Uniform Distribution of Generalized Polynomials

Dissertation

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By

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CHAPTER I

Introduction

1.1 General discussion

The classical theory of uniform distribution modulo one is concerned with the distribution of the fractional parts of real numbers in the unit interval $[0, 1]$. It grew out of the theory of diophantine approximations in the beginning of this century. In [40] H. Weyl proved that for any real-valued polynomial

$$p(x) = a_k x^k + \ldots + a_1 x + a_0,$$  \hspace{1cm} (1.1)

where at least one coefficient $a_i, i \geq 1,$ is irrational, the sequence $p(n), n = 1, 2, \ldots,$ is uniformly distributed (mod 1). This was a generalization of Hardy and Littlewood’s results from 1914 on the fractional part of $n^k \theta$, [10]. Weyl’s proof was simpler and it was Weyl who introduced the notion of uniform distribution (mod 1) and gave several equivalent formulations of it, among them the following. A sequence $x_n$ is uniformly distributed (mod 1) if and only if for all $f \in C([0, 1]),$

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\{x_n\}) = \int_0^1 f(x)dx.$$  \hspace{1cm} (1.2)

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In [34] van der Corput introduced an even simpler method to prove Weyl’s theorem, based on the fact that if the sequence \(x_{n+h} - x_n, n = 1, 2, \ldots,\) is uniformly distributed (mod 1) for all integers \(h \geq 1,\) then the sequence \(x_n, n = 1, 2, \ldots,\) is also uniformly distributed (mod 1).

A natural extension of the family of real-valued polynomials arises by adding to the arithmetic operations the operation of taking the greatest integer function \(\lfloor \cdot \rfloor\). In this way functions like \(q_1(x) = \lfloor b_1x^2 + b_2x \rfloor b_3x^2 + \lfloor b_4x \rfloor b_5\) and \(q_2(x) = \lfloor \lfloor b_1x \rfloor \lfloor b_2x^2 \rfloor b_3 \rfloor b_4x\) can be obtained. We call such functions generalized polynomials. By restricting the domain to the integers, as we will do, these representations are not unique. However, it is enough to maintain certain conventions in order to treat questions of uniform distribution. We call \(b_1, \ldots, b_5\) the coefficients of these concrete representations. The simplest generalized polynomials, next after the polynomials, are those of the form \([p(x)]\beta,\) where \(p(x)\) is a polynomial and \(\beta\) a real number. The case \([an]\beta\) is treated in ([24], p.310), and it follows from [38] that \([p(n)]\beta,\) where \(p(x)\) is a polynomial (1.1), is well distributed iff either \(a_1, \ldots, a_k\) do not lie in a singly generated additive subgroup of the reals and \(\beta\) is irrational, or there exists \(\gamma \in \mathbb{R}\) such that \(a_i = b_i\gamma, b_i \in \mathbb{Q}, i = 1, \ldots, k,\) and \(\beta\) is rationally independent of \(1, \gamma.\)

Besides being of intrinsic interest, generalized polynomials appear naturally in some questions related to multiple recurrence in ergodic theory. Let \((X, \mathcal{B}, \mu)\) be a probability space, and let \(T : X \rightarrow X\) be a measurable, measure preserving transformation, i.e., if \(A \in \mathcal{B},\) then \(\mu(T^{-1}A) = \mu(A).\) Then \((X, \mathcal{B}, \mu, T)\) is said to be a measure preserving system. \(T\) is ergodic if \(A \in \mathcal{B}\) and \(T^{-1}A = A\) implies either
\( \mu(A) = 1 \) or \( \mu(A) = 0 \). It was observed already in [2] that there is a connection between generalized polynomials and nilmanifold systems. For example, let \( N \) be the group of matrices

\[
\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\} \tag{1.3}
\]

and let \( \Gamma \) be the discrete subgroup of \( N \) of matrices with integer entries. Then \( N/\Gamma \) is a nilmanifold. A rotation by

\[
\begin{pmatrix} 1 & \beta & \gamma \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix} \tag{1.4}
\]
on \( N \) induces a transformation \( T \) on \( N/\Gamma \) so that the orbit of \( \Gamma \) in \( N/\Gamma \) is

\[
T^n(\Gamma) = \begin{pmatrix} 1 & \beta n & \gamma n + \alpha \beta n(n-1)/2 \\ 0 & 1 & \alpha n \\ 0 & 0 & 1 \end{pmatrix} \Gamma, \quad n = 1, 2, \ldots. \tag{1.5}
\]
Now, each element in \( N/\Gamma \) can be identified by a matrix in \( N \) for which \( 0 \leq x, y, z < 1 \). Therefore \( T^n(\Gamma) \) corresponds to the matrix in \( N \) which has the generalized polynomials \( \alpha n \pmod{1} \), \( \beta n \pmod{1} \) and \( \gamma n + \alpha \beta n(n-1)/2 - [\alpha n] \beta n \pmod{1} \) as entries.

A measure preserving transformation \( T \) on \( (X, \mathcal{B}, \mu) \) induces a unitary operator, which we also call \( T \), on the Hilbert space \( L^2(X, \mathcal{B}, \mu) \), by \( Tf(x) = f(Tx) \). One of the early results in ergodic theory was the mean ergodic theorem, due to von Neumann (1931), which says that the averages \( \frac{1}{N} \sum_{n=0}^{N-1} T^n f \) converge to a function \( f^* \) in \( L^2 \)-norm as \( N \to \infty \) and if \( T \) is ergodic, then \( f^* = \int f \, d\mu \). An ergodic approach to some problems of additive number theory initiated by H. Furstenberg ([15], [17]), leads to the study of expressions of the form

\[
\frac{1}{N} \sum_{n=0}^{N-1} T^n f_1 T^{2n} f_2 \cdots T^{kn} f_k, \quad f_1, \ldots, f_k \in L^\infty(X, \mathcal{B}, \mu). \tag{1.6}
\]
Von Neumann’s mean ergodic theorem is the special case $k = 1$ and it can be proved by Hilbert space techniques. However, to determine the existence of and identifying the $L^2$-limit of the averages (1.6) when $k \geq 2$, is a much harder problem and it was solved so far only for $k \leq 3$. H. Fursenberg and B. Weiss found in their study of the averages

$$\frac{1}{N} \sum_{n=0}^{N-1} T^n f_1 T^{2n} f_2 T^{3n} f_3$$

(1.7)

that nilmanifold systems come into play when determining (the existence of) the limit of such averages and hence that nilmanifold systems play an important role in the structure of ergodic systems (see [18] for a further discussion of this).

It was observed long ago that the theory of uniform distribution is intimately connected with ergodic theory. A strengthening of von Neumann’s mean ergodic theorem was established by G.B. Birkhoff in 1931.

**Theorem 1.1.1 (Birkhoff ergodic theorem)** Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system and $f \in L^1(\mu)$. Then

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = f^*(x) \quad \mu\text{-a.e.}$$

(1.8)

where $f^* \in L^1(\mu)$, $f^*(Tx) = f^*(x)$ a.e., and $f^*(x) = \int f \, d\mu$ a.e. for all $f \in L^1(\mu)$ if and only if $T$ is ergodic.

