ such that $\|Tx - x\| \le \varepsilon$ for all $x \in K$.

Since finite rank operators are compact, the MCAP is formally weaker than the MAP. It really is weaker: Willis [26] has constructed a separable reflexive Banach space with the MCAP but without the MAP.

The paper [18] (see also [17]) exposes several common features of MAP and MCAP related to the internal geometric structure of the underlying Banach spaces. On the other hand, in [14] and [13] criteria for the approximation property and for the compact approximation property were given in terms of ideals that might have been expected to be similar, but are, in fact, not similar at all.

Let us consider the trace mapping V from the projective tensor product $X^*\hat{\otimes}_\pi X$ to $\mathcal{F}(X,X)^*$, the dual space of $\mathcal{F}(X,X)$, defined by

$$(Vu)(T) = \text{trace}(Tu), \quad u \in X^*\hat{\otimes}_\pi X, \quad T \in \mathcal{F}(X,X),$$

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that is, if $u = \sum_{n=1}^{\infty} x_n^* \otimes x_n$, then $(Vu)(T) = \sum_{n=1}^{\infty} x_n^*(Tx_n)$. The starting point of our studies is the following well-known criterion of the MAP due to Grothendieck [8, Chapter I, page 179] (see also, e.g., [4, page 243] or [25, page 80]).

Theorem 1.1 (Grothendieck)**.** *A Banach space X has the MAP if and only if the trace mapping $V : X^* \hat{\otimes}_\pi X \to \mathcal{F}(X,X)^*$ is isometric.*

It is not known whether the similar result holds for the MCAP: does X have the MCAP if and only if the trace mapping $V : X^* \hat{\otimes}_\pi X \to \mathcal{K}(X,X)^*$ is isometric?

This paper aims to study the problem by establishing criteria involving the trace mapping which are similar for the cases MAP and MCAP (see Theorem 2.1 and Theorem 2.8). In particular, we shall indicate a simple reason why the "if" part works for both cases MAP and MCAP (see the proof of Corollary 2.2). Relying on this, a proof of Theorem 1.1 will be given (see Corollaries 2.2 and 2.6) different from the existing ones, which, together with other results of the article suggests that there might be counter-examples to the "only if" part in the case of MCAP.

We also study the MCAP for dual spaces (see Theorem 3.1 and Theorem 3.6). As an application, two alternative (hopefully enlightening) proofs will be given for the well-known result that X^* has the MAP (respectively, the MCAP with conjugate operators) whenever X^* has the AP (respectively, the CAP with conjugate operators) and X^* or X^{**} has the Radon-Nikodým property (see [8], [16], [7], [14] for different proofs).

Let us fix some more notation. We consider normed linear spaces (Banach spaces) over the same field of real or complex numbers. In a linear normed space X, we denote the closed unit ball by B_X. The closure of a set $A \subset X$ is denoted by $\overline{A}$ and its linear span by $\operatorname{span} A$. We shall always regard X as a subspace of X^{**}. Thus the identity operator I_X on X is also considered as the embedding, identifying I_X with the canonical embedding $j_X : X \to X^{**}$.

2. Metric approximation properties

The following three results hold for the general version of the metric approximation property defined by any operator ideal $\mathcal{A}$ (in the sense of Pietsch [21]), studied, for instance, by Reinov [24] and Grønbæk and Willis [9]. A Banach space X is said to have the *metric $\mathcal{A}$-approximation property* (M-$\mathcal{A}$-AP) if for every compact set K in X and every $\varepsilon > 0$, there is an operator $T \in B_{\mathcal{A}(X,X)}$ such that $\|Tx - x\| \leq \varepsilon$ for all $x \in K$. Clearly, the MAP coincides with the M-$\mathcal{F}$-AP and the MCAP coincides with the M-$\mathcal{K}$-AP.

Below, $\mathcal{A}(X,X)$ is always equipped with the norm topology from $\mathcal{L}(X,X)$. Thus the trace mapping $V : X^* \hat{\otimes}_\pi X \to \mathcal{A}(X,X)^*$ has norm one.

Theorem 2.1. *Let $\mathcal{A}$ be an operator ideal. A Banach space X has the M-$\mathcal{A}$-AP if and only if $I_X \in V^*(B_{\mathcal{A}(X,X)^{**}})$ for the trace mapping $V : X^* \hat{\otimes}_\pi X \to \mathcal{A}(X,X)^*$.*

Remark 2.1. The condition $I_X \in V^*(B_{\mathcal{A}(X,X)^{**}})$ clearly uses the canonical identification $(X^* \hat{\otimes}_\pi X)^* = \mathcal{L}(X, X^{**})$. When the canonical identification

$(X^*\hat{\otimes}_\pi X)^* = \mathcal{L}(X^*, X^*)$ is used, then this condition becomes equivalent to $I_{X^*} \in V^*(B_{\mathcal{A}(X,X)^{**}})$. In fact, since $\mathcal{L}(X, X^{**})$ is canonically identified with $\mathcal{L}(X^*, X^*)$ under the mapping $T \to T^* \circ j_{X^*}$, the identity operator I_X, or more precisely, $j_X \circ I_X$ identifies with $(j_X \circ I_X)^* \circ j_{X^*} = I_X^* \circ j_X^* \circ j_{X^*} = I_X^* \circ I_{X^*} = I_{X^*}$.

Proof of Theorem 2.1. By definition, X has the M-$\mathcal{A}$-AP if and only if I_X belongs to the closure of $B_{\mathcal{A}(X,X)}$ in the locally convex topology on $\mathcal{L}(X, X)$ of uniform convergence on compact subsets of X. By the identification of the dual space of the locally convex space $\mathcal{L}(X, X)$ due to Grothendieck [8, Chapter I, page 113] (see, e.g., [16, page 31]), this is well known to be equivalent to the fact that I_X belongs to the closure of $B_{\mathcal{A}(X,X)}$ in the weak topology $\sigma(\mathcal{L}(X, X), X^*\hat{\otimes}_\pi X)$, i.e.,

$$I_X \in \overline{B}_{\mathcal{A}(X,X)}^{\sigma(\mathcal{L}(X,X),X^*\hat{\otimes}_\pi X)} = \mathcal{L}(X, X) \cap \overline{B}_{\mathcal{A}(X,X)}^{\sigma(\mathcal{L}(X,X^{**}),X^*\hat{\otimes}_\pi X)}. \tag{2.1}$$