When $T$ is ergodic, this can be interpreted as to say that the orbit of almost every $x \in X$ is uniformly distributed in $X$. A consequence of this is the number theoretical result, shown earlier by Borel, that almost every number (with respect to Lebesgue measure) is normal. A number $x \in \mathbb{R}$ is normal to the base $b$, where $b \in \mathbb{N}$, $b > 1$,
if and only if the sequence $b^n x, n = 1, 2, \ldots$, is uniformly distributed (mod 1). This sequence is generated by the transformation $T$ on the unit interval, given by $Tx = bx \pmod{1}$, since $T^n x = b^n x \pmod{1}$. Since the ergodic theorem is only an almost everywhere statement, uniform distribution may fail for uncountably many $x \in X$, and it does not give information about any specific $x_0 \in X$. So in order to assure uniform distribution for every $x \in X$, $T$ must be uniquely ergodic. $T$ is said to be uniquely ergodic if $X$ is a compact metric space and there exists a unique invariant measure $\mu$ for $T$, or equivalently, for all $f \in C(X)$ and for all $x \in X$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = \int f \, d\mu.$$  \hspace{1cm} (1.9)

Consider the measure preserving system $(K, \mathcal{B}, \mu, T)$, where $T$ is the rotation by an irrational number $\alpha$ on the torus $K$, written additively $Tx = x + \alpha \pmod{1}$. $T$ is uniquely ergodic with Haar measure as the unique invariant measure. Since $T^n x = x + n\alpha \pmod{1}$, by (1.9) we have for all $f \in C([0,1])$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n 0) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\{n\alpha\}) = \int_0^1 f(x) \, dx,$$

which gives (1.2) for the sequence $x_n = n\alpha$. Hence this proves that the sequence $n\alpha, n = 1, 2, \ldots$, is uniformly distributed (mod 1).

This suggests a dynamical way of proving uniform distribution of sequences. In [16], Furstenberg demonstrated this by constructing uniquely ergodic affine transformations on $k$-dimensional tori to show that any polynomial with at least one irrational coefficient is uniformly distributed (mod 1). This approach is also useful for some classes of sequences coming from generalized polynomials. For example, any
uniformly distributed sequence of form \([an]βn\) can be shown to be uniformly distributed (mod 1) by considering affine transformations on nilmanifolds. See Chapter IV for a treatment of this.

In Chapter II we will use van der Corput’s method to show that a sequence coming from a generalized polynomial \(q(x)\) having coefficients satisfying certain independence conditions, is uniformly distributed (mod 1). However, the proof in this case is far more complicated than in the polynomial case proved by van der Corput.

It is clear that we need some conditions on the coefficients of a generalized polynomial \(q(x)\) in order that the sequence \(q(n)\) will be uniformly distributed (mod 1). For example, for any real number \(γ\) the sequence \([√2n]√3\)γn − \([√6n]γn\) fails to be uniformly distributed because \([√2n]√3\) = \([√6n]\) on a set of \(n\) of positive density.

A less obvious example of a sequence which is not uniformly distributed (mod 1) is \([√2n][√3n]\sqrt{6}\), see Proposition 3.1.3. Here \(1, √2, √3, √6\) are rationally independent, but \(√2√6 = 2√3\) and \(√3√6 = 3√2\) are rationally dependent of \(1, √2, √3\). However, there are many sequences coming from generalized polynomials having dependent coefficients which are uniformly distributed (mod 1). The van der Corput’s method fails to show that the sequences \(αn\), where \(α\) is an irrational number, is uniformly distributed (mod 1), because \(α(n+h) − αn = αh\) is a constant for each fixed \(h ∈ \mathbf{N}\). In a similar way, the van der Corput’s method may fail to work for a uniformly distributed sequence coming from a generalized polynomial having dependent coefficient, as in the case of \([√2n]^2\sqrt{2}\). However, this sequence can be seen to be uniformly distributed
(mod 1) by rewriting it,

$$[\sqrt{2n}]^2 \sqrt{2} \equiv -2\sqrt{2}n^2 + \{\sqrt{2n}\}^2 \sqrt{2} \pmod{1}, \quad (1.11)$$

and using that \((2\sqrt{2}n^2, \sqrt{2}n)\) is uniformly distributed \(\pmod{1}\) in \(\mathbb{R}^2\). For then we have

$$\frac{1}{N} \sum_{n=0}^{N-1} \exp(2\pi ik[\sqrt{2n}]^2 \sqrt{2}) = \frac{1}{N} \sum_{n=0}^{N-1} \exp\left(2\pi ik(-2\sqrt{2}n^2 + \{\sqrt{2n}\}^2 \sqrt{2})\right)$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} f(2\sqrt{2}n^2, \sqrt{2}n) \to \int_0^1 \int_0^1 f dx dy = 0 \quad (1.12)$$

where \(f(x, y) = \exp(2\pi ik(-x + \{y\}^2 \sqrt{2}))\) is a Riemann-integrable periodic \(\pmod{1}\) function. So by Weyl’s criterion for uniform distribution (see Theorem 1.2.1 for several equivalent formulations of uniform distribution), \([\sqrt{2n}]^2 \sqrt{2}\) is uniformly distributed \(\pmod{1}\).

On the other hand, \(2[\sqrt{2n}]\sqrt{2}n\) is not uniformly distributed \(\pmod{1}\) in the usual sense. This is due to the following observation made by I.Z. Ruzsa,

$$2[\sqrt{2n}]\sqrt{2}n \equiv 1 - \{\sqrt{2n}\}^2 \pmod{1}. \quad (1.13)$$

For since \(\{\sqrt{2n}\}\) is uniformly distributed \(\pmod{1}\), \(\{\sqrt{2n}\}^2\) and hence \(1 - \{\sqrt{2n}\}^2\) cannot be uniformly distributed \(\pmod{1}\). However, \(2[\sqrt{2n}]\sqrt{2}n\) has \(g(x) = \sqrt{1-x}\) as continuous asymptotic distribution function \(\pmod{1}\) (see [24], p.53), so in particular

\(2[\sqrt{2n}]\sqrt{2}n \pmod{1}\) is dense in the unit interval.

We remark that in [31], Peres shows by using spectral theory that if a sequence \(x(n)\) is uniformly distributed \(\pmod{1}\) by the van der Corput’s method, then the sequence \(x([\alpha n])\) is also uniformly distributed \(\pmod{1}\) for any non-zero \(\alpha \in \mathbb{R}\). This gives another proof of uniform distribution of \([\sqrt{2n}]^2 \sqrt{2}\).
If we let the coefficients $a_1, \ldots, a_k$ of the polynomial (1.1) be parameters, then a weak version of Weyl’s theorem says that polynomials (1.1) are uniformly distributed (mod 1) for all but countably many $k$-tuples $(a_1, \ldots, a_k) \in \mathbb{R}^k$. In the same way we can treat the coefficients $b_1, \ldots, b_k$ of a representation of a generalized polynomial $q(x)$ as parameters. We will show in Section 2.2 that $q(n)$ is uniformly distributed (mod 1) if the $k$-tuple $(b_1, \ldots, b_k) \in \mathbb{R}^k$ lies outside a set $\Gamma$, where $\Gamma$ is the union of at most countably many hypersurfaces in $\mathbb{R}^k$, see Theorem 2.2.1.

If the coefficients of $q(x)$ are not in the exceptional set $\Gamma$, we say that $q(x)$ has independent coefficients, see Definition 2.2.1 for a precise definition. In Section 2.3 we will use the result from Section 2.2 to show that if the generalized polynomial $\sum_{i=1}^k q_i(x)$ has independent coefficients then $\sum_{i=1}^k [q_i(n)]p_i(n)$ is uniformly distributed (mod 1) for all but countably many $(\gamma_1, \ldots, \gamma_k) \in \mathbb{R}^k$, where $\gamma_i$ is the leading coefficient of $p_i(n)$, see Theorem 2.3.1 and its corollary. Note that in contrast to this result, there exist integer sequences $a_n$ with linear rate of growth such that $a_n\alpha$ fails to be uniformly distributed (mod 1) for uncountably many $\alpha \in \mathbb{R}$. In fact, in [7], Boshernitzan shows that for any real sequence $b_n$ there exists a sequence $t_n$, $t_n \in \{0, 1\}$, such that if $a_n = b_n + t_n$, then the sequence $a_n\alpha$ is not uniformly distributed (mod 1) for uncountably many $\alpha \in \mathbb{R}$.

In Chapter III we give necessary and sufficient conditions for uniform distribution of sequences coming from some classes of generalized polynomials of degree 2 and 3, including all the generalized polynomials of form $[\alpha n]3n$, $[\alpha n][3n]7$ and $[[\alpha n]3n][7n]7$. Our main tools are identities like (1.11) and (1.13) and van der Corput’s method. We
also show that any generalized polynomial of form \([\alpha_1 n] [\alpha_2 n] \cdots [\alpha_k n] \beta\), \(k \geq 3\), for which \(\beta\) is irrational, is uniformly distributed (mod 1).

Chapter IV is devoted to a dynamical approach to uniform distribution of generalized polynomials. We give definitions and basic results in nilmanifold theory and we show how nilmanifold theory can be used to prove uniform distribution of some classes of generalized polynomials. For example, we show that any uniformly distributed generalized polynomial of degree two can be shown to be uniformly distributed this way. Most of the classes we obtain by this method consist of generalized polynomials having some relations between their coefficients. We also discuss properties like unique ergodicity, (point) distality, almost automorphy and Besicovitch almost periodicity of generalized polynomials.

Chapter V is devoted to a discussion of some questions and open problems related to our study of generalized polynomials. We also bring there a table which summarizes some of our findings about generalized polynomials.

### 1.2 Definitions and basic results about generalized polynomials

Denote by \([r]\) the greatest integer less than or equal to \(r\), and \(\{r\}\) the fractional part of \(r\), so that \(r = [r] + \{r\}\). We will refer to the greatest integer function on \(\mathbb{R}\) as
bracket operation. If \( f_1(x) \) and \( f_2(x) \) are real-valued functions, then
\[
\begin{align*}
g_1(x) &= [f_1(x)]f_2(x) \\
g_2(x) &= [f_1(x)][f_2(x)] \\
g_3(x) &= [[f_1(x)]f_2(x)] \\
g_4(x) &= [f_1(x)] + f_2(x) \cdot f_1(x)
\end{align*}
\]
are examples of new real-valued functions, obtained from \( f_1(x) \) and \( f_2(x) \) by the use of bracket operations, products and sums. Note that \( g_2(x) \) is a product of brackets, but that \( g_3(x) \) has nested brackets.

**Definition 1.2.1** A generalized polynomial \( q(x) \) is a real-valued function on \( \mathbb{R} \) obtained from a finite number of real-valued polynomials by the use of bracket operations, sums and products.

**Example 1** Let \( a, b, c, d, x \in \mathbb{R} \). Then
\[
\begin{align*}
q_1(x) &= [ax](bx^2 + cx) \\
q_2(x) &= [ax]bx[cx^3 + dx]^4 \\
q_3(x) &= [(ax)^3bx^2]c
\end{align*}
\]
are generalized polynomials.

We will always write a polynomial in the form \( p(x) = a_kx^k + \cdots + a_1x + a_0 \), where \( a_i \in \mathbb{R}, i = 0, \ldots, k \). With this agreement the symbolic representation of a polynomial is unique. Also, a polynomial is uniquely determined by its values on \( \mathbb{N} \). The situation is different for generalized polynomials since two different generalized polynomials can
have the same values on $\mathbb{N}$. For example, $[2x]ax^2 \neq 2ax^3$ as functions on $\mathbb{R}$, but $[2n]an^2 = 2an^3$ for any $n \in \mathbb{N}$.

Since we are only interested in uniform distribution of sequences coming from generalized polynomials, we will from now on only deal with the sequences and not the generalized polynomials themselves. For simplicity reasons we will call these sequences generalized polynomials. Furthermore, we are not interested in terms which take only integer values. So we will leave them out as long as they are not multiplied by another generalized polynomial.

A generalized polynomial may have many symbolic representations. For example, we have for any generalized polynomial $q(n)$, the identities

$$\left[[q(n)]\alpha\right] = \left[[q(n)]\alpha\beta\right] - 1 \text{ if } \alpha \text{ is irrational and } \beta < 1 \quad (1.16)$$

and

$$\left[[q(n)]\frac{b}{a_1 \cdots a_l}\right] = \left[[\cdots [[q(n)]\frac{b}{a_1} \frac{1}{a_2} \cdots] \frac{1}{a_l}\right] \text{ if } b, a_i \in \mathbb{Z} \setminus \{0\}, i = 1, \ldots, l. \quad (1.17)$$

Furthermore, we have identities like

$$[\alpha n] = \beta n^2 + [\alpha_1 n] \beta n \text{ when } \alpha = 1 + \alpha_1. \quad (1.18)$$

However, to prove uniform distribution of a sequence $q(n)$, it is enough to use some representation of $q(n)$. And that is what we will do. We will allow abuse of language, saying that $q(n)$ is a generalized polynomial when we actually mean that $q(n)$ is a fixed representation of the corresponding generalized polynomial.

Nevertheless, we will need to make certain agreements for writing a generalized polynomial because of the way we define coefficients and put conditions on them. It
will be done inductively.

Let \( R_0 \) be the set of sequences of the form

\[ p(n) = a_k n^k + \cdots + a_1 n + a_0, \quad a_i \in \mathbb{R} \setminus \mathbb{Z}, \tag{1.19} \]

and let \( R_1 \) be the set of all sequences

\[ q(n) = \sum_{i=1}^{l} \prod_{j=1}^{l_i} [q_{ij}(n)]^{k_{ij}} p_i(n) \tag{1.20} \]

where \( q_{ij}(n), p_i(n) \in R_0 \) with at least one \( q_{ij}(n) \) non-constant and such that

\[ \text{if } i_1 \neq i_2, \text{ then } \prod_{j=1}^{l_{i_1}} [q_{i_1 j}(n)]^{k_{i_1 j}} \neq \prod_{j=1}^{l_{i_2}} [q_{i_2 j}(n)]^{k_{i_2 j}}. \tag{1.21} \]

Suppose \( R_k \) is defined for \( k < K \) and let \( R_K \) be all the sequences of form (1.20), where \( p_i(n) \in R_0 \) and \( q_{ij}(n) \in R_k, k < K \), with at least one \( q_{ij}(n) \in R_{K-1} \), and such that (1.21) holds. Then

\[ G = \bigcup_{i=0}^{\infty} R_i \tag{1.22} \]

is the family of generalized polynomial sequences, and we agree to write any generalized polynomial in form (1.20).