Let us consider the dual systems $\langle\, X^*\hat{\otimes}_\pi X,\ \mathcal{L}(X, X^{**})\,\rangle$ and $\langle\, \mathcal{A}(X, X)^*, \mathcal{A}(X, X)^{**}\,\rangle$. By an easy and straightforward verification, looking at $B_{\mathcal{A}(X,X)}$ as a subset of $\mathcal{L}(X, X^{**})$, we have that

$$B^\circ_{\mathcal{A}(X,X)} = V^{-1}(B_{\mathcal{A}(X,X)^*}).$$

Therefore, by elementary facts from the duality theory and by the bipolar theorem,

$$V^{-1}(B_{\mathcal{A}(X,X)^*}) = V^{-1}(B^\circ_{\mathcal{A}(X,X)^{**}}) = (V^*(B_{\mathcal{A}(X,X)^{**}}))^\circ$$

and

$$V^*(B_{\mathcal{A}(X,X)^{**}}) = (V^*(B_{\mathcal{A}(X,X)^{**}}))^{\circ\circ} = B^{\circ\circ}_{\mathcal{A}(X,X)} = \overline{B}_{\mathcal{A}(X,X)}^{\sigma(\mathcal{L}(X,X^{**}),X^*\hat{\otimes}_\pi X)}.$$

This implies that (2.1) is equivalent to the condition

$$I_X \in V^*(B_{\mathcal{A}(X,X)^{**}})$$

as desired. □

Corollary 2.2. *Let X be a Banach space and let $\mathcal{A}$ be an operator ideal. If the trace mapping $V : X^*\hat{\otimes}_\pi X \to \mathcal{A}(X, X)^*$ is isometric, then X has the M-$\mathcal{A}$-AP.*

Proof. Since $V^* : \mathcal{A}(X, X)^{**} \to \mathcal{L}(X, X^{**})$ is the conjugate of an into isometry, for every $T \in \mathcal{L}(X, X^{**})$, in particular for $T = I_X$, there exists $\varphi \in \mathcal{A}(X, X)^{**}$ satisfying $V^*\varphi = T$ and $\|\varphi\| = \|T\|$. Hence, $I_X \in V^*(B_{\mathcal{A}(X,X)^{**}})$, meaning that X has the M-$\mathcal{A}$-AP. □

Remark 2.2. Corollary 2.2 also follows from [24, Corollary 1.4] affirming that X does not have the M-$\mathcal{A}$-AP if and only if there exist $\varepsilon > 0$ and $u \in X^*\hat{\otimes}_\pi X$ such that $\operatorname{trace}(u) = 1$ and $|(Vu)(T)| \le (1 - \varepsilon)\|T\|$ for all $T \in \mathcal{A}(X, X)$.

Proposition 2.3. *Let $\mathcal{A}$ be an operator ideal. If a Banach space X has the M-$\mathcal{A}$-AP, then the trace mapping $W : Y\hat{\otimes}_\pi X \to \mathcal{A}(X, Y^*)^*$ is isometric for every Banach space Y.*

Proof. Clearly $\|Wu\| \le \|u\|_\pi$ for all $u \in Y\hat{\otimes}_\pi X$. To show the converse, let us consider any $u = \sum_{n=1}^\infty y_n \otimes x_n \in Y\hat{\otimes}_\pi X$. Since $(Y\hat{\otimes}_\pi X)^* = \mathcal{L}(X, Y^*)$, there exists $T \in \mathcal{L}(X, Y^*)$ with $\|T\| = 1$ such that

$$\|u\|_\pi = \operatorname{trace}(Tu) = \sum_{n=1}^\infty (Tx_n)(y_n).$$

We may assume $x_n \to 0$ and $\sum_{n=1}^\infty \|y_n\| = 1$. Let $\varepsilon > 0$. Since $\{0, x_1, x_2, \ldots\}$ is a compact set, there exists $S \in B_{\mathcal{A}(X,X)}$ such that

$$\|Sx_n - x_n\| \le \varepsilon \quad \forall n \in \mathbb{N},$$

and therefore

$$\|Tx_n - TSx_n\| \le \varepsilon \quad \forall n \in \mathbb{N}.$$

We have $T \circ S \in B_{\mathcal{A}(X,Y^*)}$ and

$$|\|u\|_\pi - (Wu)(T\circ S)| = |\sum_{n=1}^\infty (Tx_n - TSx_n)(y_n)| \le \sum_{n=1}^\infty \|Tx_n - TSx_n\|\|y_n\| \le \varepsilon,$$

implying

$$\|u\|_\pi \le |(Wu)(T \circ S)| + \varepsilon \le \|Wu\| + \varepsilon.$$

Hence, $\|u\|_\pi \le \|Wu\|$ and W is an isometric embedding. □

Recall that an operator ideal $\mathcal{A}$ is *symmetric* if $T^* \in \mathcal{A}(Y^*, X^*)$ whenever $T \in \mathcal{A}(X, Y)$.

Corollary 2.4. *Let $\mathcal{A}$ be a symmetric operator ideal. If a Banach space X fails the M-$\mathcal{A}$-AP but its dual X^* has the M-$\mathcal{A}$-AP, then the trace mapping $W : X^*\hat{\otimes}_\pi X \to \mathcal{A}(X, X^{**})^*$ is isometric but the trace mapping $V : X^*\hat{\otimes}_\pi X \to \mathcal{A}(X, X)^*$ is not isometric.*

Proof. If X fails the M-$\mathcal{A}$-AP, then V is not isometric by Corollary 2.2.

Assume that X^* has the M-$\mathcal{A}$-AP. By Proposition 2.3, the trace mapping from $X^*\hat{\otimes}_\pi X$ to $\mathcal{A}(X^*, X^*)$ is isometric. Since $\mathcal{A}$ is a symmetric operator ideal, $\mathcal{A}(X, X^{**})$ is canonically identified with $\mathcal{A}(X^*, X^*)$ under the mapping $T \to T^* \circ j_{X^*}$. It follows that also W is isometric. □

Remark 2.3. Casazza and Jarchow [2] (relying on the example due to Willis [26]) have given an example of a Banach space X such that X does not have the MCAP, but X^* has the MCAP. Thus Corollary 2.4 applies in the case when $\mathcal{A} = \mathcal{K}$. It cannot be applied in the case of the symmetric operator ideal $\mathcal{A} = \mathcal{F}$ since the MAP always passes from X^* to X (this is a well-known result of Grothendieck [8, Chapter I, page 180] which is almost immediate from Theorem 1.1).