**Definition 1.2.2** If \( q(n) \in R_k \) then we will say that \( q(n) \) has a sequence of nested brackets of length \( k \), i.e., \( q(n) \) has a term like \([\cdots [[q_1(n)]q_2(n)]\cdots]q_{k+1}(n)\), and we write \( B(q) = k \).

In Example 1, \( B(q_1) = 1 \) and \( B(q_2) = B(q_3) = 2 \).
Definition 1.2.3 Define the set $S(q)$ of coefficients of (a representation of) a generalized polynomial $q(n)$ inductively. If $q(n) = a_k n^k + \cdots + a_1 n + a_0$ is a polynomial, then as usual, $S(q) = \{a_i \mid a_i \neq 0\}$. If

$$q(n) = \sum_i \prod_j [q_{ij}(n)]^{k_{ij}} p_i(n) \in R_K,$$

where $q_{ij}(n) \in R_k, k < K, p_i(n) \in R_0$, then

$$S(q) = \bigcup_{i,j} S(q_{ij}) \cup \bigcup_i S(p_i).$$

(1.23)

A real number $\alpha$ is a coefficient of $q(n)$ if $\alpha \in S(q)$. A coefficient $\alpha$ is an outer coefficient if $\alpha \in \bigcup_i S(p_i)$.

Definition 1.2.4 $q(n)$ is a simple generalized polynomial if its representation does not contain any sums.

In other words, a simple generalized polynomial is a real-valued function on $\mathbb{R}$ obtained from a finite number of real-valued monomials by using bracket operations and products. For example, $q_3(x)$ in Example 1 is simple, but $q_1(x)$ and $q_2(x)$ are not.

Definition 1.2.5 Let $q(n)$ and $r(n)$ be simple generalized polynomials.

$r(n)$ is an inner subpolynomial of $q(n)$ if $[r(n)]$ occurs in the representation of $q(n)$.

$r(n)$ is an outer subpolynomial of $q(n)$ if $r(n)$ is the remaining part of $q(n)$ when one or more inner subpolynomials are removed from $q(n)$. 
$r(n)$ is an induced inner subpolynomial of $q(n)$ if $r(n)$ is an inner subpolynomial of an outer subpolynomial of $q(n)$.

Finally, $r(n)$ is a subpolynomial of $q(n)$ if it is an inner, outer or induced inner subpolynomial of $q(n)$.

If $q(n)$ is a sum of simple generalized polynomials, then we will say that $r(n)$ is a subpolynomial of $q(n)$ if $r(n)$ is a subpolynomial of a simple term of $q(n)$.

Note that an inner subpolynomial is also an induced inner subpolynomial, and that if $[q_1(n)q_2(n)]$ occurs in the representation of $q(n)$, then $q_2(n)$ is an induced inner subpolynomial of $q(n)$ since $q_1(n)$ is an inner subpolynomial which can be removed.

**Example 2** If

$$q(n) = \left[(\alpha n)\beta n^2\right]\gamma + [\delta n][\rho n]\lambda n,$$

then $\alpha n, \delta n, \rho n, [\alpha n]\beta n^2$ are inner subpolynomials of $q(n)$, $\beta n^2$ is an induced inner subpolynomial of $q(n)$ and $[\beta n^2]\gamma, \gamma, [\delta n]\lambda n, [\rho n]\lambda n, \lambda n$ are outer subpolynomials of $q(n)$.

Note also that if $x(n)$ is a subpolynomial of $y(n)$ and $y(n)$ is a subpolynomial of $z(n)$, then $x(n)$ is also a subpolynomial of $z(n)$.

**Definition 1.2.6** Define the degree of a generalized polynomial $q(n)$, denoted by $\deg(q)$, inductively. If $q(n) = a_kn^k + \cdots + a_1n + a_0$ is a polynomial, then as usual,
\[\deg(q) = k. \text{ If} \]
\[q(n) = \sum_i \prod_j [q_{ij}(n)]^{k_{ij}} p_i(n) \in R_K, \tag{1.26}\]
where \(q_{ij}(n) \in R_k, k < K, p_i(n) \in R_0, \text{ then} \]
\[\deg(q) = \max_{i,j} \left( k_{ij} \deg(q_{ij}) + \deg(p_i) \right). \tag{1.27}\]
So if for example \(q_1(n) = \left[\lfloor \alpha n^2 \beta n \rfloor \right]^3 \gamma \) and \(q_2(n) = \lfloor \alpha n^2 \beta n - \alpha \beta n^3 \rfloor \), then \(\deg(q_1) = 9\) and \(\deg(q_2) = 3\).

Uniform distribution (mod 1) of sequences in \(R^l, l \geq 1\), is defined in the following way.

**Definition 1.2.7** A sequence \((x_1(n), \ldots, x_l(n))\), \(n = 1, 2, 3, \ldots\), is uniformly distributed (mod 1) in \(R^l\) if for any real numbers \(0 \leq a_i < b_i \leq 1, i = 1, \ldots, l, \]
\[
\lim_{N \to \infty} \frac{1}{N} \text{card} \left( \{1 \leq n \leq N \mid (\{x_1(n)\}, \ldots, \{x_l(n)\}) \in \prod_{i=1}^{l} [a_i, b_i] \} \right) = \prod_{i=1}^{l} (b_i - a_i). 
\]
There are many equivalent formulations which we will use freely.

**Theorem 1.2.1 ([24])** Let \(\varphi(n) = (x_1(n), \ldots, x_l(n))\) be a sequence in \(R^l\). Then the following statements are equivalent:

(i) \(\varphi(n)\) is uniformly distributed (mod 1) in \(R^l\).

(ii) \(\sum_{i=1}^{l} k_i x_i(n)\) is uniformly distributed (mod 1) in \(R\) for all \(l\)-tuples \((k_1, \ldots, k_l) \neq (0, \ldots, 0)\) of integers.
(iii) Weyl’s criterion:

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \exp\left(2\pi i \sum_{j=1}^{l} k_j x_j(n)\right) = 0
\]  \hspace{1cm} (1.28)

for all l-tuples \((k_1, \ldots, k_l) \neq (0, \ldots, 0)\) of integers.

(iv) For every Riemann-integrable function \(f\) on \([0,1]^l\),

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\{x(n)\}) = \int_0^1 \cdots \int_0^1 f(\mathbf{x}) d\mathbf{x}_1 \cdots d\mathbf{x}_l.
\]  \hspace{1cm} (1.29)

Let \(q(n)\) be a generalized polynomial and \(\beta\) a real number. By Lemma 1.2.2, below, the new generalized polynomial \([q(n)]\beta\) is uniformly distributed (mod 1) in \(\mathbb{R}\) if \((q(n)_\beta, q(n))\) is uniformly distributed (mod 1) in \(\mathbb{R}^2\). Note that the converse is not true. If for example \(q(n) = n\), then \([q(n)]\beta = n\beta\) is uniformly distributed (mod 1) if \(\beta\) is irrational, but \((q(n)_\beta, q(n)) = (n\beta, n)\) is certainly not uniformly distributed (mod 1). However, if \(q(n)\) is a polynomial with at least one irrational coefficient other than the constant term, then \([q(n)]\beta\) is uniformly distributed (mod 1) if and only if \((q(n)_\beta, q(n))\) is uniformly distributed (mod 1). This follows from Proposition 1.2.3 below and is also shown in [24, p. 310] for the special case \(q(n) = \alpha n\).

If \(x(n)\) is a sequence in \(\mathbb{R}^k\) and \(y(n)\) a sequence in \(\mathbb{R}^l\) such that \(x(n)\) is uniformly distributed (mod 1) if \(y(n)\) is, then we will write

\[x(n) \overset{\beta}{\leftrightarrow} y(n).\]  \hspace{1cm} (1.30)

We have already seen that \([q(n)]\beta \overset{\alpha}{\leftrightarrow} (q(n)_\beta, q(n))\).

It follows from the above discussion that for certain generalized polynomials we can, by going up in dimension, get rid of a bracket which has only a constant on the outside. In the same way we could remove brackets from the new terms by going
further up in dimension. This can be repeated until all the terms are of one of the forms $[q_1(n)][q_2(n)]\beta$ or $[q_1(n)]p(n)$, where $q_1(n)$ and $q_2(n)$ are generalized polynomials of positive degrees and $p(n)$ is a non-constant polynomial.

If $q(n) = \cdots [q_1(n)]\lambda \cdots \lambda_k$ is a simple generalized polynomial where $q_1(n)$ is of one of the above mentioned forms, then we will use the following notation:

$$q^i(n) = [\cdots [q_1(n)]\lambda_i \cdots ]\lambda_i, \quad 0 \leq i < k,$$

$$q^i(n) = q_1(n)\lambda_1 \cdots \lambda_i, \quad 0 \leq i < k$$

and

$$q(n) = q_1(n)\lambda_1 \cdots \lambda_k.$$ (1.33)

Note that $q^i(n)$ is an inner subpolynomial of $q(n)$. It follows from the next lemma that

$$q(n) \leftrightarrow (q(n), q^{k-1}(n), \ldots, q^0(n)).$$ (1.34)

**Lemma 1.2.2** Let $q_i(n) = [\cdots [q_{i1}(n)]\lambda_{i1} \cdots ]\lambda_{ik}, \ i = 1, \ldots, s$, be simple generalized polynomials and $q(n) = \sum_{i=1}^s a_i q_i(n), \ a_i \in \mathbb{Z}$. Let $r_1(n), \ldots, r_m(n)$ be all the distinct elements from the set $\{q_{il}(n) \mid l = 0, \ldots, k_i - 1, i = 1, \ldots, s\}$. Then

$$q(n) \leftrightarrow \left(\sum_{i=1}^s a_i q_i(n), r_1(n), \ldots, r_m(n)\right).$$ (1.35)

**Proof:** We will first prove by induction on $k$ that if

$$q(n) = [\cdots [q_1(n)]\lambda_1 \cdots ]\lambda_k,$$ (1.36)
where \( q_1(n) \) is a simple generalized polynomial such that \( \overline{q}_1(n) = q_1(n) \), then

\[
q(n) = \overline{q}(n) - \{ q^0(n) \} \lambda_1 \cdots \lambda_k - \{ q^1(n) \} \lambda_2 \cdots \lambda_k - \cdots - \{ q^{k-1}(n) \} \lambda_k
\]  

(1.37)

and

\[
q^i(n) = \overline{q}^i(n) - \{ q^0(n) \} \lambda_1 \cdots \lambda_i - \cdots - \{ q^{i-1}(n) \} \lambda_i.
\]

(1.38)

If \( k = 1 \), then

\[
q(n) = q_1(n) \lambda_1 - \{ q_1(n) \} \lambda_1 = \overline{q}(n) - \{ q^0(n) \} \lambda_1
\]

(1.39)

which proves (1.37) in this case. Assume now that (1.37) is true for \( k \), and let

\[
q(n) = \overline{q}(n) + \sum_{i=1}^s a_i q_i(n)
\]

(1.40)

then

\[
q(n) = \sum_{i=1}^s a_i \overline{q}_i(n) - \sum_{i=1}^s \sum_{j=1}^{k_i} \{ q_{ij}^{-1}(n) \} \lambda_{ij} \cdots \lambda_{ik_i}
\]

(1.41)

which we wanted to prove.

Note that this gives us that \( q(n) \) is a function of \( \overline{q}(n) \) and the \( \overline{q}^i(n) \)'s. If \( q(n) = \sum_{i=1}^s a_i q_i(n) \) and

\[
q_i(n) = \{ \cdots [ q_{i1}(n) ] \lambda_{i1} \cdots \} \lambda_{ik_i}
\]

(1.42)

then

\[
q(n) = \sum_{i=1}^s a_i \overline{q_i}(n) - \sum_{i=1}^s \sum_{j=1}^{k_i} \{ q_{ij}^{-1}(n) \} \lambda_{ij} \cdots \lambda_{ik_i}
\]

(1.43)

Let \( r_i(n), i = 1, \ldots, m, \) be all the distinct elements from the set

\[
\{ \overline{q}_j(n) \mid j = 0, \ldots, k_i - 1, i = 1, \ldots, s \}
\]

and for each \( b \in \mathbb{Z} \setminus \{0\} \), define a function

\[
f_b(x_0, x_1, \ldots, x_m) = \exp 2\pi i b \left( x_0 - \sum_{i=1}^s \sum_{j=1}^{k_i} \{ f_{ij}(x_1, \ldots, x_m) \} \lambda_{ij} \cdots \lambda_{ik_i} \right)
\]
where \( f_{ij}(x_1, \ldots, x_m) \) is defined such that
\[
f_{ij}(r_1(n), \ldots, r_m(n)) = \tilde{q}_{i}^{j-1}(n) - \sum_{l=1}^{j-1}\{q_{i}^{l-1}(n)\} \lambda_{il} \ldots \lambda_{ij-1}. \tag{1.44}
\]

It follows that for any \( b \), \( f_b \) is a Riemann-integrable, periodic \((\mod 1)\) function with integral equal 0. Hence, if \( \left( \sum_{i=1}^{s} a_i \tilde{q}_i(n), r_1(n), \ldots, r_m(n) \right) \) is uniformly distributed \((\mod 1)\), then
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi ibq(n)} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f_b \left( \sum_{i=1}^{s} a_i \tilde{q}_i(n), r_1(n), \ldots, r_m(n) \right) = \int f_b(x_0, \ldots, x_m) dx_0 \cdots dx_m = 0 \tag{1.45}
\]

which shows that \( q(n) \) is uniformly distributed \((\mod 1)\).

If \( q(n) \) in Lemma 1.2.2 is such that \( s = 1 \) and \( q_{11}(n) \) is a polynomial, then we have the following stronger result.