Remark 2.4. If $B_{\mathcal{A}(X,Y)}$ is dense in the set $\{T \circ S : S \in B_{\mathcal{A}(X,X)}, T \in B_{\mathcal{L}(X,Y^{**})}\}$ in the topology of pointwise convergence, then the proof of Proposition 2.3 yields that $V : Y^*\hat{\otimes}_\pi X \to \mathcal{A}(X, Y)^*$ is isometric, in particular, $V : X^*\hat{\otimes}_\pi X \to \mathcal{A}(X, X)^*$ is isometric.

(In fact, choose $N \in \mathbb{N}$ such that $\sup_k \|x_k\| \sum_{n>N} \|y_n^*\| \le \varepsilon$ (we use the notation y_n^* instead of y_n here). Choose $R \in B_{\mathcal{A}(X,Y)}$ such that $\|TSx_n - Rx_n\| \le \varepsilon$ for $n = 1, 2, \ldots, N$. Then $|\|u\|_\pi - (Vu)(R)| \le |\|u\|_\pi - (Wu)(T \circ$

$S)|+|(Wu)(T\circ S)-(Vu)(R)|\leq \varepsilon+\sum_{n=1}^{\infty}\|TSx_n-Rx_n\|\|y_n^*\|\leq \varepsilon+\varepsilon+2\varepsilon = 4\varepsilon$.)

The assumption from Remark 2.4 is clearly satisfied in the case of $\mathcal{A}=\mathcal{F}$ thanks to the Grothendieck's result [8, Chapter I, page 129] that $B_{\mathcal{F}(X,Y)}$ is always dense in $B_{\mathcal{F}(X,Y^{**})}$ in the topology of pointwise convergence. This gives a proof of the "only if" part of Theorem 1.1. However, we would like to indicate another reason - this is Proposition 2.5 below - why the "only if" part of Theorem 1.1 is immediate from Proposition 2.3.

Let E be a subspace of a normed linear space F. A linear operator $\Phi : E^* \to F^*$ is called a *Hahn-Banach extension operator* if $(\Phi e^*)(e) = e^*(e)$ and $\|\Phi e^*\| = \|e^*\|$ for all $e \in E$ and all $e^* \in E^*$. In [12, Theorem 3], using the principle of local reflexivity, it was proved that there always exists a Hahn-Banach extension operator $\Phi : \mathcal{F}(X,Y)^* \to \mathcal{F}(X,Y^{**})^*$ (see also [22] and [15, Corollary 2.3]). The next result shows that, thanks to Grothendieck, a very good Hahn-Banach extension operator can be chosen here (note that the principle of local reflexivity will not be used).

Proposition 2.5. *Let X and Y be Banach spaces. Then there exists a Hahn-Banach extension operator $\Phi : \mathcal{F}(X,Y)^* \to \mathcal{F}(X,Y^{**})^*$ such that $W = \Phi\circ V$, where $V : Y^*\hat{\otimes}_\pi X^{**} \to \mathcal{F}(X,Y)^*$ and $W : Y^*\hat{\otimes}_\pi X^{**} \to \mathcal{F}(X,Y^{**})^*$ are trace mappings. In particular, $\|Wu\| = \|Vu\|$ for all $u \in Y^*\hat{\otimes}_\pi X^{**}$.*

Proof. Let I_1 be the natural isometry from $\mathcal{F}(X,Y)^*$ onto $\mathcal{I}(X^*,Y^*)$, the Banach space of integral operators from X^* to Y^* (see [8, Chapter I, pages 124-125] or, e.g., [4, pages 231-232] or [25, page 58]). Let $I_2 : \mathcal{F}(X,Y^{**})^* \to \mathcal{I}(X^*,Y^{***})$ also be the natural isometry. Let $J : \mathcal{I}(X^*,Y^*) \to \mathcal{I}(X^*,Y^{***})$ be the natural embedding (defined by $J(T) = j_{Y^*}\circ T$, $T \in \mathcal{I}(X^*,Y^*)$). Then J is isometric (see [8, Chapter I, page 128] or, e.g., [4, page 233] or [25, page 65]). The map $\Phi : \mathcal{F}(X,Y)^* \to \mathcal{F}(X,Y^{**})^*$ defined by $\Phi = I_2^{-1}\circ J\circ I_1$ is clearly a Hahn-Banach extension operator, since for all $f \in \mathcal{F}(X,Y)^*$, $x^* \in X^*$, and $y \in Y$,

$$(\Phi f)(x^*\otimes y) = ((J(I_1 f))x^*)(y) = ((I_1 f)x^*)(y) = f(x^*\otimes y).$$

For all $y^* \in Y^*$, $x^{**} \in X^{**}$, and $y^{**} \in Y^{**}$, we also have

$$(I_2(W(y^*\otimes x^{**}))x^*)(y^{**}) = y^{**}(x^{**}(x^*)y^*)$$

and

$$((J(I_1(V(y^*\otimes x^{**})))x^*)(y^{**}) = y^{**}(I_1(V((y^*\otimes x^{**}))x^*) = y^{**}(x^{**}(x^*)y^*),$$

yielding that $I_2\circ W = J\circ I_1\circ V$. Hence $W = \Phi\circ V$. □

Corollary 2.6. *Let X and Y be Banach spaces. If X has the MAP, then the trace mapping $V : Y^*\hat{\otimes}_\pi X \to \mathcal{F}(X,Y)^*$ is isometric.*

Proof. This is immediate from Propositions 2.3 and 2.5. □

As we mentioned, the principle of local reflexivity was not used in the proof of Proposition 2.5 above. It is interesting to observe that Proposition 2.5 implies the following result that might be called a version of the principle of local reflexivity. In fact, as the forthcoming paper [20] shows, it applies to give a short proof of the principle of local reflexivity.