**Proposition 1.2.3** If \( p(x) = a_l x^l + \cdots + a_1 x + a_0 \) is a polynomial with real coefficients, then the following are equivalent:

(i) \( \left( p(n), [p(n)]\lambda_1, \ldots, [\cdots [p(n)]\lambda_1 \cdots \lambda_k] \right) \) is uniformly distributed \((\mod 1)\) in \( \mathbb{R}^{k+1} \).

(ii) \( \left( p(n), p(n)\lambda_1, \ldots, p(n)\lambda_1 \cdots \lambda_k \right) \) is uniformly distributed \((\mod 1)\) in \( \mathbb{R}^{k+1} \).

(iii) If there exist \( b_i \in \mathbb{Q} \) and an irrational number \( \alpha \) such that the coefficients \( a_i = b_i\alpha, \ i = 1, \ldots, l \), then \( 1, \alpha, \alpha \lambda_1, \ldots, \alpha \lambda_1 \cdots \lambda_k \) are rationally independent.

Or if there exist coefficients \( a_i \) and \( a_j, \ i, j \neq 0 \), such that \( a_i/a_j \notin \mathbb{Q} \), then \( 1, \lambda_1, \ldots, \lambda_1 \cdots \lambda_k \) are rationally independent.
We will make use of Weyl’s theorem ([40]) in the proof.

**Theorem 1.2.4 (Weyl)** If $p(x)$ is a real polynomial with at least one coefficient other than the constant term irrational, then the sequence $p(n)$ is uniformly distributed (mod 1).

**Proof of Proposition 1.2.3:** We have (ii) $\iff$ (iii), for by Theorem 1.2.4 and Theorem 1.2.1, (iii) implies (ii), and since it is clear that a polynomial with only rational coefficients is not uniformly distributed (mod 1), (ii) implies (iii).

(ii) $\Rightarrow$ (i) follows from Lemma 1.2.2. For by Theorem 1.2.1, 

\[
\left(p(n), [p(n)]\lambda_1, \ldots, [\cdots [p(n)]\lambda_1 \cdots ]\lambda_k\right)
\]

is uniformly distributed (mod 1) in $\mathbb{R}^{k+1}$ if for any $d = (d_0, \ldots, d_k) \in \mathbb{Z}^{k+1}\{ (0, \ldots, 0) \}$, $g_d(n) = d_0 p(n) + \sum_{i=1}^k d_i [\cdots [p(n)]\lambda_1 \cdots ]\lambda_i$ is uniformly distributed (mod 1). Without loss of generality we may assume $d_k \neq 0$. Then by Lemma 1.2.2, $g_d(n)$ is uniformly distributed (mod 1) if 

\[
\left(d_0 p(n) + \sum_{i=1}^k d_i p(n)\lambda_1 \cdots \lambda_i, p(n), p(n)\lambda_1, \ldots, p(n)\lambda_1 \cdots \lambda_{k-1}\right)
\]

is uniformly distributed (mod 1) in $\mathbb{R}^{k+1}$, hence if 

\[
\left(p(n), p(n)\lambda_1, \ldots, p(n)\lambda_1 \cdots \lambda_{k-1}, p(n)\lambda_1 \cdots \lambda_{k}\right)
\]

is uniformly distributed (mod 1) in $\mathbb{R}^{k+1}$.

We will use induction on $k$ to show (i) $\Rightarrow$ (iii). If $k = 0$, then it is trivial since (ii)$\iff$(iii). Suppose (i) $\Rightarrow$ (iii) is true for $k-1$ for some $k \geq 1$. Let us first consider the
case when $a_i = c_i \alpha, i = 1, \ldots, l$, where $c_i \in \mathbb{Q}$ and $\alpha$ is an irrational number. Then $p(n) = \alpha p_1(n)$, where $p_1(n)$ is a polynomial with rational coefficients. Let $a \in \mathbb{Z}$ be such that $ap_1(n)$ has integer coefficients. Since we are assuming that

$$\left( p(n), [p(n)]_{\lambda_1}, \ldots, [\cdots [p(n)]_{\lambda_1} \cdots]_{\lambda_k} \right) \quad (1.49)$$

is uniformly distributed (mod 1) in $\mathbb{R}^{k+1}$, it follows by the induction hypothesis that $1, \alpha, \alpha \lambda_1, \ldots, \alpha \lambda_1 \cdots \lambda_{k-1}$ are rationally independent. Suppose $\alpha \lambda_1 \cdots \lambda_k$ is rationally dependent of $1, \alpha, \alpha \lambda_1, \ldots, \alpha \lambda_1 \cdots \lambda_{k-1}$ and that

$$b\alpha \lambda_1 \cdots \lambda_k = b_k + b_0 \alpha + b_1 \alpha \lambda_1 + \cdots + b_{k-1} \alpha \lambda_1 \cdots \lambda_{k-1} \quad (1.50)$$

for some integers $b \neq 0$ and $b_i, i = 0, \ldots, k$. Let

$$\beta_{k-i} = \frac{b_k + b_0 \alpha + b_1 \alpha \lambda_1 + \cdots + b_{k-i-1} \alpha \lambda_1 \cdots \lambda_{k-i-1}}{\alpha \lambda_1 \cdots \lambda_{k-i}}, \quad i = 1, \ldots, k \quad (1.51)$$

where we let $\lambda_0 = 1$ and $\lambda_{-1} = 0$ so that $\beta_0 = \frac{b_k}{\alpha}$. Then

$$b\lambda_k = \frac{b_k + b_0 \alpha + b_1 \alpha \lambda_1 + \cdots + b_{k-2} \alpha \lambda_1 \cdots \lambda_{k-2}}{\alpha \lambda_1 \cdots \lambda_{k-1}} + b_{k-1} = \beta_{k-1} + b_{k-1} \quad (1.52)$$

and $\lambda_{k-i} \beta_{k-i} = \beta_{k-i-1} + b_{k-i-1}, \quad i = 1, \ldots, k-1$. Note that $\beta_{k-i}$ is irrational for any $i$. We now have

$$ab\left[ [\cdots [p(n)]_{\lambda_1} \cdots]_{\lambda_{k-1}} \right]_{\lambda_k}$$

$$= a\left[ [\cdots [p(n)]_{\lambda_1} \cdots]_{\lambda_{k-1}} \right] (\beta_{k-1} + b_{k-1})$$

$$\equiv a\left[ [\cdots [p(n)]_{\lambda_1} \cdots]_{\lambda_{k-1}} \right] \beta_{k-1} \quad (\text{mod 1})$$

$$= a\left[ [\cdots [p(n)]_{\lambda_1} \cdots]_{\lambda_{k-2}} \right] \lambda_{k-1} \beta_{k-1} - a\left[ [\cdots [p(n)]_{\lambda_1} \cdots]_{\lambda_{k-1}} \right] \beta_{k-1}$$

$$\equiv a\left[ [\cdots [p(n)]_{\lambda_1} \cdots]_{\lambda_{k-2}} \right] \beta_{k-2} - a\left[ [\cdots [p(n)]_{\lambda_1} \cdots]_{\lambda_{k-1}} \right] \beta_{k-1} \quad (\text{mod 1}). (1.53)$$
Since the process of reducing the number of brackets in the first term can be continued until we get down to the term

\[ a[p(n)]β_0 = a[p(n)]\frac{b_k}{α} \]

\[ = ap_1(n)b_k - a\{p(n)\} \frac{b_k}{α} \]

\[ ≡ -a\{p(n)\} \frac{b_k}{α} \quad \text{(mod 1)}, \quad (1.54) \]

it follows that

\[ ab[\cdots [p(n)]\lambda_{k-1}]\lambda_k ≡ -a \sum_{i=1}^{k} \{\cdots [p(n)]\lambda_i \cdots \} \beta_{k-i} \quad \text{(mod 1)}. \quad (1.55) \]