Corollary 2.7. *Let X and Y be Banach spaces. If $T \in \mathcal{F}(X, Y^{**})$, then there exists a net $(T_\alpha) \subset \mathcal{F}(X, Y)$ such that $\sup_\alpha \|T_\alpha\| \le \|T\|$, $T_\alpha x \to Tx$ for all $x \in T^{-1}(Y)$, and $T_\alpha^* y^* \to T^* y^*$ for all $y^* \in Y^*$.*

Proof. Let $W = \Phi \circ V$ be as in Proposition 2.5. Then $\Phi^*(T) \in \|T\| B_{\mathcal{F}(X,Y)^{**}}$. By Goldstine's theorem, there is a net $(T_\alpha) \subset \mathcal{F}(X, Y)$ such that $\sup_\alpha \|T_\alpha\| \le \|T\|$ and (T_α) converges weak* to $\Phi^*(T)$. Thus, for any $u \in Y^* \hat{\otimes}_\pi X^{**}$, we have

$$(Wu)(T) = (\Phi V u)(T) = (\Phi^*(T))(Vu) = \lim_\alpha (Vu)(T_\alpha).$$

In particular, taking $u = y^* \otimes x^{**}$ yields that

$$x^{**}(T_\alpha^* y^*) \underset{\alpha}{\longrightarrow} x^{**}(T^* y^*) \quad \forall\, y^* \in Y^*,\ \forall\, x^{**} \in X^{**}.$$

This means that $T_\alpha^* \to T^*|_{Y^*}$ in the weak operator topology of $\mathcal{L}(Y^*, X^*)$. Since the weak and strong operator topologies yield the same dual space (see, e.g., [5, Theorem VI.1.4]), after passing to convex combinations, we may assume that $T_\alpha^* y^* \to T^* y^*$ for all $y^* \in Y^*$. This shows that $T_\alpha \to T$ in the weak operator topology of $\mathcal{L}(T^{-1}(Y), Y)$. Therefore, after passing to a net of convex combinations, we may also assume that $T_\alpha x \to Tx$ for all $x \in T^{-1}(Y)$. □

In contrast to Proposition 2.5, let us recall that if X is the space defined by Casazza and Jarchow [2], then (as it was proved in [15, Example 1.2]) there does *not* exist any Hahn-Banach extension operator $\Phi : \mathcal{K}(X, X)^* \to \mathcal{K}(X, X^{**})^*$. Recall also that this X fails the MCAP, but all its duals have the MCAP.

If a Banach space X has the MCAP, then developing an idea due to J. Johnson (see the proof of Lemma 1 in [10]), a "relatively" good Hahn-Banach extension operator $\Phi : \mathcal{K}(Y, X)^* \to \mathcal{K}(Y, X^{**})^*$ can be constructed for all Banach spaces Y. This will be clear from the following result (see also Remark 2.7) which gives some criteria of the M-$\mathcal{A}$-AP whenever $\mathcal{A}$ is $\mathcal{F}$ or $\mathcal{K}$ or the operator ideal $\mathcal{W}$ of weakly compact operators.

Theorem 2.8. *Let X be a Banach space and let $\mathcal{A}$ be $\mathcal{F}$, $\mathcal{K}$, or $\mathcal{W}$. The following statements are equivalent.*

(a) *X has the M-$\mathcal{A}$-AP.*

(b) *For every Banach space Y, there exists a norm one operator $\Phi : \mathcal{A}(Y, X)^* \to \mathcal{L}(Y, X^{**})^*$, which is a Hahn-Banach extension operator whenever $\mathcal{A}$ is $\mathcal{F}$ or $\mathcal{K}$, such that $V^*(\Phi^*(T)) = T$ for all $T \in \mathcal{L}(Y, X)$, where $V : X^* \hat{\otimes}_\pi Y \to \mathcal{A}(Y, X)^*$ is the trace mapping.*

(c) *For every Banach space Y, there exists a norm one operator $\Phi : \mathcal{A}(X, Y)^* \to \mathcal{L}(X, Y)^*$ such that $V^*(\Phi^*(T)) = T$ for all $T \in \mathcal{L}(X, Y)$, where $V : Y^* \hat{\otimes}_\pi X \to \mathcal{A}(X, Y)^*$ is the trace mapping.*

(b′) *For every Banach space Y, there exists an into isometry $U : \mathcal{L}(Y, X) \to \mathcal{A}(Y, X)^{**}$ such that $V^*(U(T)) = T$ for all $T \in \mathcal{L}(Y, X)$, where $V : X^* \hat{\otimes}_\pi Y \to \mathcal{A}(Y, X)^*$ is the trace mapping. Moreover, if $\mathcal{A}$ is $\mathcal{F}$ or $\mathcal{K}$, then $U(T) = T$ for all $T \in \mathcal{A}(Y, X)$.*

(c′) *For every Banach space Y, there exists an into isometry $U : \mathcal{L}(X, Y) \to \mathcal{A}(X, Y)^{**}$ such that $V^*(U(T)) = T$ for all $T \in \mathcal{L}(X, Y)$, where $V : Y^* \hat{\otimes}_\pi X \to \mathcal{A}(X, Y)^*$ is the trace mapping.*

(d) *There exists a norm one operator* $U : \operatorname{span}\{I_X\} \to \mathcal{A}(X,X)^{**}$ *such that* $V^*(U(I_X)) = I_X$*, where* $V : X^*\hat{\otimes}_\pi X \to \mathcal{A}(X,X)^*$ *is the trace mapping.*

Remark 2.5. If $\Phi : \mathcal{A}(Y,X)^* \to \mathcal{L}(Y,X^{**})^*$ is a Hahn-Banach extension operator, and $\mathcal{J}_{\mathcal{L}} : \mathcal{L}(Y,X) \to \mathcal{L}(Y,X^{**})$ and $\mathcal{J}_{\mathcal{A}} : \mathcal{A}(Y,X^{**}) \to \mathcal{L}(Y,X^{**})$ are natural embeddings (defined by $\mathcal{J}_{\mathcal{L}}(T) = j_X \circ T$ and $\mathcal{J}_{\mathcal{A}}(T) = T$), then clearly $\mathcal{J}_{\mathcal{L}}^* \circ \Phi : \mathcal{A}(Y,X)^* \to \mathcal{L}(Y,X)^*$ and $\mathcal{J}_{\mathcal{A}}^* \circ \Phi : \mathcal{A}(Y,X)^* \to \mathcal{A}(Y,X^{**})^*$ are Hahn-Banach extension operators.

Proof of Theorem 2.8. (a) ⇒ (b) & (c). Let $(S_\alpha) \subset B_{\mathcal{A}(X,X)}$ be a net converging to I_X uniformly on compact subsets of X. Since (S_α) is contained in $B_{\mathcal{A}(X,X)^{**}}$, which is weak* compact, after passing to a subnet, we may assume that the limit $\lim_\alpha f(S_\alpha)$ exists for all $f \in \mathcal{A}(X,X)^*$. (This is precisely J. Johnson's argument [10, proof of Lemma 1] that we followed here.)

To prove (b), let us observe that each pair of $f \in \mathcal{A}(Y,X)^*$ and $T \in \mathcal{L}(Y,X^{**})$ defines a functional f_T by $f_T(S) = f(S^{**} \circ T)$, $S \in \mathcal{A}(X,X)$. (Here we used that $\operatorname{ran} S^{**} \subset X$ whenever $S \in \mathcal{W}(X,X)$.) Since $f_T \in \mathcal{A}(X,X)^*$, we can define a linear norm one operator $\Phi : \mathcal{A}(Y,X)^* \to \mathcal{L}(Y,X^{**})^*$ by

$$(\Phi f)(T) = \lim_\alpha f_T(S_\alpha) = \lim_\alpha f(S_\alpha^{**} \circ T), \quad f \in \mathcal{A}(Y,X)^*, \quad T \in \mathcal{L}(Y,X^{**}).$$

If $\mathcal{A} \subset \mathcal{K}$ and $T \in \mathcal{A}(Y,X)$, then $\|S_\alpha^{**} \circ T - T\| \to 0$ because $S_\alpha \to I_X$ uniformly on the relatively compact subset $T(B_Y)$ of X. This implies that $(\Phi f)(T) = f(T)$ whenever $T \in \mathcal{A}(Y,X)$ and $f \in \mathcal{A}(Y,X)^*$. Hence, $\|\Phi f\| \geq \|f\|$ and, since $\|\Phi\| = 1$, $\|\Phi f\| = \|f\|$ for all $f \in \mathcal{A}(Y,X)^*$. This means that Φ is a Hahn-Banach extension operator.

Finally, let $T \in \mathcal{L}(Y,X)$. Since for all $u = x^* \otimes y$ with $x^* \in X^*$ and $y \in Y$,

$$\begin{aligned}\langle V^*(\Phi^*(T)), u\rangle &= (\Phi(Vu))(T) = \lim_\alpha (x^* \otimes y)(S_\alpha \circ T)\\ &= \lim_\alpha x^*(S_\alpha T y) = x^*(Ty) = \langle T, u\rangle,\end{aligned}$$

we also have that $V^*(\Phi^*(T)) = T$ for all $T \in \mathcal{L}(Y,X)$.

To prove (c), one argues similarly, considering $f_T(S) = f(T \circ S)$, $S \in \mathcal{A}(X,X)$, for each pair $f \in \mathcal{A}(X,Y)^*$ and $T \in \mathcal{L}(X,Y)$. Then

$$(\Phi f)(T) = \lim_\alpha f_T(S_\alpha) = \lim_\alpha f(T \circ S_\alpha), \quad f \in \mathcal{A}(X,Y)^*, \quad T \in \mathcal{L}(X,Y),$$

defines the required norm one operator $\Phi : \mathcal{A}(X,Y)^* \to \mathcal{L}(X,Y)^*$.

(b) ⇒ (b′) and (c) ⇒ (c′). Take $U = \Phi^*|_{\mathcal{L}(Y,X)}$ and $U = \Phi^*|_{\mathcal{L}(X,Y)}$, respectively. For (b′), recall that if Φ is a Hahn-Banach extension operator, then Φ^* is a norm one projection from $\mathcal{L}(Y,X^{**})^{**}$ onto $\mathcal{A}(Y,X)^{**}$. Hence $\Phi^*|_{\mathcal{A}(Y,X)}$ is the canonical embedding.

(b′) ∨ (c′) ⇒ (d). This is obvious.

(d) ⇒ (a). This is immediate from Theorem 2.1. □

Remark 2.6. The proof of Theorem 2.8 shows that (a) ⇔ (c) ⇔ (c′) ⇔ (d) for any operator ideal $\mathcal{A}$.

Remark 2.7. J. Johnson [10, Lemma 2] has proved that if X has the MAP, then there exists an into isometry $U : \mathcal{L}(Y,X) \to \mathcal{K}(Y,X)^{**}$ such that $U(T) = T$ for all $T \in \mathcal{K}(Y,X)$. The implication (a) ⇒ (b′) shows, in particular, that this is true whenever X has the MCAP.

Comparing with the situation in Proposition 2.5, it should be noted that if Φ is defined as in (b) of Theorem 2.8, then the equality $W = \Phi \circ V$ fails in general even for W and V restricted to $X^* \otimes X$, because it would easily imply that X^* has the M-$\mathcal{A}$-AP with conjugate operators. We shall discuss the latter property in the next section.

3. Metric approximation properties with conjugate operators

It is convenient to extend the notion of the (M)CAP with conjugate operators from $\mathcal{K}$ to any operator ideal $\mathcal{A}$ as follows. We say that the dual space X^* of a Banach space X has the *$\mathcal{A}$-approximation property ($\mathcal{A}$-AP) with conjugate operators* if I_{X^*} belongs to the closure of the subset $\{T^* : T \in \mathcal{A}(X,X)\}$ of $\mathcal{L}(X^*, X^*)$ with respect to the topology of uniform convergence on compact subsets of X^*. If $\mathcal{A}(X,X)$ can be replaced by $B_{\mathcal{A}(X,X)}$, then we say that X^* has the M-$\mathcal{A}$-AP *with conjugate operators.*

An example due to Grønbœk and Willis [9] shows that the CAP of X^* does not imply the CAP with conjugate operators. Moreover, Casazza and Jarchow [2] have shown that there is a Banach space X failing the MCAP such that all its duals X^*, X^{**}, ... have the MCAP.

By Theorem 2.1, X^* has the M-$\mathcal{A}$-AP if and only if $I_{X^*} \in V^*(B_{\mathcal{A}(X^*,X^*)^{**}})$ for the trace mapping $V : X^{**}\hat{\otimes}_\pi X^* \to \mathcal{A}(X^*,X^*)^*$. Note that $\mathcal{A}(X^*,X^*)$ can be canonically identified with $\mathcal{A}(X,X^{**})$ (under the mapping $T \to T^*|_X$) whenever $\mathcal{A}$ is $\mathcal{F}$, $\mathcal{K}$ or $\mathcal{W}$. Hence, for these operator ideals, X^* has the M-$\mathcal{A}$-AP if and only if $I_{X^*} \in V^*(B_{\mathcal{A}(X,X^{**})^{**}})$ for the trace mapping $V : X^*\hat{\otimes}_\pi X^{**} \to \mathcal{A}(X,X^{**})^*$. This should be compared with the following criterion.