Hence,

\[ \frac{1}{N} \sum_{n=0}^{N-1} \exp\left(2πiab[\cdots [p(n)]\lambda_1 \cdots ]\lambda_{k-1}]\lambda_k\right) \]

\[ = \frac{1}{N} \sum_{n=0}^{N-1} \exp\left(-2πiα \sum_{i=1}^{k} \{\cdots [p(n)]\lambda_i \cdots \} \beta_{k-i}\right) \]

\[ = \frac{1}{N} \sum_{n=0}^{N-1} f\left(p(n), [p(n)]\lambda_1, \ldots, [[\cdots [p(n)]\lambda_{k-2}]\lambda_{k-1}\right), \quad (1.56) \]

where

\[ f(x_0, x_1, \ldots, x_{k-1}) = \exp\left(2πiα \sum_{i=1}^{k}\{x_{k-i}\} \beta_{k-i}\right) \quad (1.57) \]

is a Riemann-integrable, periodic (mod 1) function and

\[ \int fdx_0 \cdots dx_{k-1} = \prod_{i=1}^{k} \int_0^1 \exp(2πiax_{k-i}β_{k-i})dx_{k-i} \]

\[ = \prod_{i=1}^{k} \frac{1 - \exp(-2πiaβ_{k-i})}{2πiaβ_{k-i}} ≠ 0 \quad (1.58) \]

since $β_{k-i}, i = 1, \ldots, k,$ are irrationals. Now,

\[ (p(n), [p(n)]\lambda_1, \ldots, [[\cdots [p(n)]\lambda_{k-2}]\lambda_{k-1}) \quad (1.59) \]
is uniformly distributed (mod 1) so by Theorem 1.2.1,
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(p(n), [p(n)]\lambda_1, \ldots, [[\ldots [p(n)]\lambda_1 \ldots \lambda_{k-2}]\lambda_{k-1}) = \int f dx_0 \cdots dx_{k-1} \neq 0,
\]
(1.60)
which shows that \([[[\ldots [p(n)]\lambda_1 \ldots \lambda_{k-1}]\lambda_k \text{ is not uniformly distributed (mod 1), contradicting our assumption. Hence (i)⇒(iii) in this case.}

The proof in the case where there exist \(i, j\) such that \(\frac{a_i}{a_j} \notin \mathbb{Q}\) is very similar. For by the induction hypothesis, \(1, \lambda_1, \ldots, \lambda_{k-1}\) are rationally independent and if we assume there exist integers \(b \neq 0, b_i\) such that
\[
b\lambda_1 \cdots \lambda_k = b_k + b_1 \lambda_1 + \cdots b_{k-1} \lambda_1 \cdots \lambda_{k-1}
\]
(1.61)
and let
\[
\beta_{k-i} = \frac{b_k + b_1 \lambda_1 + \cdots b_{k-i-1} \lambda_1 \cdots \lambda_{k-i-1}}{\lambda_1 \cdots \lambda_{k-i}}, \quad i = 1, \ldots, k-1,
\]
(1.62)
then
\[
b[[[\ldots [p(n)]\lambda_1 \ldots \lambda_{k-1}]\lambda_k \equiv -\sum_{i=1}^{k-1} \left\{[[[\ldots [p(n)]\lambda_1 \ldots \lambda_{k-i}]\beta_{k-i} \pmod{1}
\]
(1.63)
so that
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \exp(2\pi ib[[[\ldots [p(n)]\lambda_1 \ldots \lambda_{k-1}] \beta_{k-1}) \neq 0
\]
(1.64)
as in the first case. Hence (i)⇒(iii).

Let \(l\) be a positive integer. Then for any subset \(C\) of \([0,1)^l\), we can define an indicator function \(1_C\) on \(\mathbb{R}^l\) by
\[
1_C(x_1, \ldots, x_l) = \begin{cases} 1 & \text{if } \{x_1, \ldots, x_l\} \in C \\ 0 & \text{otherwise.} \end{cases}
\]
(1.65)
We will call \(x_1, \ldots, x_l\) the arguments of the indicator function \(1_C\). When it is not necessary to specify the arguments of the indicator function \(1_C\), we will write \(1_C(*)\).

If \(x\) and \(y\) are real numbers, then

\[
[x + y] = \begin{cases} 
[x] + [y] + 1 & \text{if } \{x\} + \{y\} \geq 1 \\
[x] + [y] & \text{if } \{x\} + \{y\} < 1 
\end{cases}
\]  

(1.66)

So if we let

\[
A = \{(x, y) \in [0, 1)^2 \mid x + y \geq 1\}
\]  

(1.67)

then we have

\[
[x + y] = [x] + [y] + 1_A(x, y).
\]  

(1.68)

**Lemma 1.2.5** Let \(x, x_1, \ldots, x_k\) be real numbers and \(h\) a positive integer. Then

(i) \(\left[ \sum_{i=1}^{k} x_i \right] = \sum_{i=1}^{k} \left( [x_i] + 1_A(x_i, \sum_{j=1}^{k-i} x_{i+j}) \right) \)

(ii) \([hx] = h[x] + \sum_{i=0}^{h-1} i 1_{\left( \left[ \frac{i}{h} \right] + 1 \right)}(x)\).

**Proof:** (i) will be proved by induction on \(k\). The case \(k = 2\) is done above. Suppose (i) is true for \(k\). Then

\[
\begin{align*}
\left[ \sum_{i=1}^{k+1} x_i \right] &= [x_1] + \left[ \sum_{i=2}^{k+1} x_i \right] + 1_A(x_1, \sum_{i=2}^{k+1} x_i) \\
&= [x_1] + \sum_{i=2}^{k+1} \left( [x_i] + 1_A(x_i, \sum_{j=1}^{k+1-i} x_{i+j}) \right) + 1_A(x_1, \sum_{j=1}^{k} x_{1+j}) \\
&= \sum_{i=1}^{k+1} \left( [x_i] + 1_A(x_i, \sum_{j=1}^{k+1-i} x_{i+j}) \right). 
\end{align*}
\]  

(1.69)

Hence, by induction, (i) is true for any natural number \(k\).
To prove (ii), observe first that since \( h\{x\} = hx - h[x] \) and \( \{hx\} = hx - [hx] \), \( h\{x\} \) differs from \( \{hx\} \) by an integer. Now, if
\[
\frac{i}{h} \leq \{x\} < \frac{i+1}{h},
\]
then \( i \leq h\{x\} < i + 1 \). So \( h\{x\} = \{hx\} + i \), and therefore \( [hx] = h[x] + i \), which was to be proved.

\[\Box\]

The previous lemma is fundamental for what we will be doing, because it allows us to write generalized polynomials as a sum of simple generalized polynomials together with indicator functions. It also tells us that we won’t do much harm by bringing out positive integers from brackets.