Theorem 3.1. *Let $\mathcal{A}$ be an operator ideal and let X be a Banach space. Consider the trace mapping $V : X^*\hat{\otimes}_\pi X^{**} \to \mathcal{A}(X,X)^*$. Then*

(a) *X^* has the $\mathcal{A}$-AP with conjugate operators if and only if $I_{X^*} \in V^*(\mathcal{A}(X,X)^{**})$, or equivalently, $I_{X^{**}} \in V^*(\mathcal{A}(X,X)^{**})$.*

(b) *X^* has the M-$\mathcal{A}$-AP with conjugate operators if and only if $I_{X^*} \in V^*(B_{\mathcal{A}(X,X)^{**}})$, or equivalently, $I_{X^{**}} \in V^*(B_{\mathcal{A}(X,X)^{**}})$.*

Proof. (a) The proof is similar to the proof of (b) below: one only has to replace the unit balls by the corresponding spaces.

(b) The proof is similar to the proof of Theorem 2.1. Let $B = \{T^* : T \in B_{\mathcal{A}(X,X)}\}$. First, we observe that X^* has the M-$\mathcal{A}$-AP with conjugate operators if and only if

$$I_{X^*} \in \overline{B}^{\sigma(\mathcal{L}(X^*,X^*),X^*\hat{\otimes}_\pi X^{**})} = \mathcal{L}(X^*,X^*) \cap \overline{B}^{\sigma(\mathcal{L}(X^*,X^{***}),X^*\hat{\otimes}_\pi X^{**})}. \tag{3.1}$$

Then, considering the dual systems $\langle X^*\hat{\otimes}_\pi X^{**}, \mathcal{L}(X^*,X^{***})\rangle$ and $\langle \mathcal{A}(X,X)^*, \mathcal{A}(X,X)^{**}\rangle$ and looking at B as a subset of $\mathcal{L}(X^*,X^{***})$, we see (similarly to the proof of Theorem 2.1) that

$$B^\circ = V^{-1}(B_{\mathcal{A}(X,X)^*}) = (V^*(B_{\mathcal{A}(X,X)^{**}}))^\circ.$$

Hence,

$$V^*(B_{\mathcal{A}(X,X)^{**}}) = B^{\circ\circ} = \overline{B}^{\sigma(\mathcal{L}(X^*,X^{***}),X^*\hat{\otimes}_\pi X^{**})},$$

implying that (3.1) is equivalent to the condition

$$I_{X^*} \in V^*(B_{\mathcal{A}(X,X)^{**}}).$$

When the canonical identification $(X^*\hat{\otimes}_\pi X^{**})^* = \mathcal{L}(X^{**}, X^{**})$ is used, then (see Remark 2.1) this condition reads as

$$I_{X^{**}} \in V^*(B_{\mathcal{A}(X,X)^{**}}).$$

□

By an important result of Grothendieck [8, Chapter I, proof of Corollary 2 on page 182 together with Corollary 3, pages 134-135], separable dual spaces with the AP have the MAP. The proof of this result “has always been a little mysterious” (see [1, page 289]). Our two different proofs of the following more general result rely on criteria from Theorem 3.1.

Corollary 3.2 (see [4, page 246] and [23, Theorem 4] for the MAP and [7, Corollary 1.6] for the MCAP)**.** *Let X be a Banach space such that X^* or X^{**} has the Radon-Nikodým property. Let $\mathcal{A}$ be $\mathcal{F}$ or $\mathcal{K}$. If X^* has the $\mathcal{A}$-AP with conjugate operators, then X^* has the M-$\mathcal{A}$-AP with conjugate operators.*

Remark 3.1. Let us recall that if X^* has the AP, then X^* has the AP with conjugate operators (this is clear from the principle of local reflexivity).

Proof 1 *of Corollary* 3.2. Let us consider the trace mapping $V : X^*\hat{\otimes}_\pi X^{**} \to \mathcal{A}(X,X)^*$. By Theorem 3.1, (a), X^* has the $\mathcal{A}$-AP with conjugate operators if and only if $I_{X^*} \in V^*(\mathcal{A}(X,X)^{**})$. Let $I_{X^*} = V^*\varphi$ for some $\varphi \in \mathcal{A}(X,X)^{**}$. By the description of $\mathcal{K}(X,X)^*$ due to Feder and Saphar [6, Theorem 1] (which assumes that X^* or X^{**} has the Radon-Nikodým property), V is a quotient mapping. Therefore V^* is an into isometry, implying that $\|\varphi\| = \|V^*\varphi\| = \|I_{X^*}\| = 1$. Hence $I_{X^*} \in V^*(B_{\mathcal{A}(X,X)^{**}})$, meaning that (see Theorem 3.1, (b)) X^* has the M-$\mathcal{A}$-AP with conjugate operators. □

Our second proof below will provide an explicit form of φ showing, in essence, why $I_{X^*} \in V^*(B_{\mathcal{A}(X,X)^{**}})$. We shall not use the part (a) of Theorem 3.1. Instead, we shall depart from the definition of the $\mathcal{A}$-AP with conjugate operators and apply the Feder-Saphar theorem to construct a norm one operator $\Phi : \mathcal{A}(X,X)^* \to \mathcal{L}(X,X)^*$ such that $\varphi = \Phi^*(I_X)$.

Proof 2 *of Corollary* 3.2. We assume that (S_α^*) with $S_\alpha \in \mathcal{A}(X,X)$ converges to I_{X^*} uniformly on compact subsets of X^*. Similarly to the proofs of Theorems 1.2 and 5.1 in [14] or of Theorem 3 in [19], we can define a norm one operator $\Phi : \mathcal{A}(X,X)^* \to \mathcal{L}(X,X)^*$ by

$$(\Phi f)(T) = \lim_\alpha f(T \circ S_\alpha), \qquad f \in \mathcal{A}(X,X)^*, \quad T \in \mathcal{L}(X,X). \tag{3.2}$$

[For completeness, we shall give the construction of Φ. It relies on the description of $\mathcal{K}(X,X)^*$ due to Feder and Saphar [6, Theorem 1]. Let $f \in \mathcal{A}(X,X)^*$. Since $\mathcal{A}(X,X) \subset \mathcal{K}(X,X)$, the description of $\mathcal{K}(X,X)^*$ yields (through the Hahn-Banach theorem, if $\mathcal{A} = \mathcal{F}$) that there exists $u \in X^*\hat{\otimes}_\pi X^{**}$ such that $\|f\| = \|u\|_\pi$ and $f(S) = \mathrm{trace}(S^*u)$, for all

$S \in \mathcal{A}(X,X)$. We may assume that $u = \sum_{n=1}^{\infty} x_n^* \otimes x_n^{**}$ with $x_n^* \to 0$ and $\sum_{n=1}^{\infty} \|x_n^{**}\| = 1$. If $T \in \mathcal{L}(X,X)$, then $T \circ S_\alpha \in \mathcal{A}(X,X)$ and

$$\begin{aligned} |\mathrm{trace}(T^*u) - f(T \circ S_\alpha)| =& |\sum_{n=1}^{\infty} x_n^{**}(T^* x_n^*) - \sum_{n=1}^{\infty} x_n^{**}((S_\alpha^* \circ T^*)x_n^*)| \\ \le& \sum_{n=1}^{\infty} |x_n^{**}((I_{X^*} - S_\alpha^*)T^* x_n^*)| \\ \le& \sup_n \|(I_{X^*} - S_\alpha^*)T^* x_n^*\| \underset{\alpha}{\longrightarrow} 0, \end{aligned}$$

because the set $\{0, T^*x_1^*, T^*x_2^*, \ldots\}$ is compact. Hence $\lim_\alpha f(T \circ S_\alpha) = \mathrm{trace}(T^*u)$ defines the norm one operator Φ.]

Let us consider the trace mapping $V : X^*\hat{\otimes}_\pi X^{**} \to \mathcal{A}(X,X)^*$. Observe that, by (3.2), for all $u = x^* \otimes x^{**} \in X^*\hat{\otimes}_\pi X^{**}$, we have

$$\begin{aligned} \langle V^*\Phi^* I_X, u\rangle =& (\Phi(Vu))(I_X) = \lim_\alpha (Vu)(S_\alpha) = \lim_\alpha x^{**}(S_\alpha^* x^*) \\ =& x^{**}(x^*) = \langle I_{X^*}, u\rangle. \end{aligned}$$

This clearly implies that $I_{X^*} = V^*\Phi^* I_X$. Hence $I_{X^*} \in V^*(B_{\mathcal{A}(X,X)^{**}})$ meaning that (see Theorem 3.1, (b)) X^* has the M-$\mathcal{A}$-AP with conjugate operators. □

Remark 3.2. There are several different proofs of versions of Corollary 3.2. The proofs in [4], [23], and [7] were modelled after Grothendieck's proofs in [8]. For separable dual spaces, an alternative proof is due to Lindenstrauss and Tzafriri [16, pages 39-40]. The latter proof was adapted in [3] under the assumption that X^* has the Radon-Nikodým property. The proofs in [7] and [14, Corollary 5.3], like the two ours above, use the description of $\mathcal{K}(X,X)^*$ due to Feder and Saphar [6].

Similarly to Corollary 2.2, Theorem 3.1 immediately implies the following result that should be compared with Corollary 2.2.

Corollary 3.3. *Let $\mathcal{A}$ be an operator ideal and let X be a Banach space. If the trace mapping $V : X^*\hat{\otimes}_\pi X^{**} \to \mathcal{A}(X,X)^*$ is isometric, then X^* has the M-$\mathcal{A}$-AP with conjugate operators.*

Connecting Theorems 2.1 and 3.1 through Proposition 2.5 yields the following alternative proof of the well-known result below. Note that the principle of local reflexivity is not used in the proof.

Corollary 3.4 (see [11, Lemma 2] and, e.g., [1, Proposition 3.5])**.** *The dual space X^* of a Banach space X has the MAP if and only if X^* has the MAP with conjugate operators.*

Proof. Consider the trace mappings $V : X^*\hat{\otimes}_\pi X^{**} \to \mathcal{F}(X,X)^*$ and $W : X^*\hat{\otimes}_\pi X^{**} \to \mathcal{F}(X,X^{**})^*$. By Theorem 2.1, X^* has the MAP if and only if

$$I_{X^*} = W^*\psi \quad \text{for some } \psi \in B_{\mathcal{F}(X,X^{**})^{**}}. \tag{3.3}$$

By Theorem 3.1, (b), X^* has the MAP with conjugate operators if and only if

$$I_{X^*} = V^*\varphi \quad \text{for some } \varphi \in B_{\mathcal{F}(X,X)^{**}}. \tag{3.4}$$

Since $V^* = W^* \circ \Phi^*$, where $\Phi : \mathcal{F}(X,X)^* \to \mathcal{F}(X,X^{**})^*$ is an into isometry (see Proposition 2.5), conditions (3.3) and (3.4) are clearly equivalent. □

By Proposition 2.3, if the dual space X^* of a Banach space X has the M-$\mathcal{A}$-AP, then the trace mapping $V : X^*\hat{\otimes}_\pi Y \to \mathcal{A}(Y,X^{**})^*$ is isometric for every Banach space Y, whenever $\mathcal{A}$ is $\mathcal{F}$, $\mathcal{K}$, or $\mathcal{W}$.

Proposition 3.5. *Let $\mathcal{A}$ be $\mathcal{F}$, $\mathcal{K}$, or $\mathcal{W}$. If the dual Banach space X^* of a Banach space X has the M-$\mathcal{A}$-AP with conjugate operators, then the trace mapping $V : X^*\hat{\otimes}_\pi Y \to \mathcal{A}(Y,X)^*$ is isometric for every Banach space Y.*

Proof. Clearly $\|Vu\| \le \|u\|_\pi$ for all $u \in X^*\hat{\otimes}_\pi Y$. To show the converse, we shall argue similarly to the proof of Proposition 2.3. Let us consider any $u = \sum_{n=1}^\infty x_n^* \otimes y_n \in X^*\hat{\otimes}_\pi Y$. We may assume that $x_n^* \to 0$ and $\sum_{n=1}^\infty \|y_n\| = 1$. Since $(X^*\hat{\otimes}_\pi Y)^* = \mathcal{L}(Y,X^{**})$, there is $T \in \mathcal{L}(Y,X^{**})$ with $\|T\| = 1$ such that

$$\|u\|_\pi = \operatorname{trace}(Tu) = \sum_{n=1}^\infty (Ty_n)(x_n^*).$$

Let $\varepsilon > 0$. Since $\{0, x_1^*, x_2^*, \ldots\}$ is a compact subset of X^*, there is $S \in B_{\mathcal{A}(X,X)}$ such that

$$\|x_n^* - S^*x_n^*\| \le \varepsilon, \quad \forall n \in \mathbb{N}.$$

We have $S^{**} \circ T \in B_{\mathcal{A}(Y,X)}$ (recall that $\operatorname{ran} S^{**} \subset X$ because S is weakly compact) and

$$\begin{aligned} |\|u\|_\pi - (Vu)(S^{**} \circ T)| &= |\sum_{n=1}^\infty (Ty_n)(x_n^*) - (Ty_n)(S^*x_n^*)| \\ &\le \sum_{n=1}^\infty \|Ty_n\|\|x_n^* - S^*x_n^*\| \le \varepsilon. \end{aligned}$$

Hence, $\|u\|_\pi \le \|Vu\|$, and V is an isometric embedding. □

Theorem 3.6. *Let X be a Banach space and let $\mathcal{A}$ be $\mathcal{F}$, $\mathcal{K}$, or $\mathcal{W}$. The following statements are equivalent.*

(a) *X^* has the M-$\mathcal{A}$-AP with conjugate operators.*

(b) *For every Banach space Y, there exists a norm one operator $\Phi : \mathcal{A}(Y,X)^* \to \mathcal{L}(Y,X^{**})^*$, which is a Hahn-Banach extension operator whenever $\mathcal{A} = \mathcal{F}$ or $\mathcal{K}$, such that $\Phi \circ V = W$, where $V : X^*\hat{\otimes}_\pi Y^{**} \to \mathcal{A}(Y,X)^*$ and $W : X^*\hat{\otimes}_\pi Y^{**} \to \mathcal{L}(Y,X^{**})$ are trace mappings.*

(c) *For every Banach space Y, there exists a norm one operator $\Phi : \mathcal{A}(X,Y)^* \to \mathcal{L}(X,Y)^*$, which is a Hahn-Banach extension operator whenever $\mathcal{A} = \mathcal{F}$ or $\mathcal{K}$, such that $\Phi \circ V = W$, where $V : Y^*\hat{\otimes}_\pi X^{**} \to \mathcal{A}(X,Y)^*$ and $W : Y^*\hat{\otimes}_\pi X^{**} \to \mathcal{L}(X,Y)^*$ are trace mappings.*

(b′) *For every Banach space Y, there exists an into isometry $U : \mathcal{L}(Y,X^{**}) \to \mathcal{A}(Y,X)^{**}$ such that $V^*(U(T)) = (j_{X^*})^* \circ T^{**}$ for all $T \in \mathcal{L}(Y,X^{**})$, where $V : X^*\hat{\otimes}_\pi Y^{**} \to \mathcal{A}(Y,X)^*$ is the trace mapping. Moreover, if $\mathcal{A}$ is $\mathcal{F}$ or $\mathcal{K}$, then $U(T) = T$ for all $T \in \mathcal{A}(Y,X)$.*

(c′) *For every Banach space Y, there exists an into isometry $U : \mathcal{L}(X,Y) \to \mathcal{A}(X,Y)^{**}$ such that $V^*(U(T)) = T^{**}$ for all $T \in \mathcal{L}(X,Y)$, where*

$V : Y^*\hat{\otimes}_\pi X^{**} \to \mathcal{A}(X,Y)^*$ *is the trace mapping. Moreover, if* $\mathcal{A}$ *is* $\mathcal{F}$ *or* $\mathcal{K}$*, then* $U(T) = T$ *for all* $T \in \mathcal{A}(X,Y)$*.*

(d) *There exists a norm one operator* $U : \mathrm{span}\{I_X\} \to \mathcal{A}(X,X)^{**}$ *such that* $V^*(U(I_X)) = I_{X^{**}}$*, where* $V : X^*\hat{\otimes}_\pi X^{**} \to \mathcal{A}(X,X)^*$ *is the trace mapping.*

Proof. The proof is similar to the proof of Theorem 2.8.

(a) ⇒ (b) & (c). Let $(S_\alpha) \subset B_{\mathcal{A}(X,X)}$ be a net such that (S_α^*) converges to I_{X^*} uniformly on the compact subsets of X^*. As in the proof of Theorem 2.8, we may assume that the limit $\lim_\alpha f(S_\alpha)$ exists for all $f \in \mathcal{A}(X,X)^*$. We define the norm one operators Φ by the same formulas as in the corresponding parts of the proof of Theorem 2.8. By this proof, if $\mathcal{A} \subset \mathcal{K}$ in (b), then Φ is a Hahn-Banach extension operator. The same is true, by the same argument, if $\mathcal{A} \subset \mathcal{K}$ in (c), because $\|T \circ S_\alpha - T\| = \|S_\alpha^* \circ T^* - T^*\| \to 0$ for all $T \in \mathcal{A}(X,Y)$. To complete the proof of (b), let $u = x^* \otimes y^{**} \in X^*\hat{\otimes}_\pi Y^{**}$ and $T \in \mathcal{L}(Y, X^{**})$. Then

$$\begin{aligned}(\Phi(Vu))(T) =& \lim_\alpha (x^* \otimes y^{**})(S_\alpha^{**} \circ T) = \lim_\alpha y^{**}(T^* S_\alpha^* x^*)\\ =& y^{**}(T^*x^*) = (Wu)(T).\end{aligned}$$

Hence, $\Phi \circ V = W$. The proof of (c) can be completed similarly.

(b) ⇒ (b′). Put $U = \Phi^*|_{\mathcal{L}(Y,X^{**})}$ and let $T \in \mathcal{L}(Y, X^{**})$. Then $V^*(U(T)) = W^*(T) = (j_{X^*})^* \circ T^{**}$, since for all $u = x^* \otimes y^{**}$,

$$\langle W^*(T), u\rangle = (Wu)(T) = (T^{**}y^{**})(j_{X^*}x^*) = \langle((j_{X^*})^* \circ T^{**}, u\rangle.$$

The operator U is an into isometry, because

$$\|T\| = \|T^* \circ j_{X^*}\| = \|(j_{X^*})^* \circ T^{**}\| = \|V^*(U(T))\| \le \|U(T)\| \le \|T\|.$$

If $\mathcal{A} \subset \mathcal{K}$, then $U(T) = T$ for all $T \in \mathcal{A}(Y,X)$, because Φ is a Hahn-Banach extension operator.

(c) ⇒ (c′). Put $U = \Phi^*|_{\mathcal{L}(X,Y)}$ and argue as above.

(b′) ∨ (c′) ⇒ (d). This is obvious, since $(j_X)^* \circ j_{X^*} = I_{X^*}$.

(d) ⇒ (a). This is immediate from Theorem 3.1, (b).

□

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Department of Mathematics, Agder University College, Serviceboks 422, 4604 Kristiansand, Norway

E-mail address: Asvald.Lima@hia.no

Faculty of Mathematics and Computer Science, Tartu University, J. Liivi 2, EE-50409 Tartu, Estonia

E-mail address: eve.oja@ut.ee