**Definition 1.2.8** A sequence \((x_1(n), \ldots, x_l(n))\), \( n = 1, 2, 3, \ldots \), is well-distributed in \( \mathbb{R}^l \) if for any real numbers \( 0 \leq a_i < b_i \leq 1 \), \( i = 1, \ldots, l \),
\[
\lim_{N \to \infty} \frac{1}{N} \text{card} \left( \{ k + 1 \leq n \leq N + k \mid (\{x_1(n)\}, \ldots, \{x_l(n)\}) \in \prod_{i=1}^l [a_i, b_i] \} \right) = \prod_{i=1}^l (b_i - a_i)
\]
uniformly in \( k = 1, 2, \ldots \).

Clearly any well-distributed sequence is uniformly distributed, but the converse is not true. For example, the sequence \( 2^n \alpha, n = 1, 2, \ldots \), is uniformly distributed (mod 1) for almost all real numbers \( \alpha \), but it is not well-distributed for any \( \alpha \in \mathbb{R} \) ([24, p. 42]). However, any uniformly distributed polynomial is well-distributed as well, and we will see that our proofs of uniform distribution of generalized polynomials also show well-distribution of these generalized polynomials.
CHAPTER II

Generalized polynomials with independent coefficients

2.1 Some lemmas

Call a subset $C$ of $[0,1)^l$ admissible if $C$ is a finite union of polytopes, where each constituent polytope is bounded by $(l-1)$-dimensional planes. In dealing with probabilities of hitting $C$ by a sequence we shall not care about the boundary of $C$ since the union of boundaries of finitely many polytopes has zero $l$-dimensional Lebesgue measure.

**Lemma 2.1.1** A sequence

$$G(n) = q_0(n) + \sum_{i=1}^{L} q_i(n) 1_{C_i}(r_{i1}(n), \ldots, r_{il}(n))$$  \hspace{1cm} (2.1)

where $q_i(n)$, $r_{ij}(n)$, $j = 1, \ldots, l$, $i = 1, \ldots, L$, are generalized polynomials and $C_i$ is an admissible subset of $[0,1)^l$, $i = 1, \ldots, L$,

is uniformly distributed (mod 1) if there exist generalized polynomials $t_1(n), \ldots, t_k(n)$
such that each \( r_{ij}(n) \) is a linear combination over \( \mathbb{Z} \) of \( t_1(n), \ldots, t_k(n) \) and such that for all subsets \( V \) of the set \( \{q_1, \ldots, q_L\} \) of generalized polynomials,

\[
\left( q_0(n) + \sum_{q \in V} q(n), t_1(n), \ldots, t_k(n) \right)
\]  

is uniformly distributed (mod 1) in \( \mathbb{R}^{k+1} \).

**Proof:** We will rewrite \( G(n) \) in terms of new indicator functions. For each \( i = 1, \ldots, L \), let \( C_i^c \) be the complement of \( C_i \) in \([0, 1)^l_i\). Then \( C_i \) and \( C_i^c \) give a partition of \([0, 1)^l_i\) consisting of two admissible sets, and \( E_1 \times \ldots \times E_l \), where \( E_i \) is either \( C_i \) or \( C_i^c \), give a partition of \([0, 1)^l\) consisting of \( 2^L \) admissible sets, where \( l = \sum l_i \). Call the family of these sets \( \mathcal{C} \), and let \( V_C \) be the subset of \( \{q_1(n), \ldots, q_L(n)\} \) that corresponds to \( C \in \mathcal{C} \), i.e., if \( C = \prod_{i=1}^L E_i \), where \( E_i = C_i \) or \( C_i^c \), then \( V_C = \{q_i(n) \mid E_i = C_i\} \). Then

\[
G(n) = \sum_{C \in \mathcal{C}} \left( q_0(n) + \sum_{q \in V_C} q(n) \right) 1_C\left(r_{11}(n), \ldots, r_{1l_i}(n), \ldots, r_{L1}(n), \ldots, r_{Ll_L}(n)\right). 
\]  

(2.3)

Since each \( r_{ij}(n) \) is a linear combination over \( \mathbb{Z} \) of \( 1, t_1(n), \ldots, t_k(n) \), the partition \( \mathcal{C} \) of \([0, 1)^l\) induces a partition \( \mathcal{D} \) of \([0, 1)^k\) also consisting of \( 2^L \) admissible sets, so that

\[
G(n) = \sum_{D \in \mathcal{D}} \left( q_0(n) + \sum_{q \in V_D} q(n) \right) 1_D(t_1(n), \ldots, t_k(n)). 
\]  

(2.4)

Let \( b \in \mathbb{Z} \setminus \{0\} \). Then

\[
\frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi ib G(n)} = \frac{1}{N} \sum_{n=0}^{N-1} \exp\left(2\pi ib \sum_{D \in \mathcal{D}} \left(q_0(n) + \sum_{q \in V_D} q(n) \right) 1_D(t_1(n), \ldots, t_k(n))\right) \\
= \sum_{D \in \mathcal{D}} \frac{1}{N} \sum_{n=0}^{N-1} \exp\left(2\pi ib \left(q_0(n) + \sum_{q \in V_D} q(n) \right) \right) 1_D(t_1(n), \ldots, t_k(n)) \\
= \sum_{D \in \mathcal{D}} \frac{1}{N} \sum_{n=0}^{N-1} f_D\left(q_0(n) + \sum_{q \in V_D} q(n), t_1(n), \ldots, t_k(n)\right) 
\]  

(2.5)
where \( f_D(x, y_1, \ldots, y_k) = e^{2\pi ibx}1_D(y_1, \ldots, y_k) \) is a Riemann-integrable periodic \((\text{mod} 1)\) function having integral equal 0. Therefore, since \( (q_0(n) + \sum_{q \in V_D} q(n), t_1(n), \ldots, t_k(n)) \) is uniformly distributed \((\text{mod} 1)\) for all \( D \in \mathcal{D} \),

\[
\sum_{D \in \mathcal{D}} \frac{1}{N} \sum_{n=0}^{N-1} f_D(q_0(n) + \sum_{q \in V_D} q(n), t_1(n), \ldots, t_k(n)) \rightarrow 0 \tag{2.6}
\]
as \( N \to \infty \) by Theorem 1.2.1. Hence \( G(n) \) is uniformly distributed \((\text{mod} 1)\).

\[\square\]

We will not always be able to write the \( r_{ij}(n) \)'s as a sum of \( t_l(n) \)'s as nicely as in this lemma. But by using Lemma 1.2.5 on the \( r_{ij}(n) \)'s, we can break them down to a linear combination of simple generalized polynomials together with new indicator functions which can be taken care of in a similar manner.

**Lemma 2.1.2** Let \( q(n) \) be a generalized polynomial. Then there exist simple generalized polynomials \( q_i(n), i = 1, \ldots, L, \) such that

\[
q(n) = \sum_{i=1}^{L} q_i(n) + \sum_{i=1}^{L} \sum_{j=1}^{l_i} 1_A(r_{1i}(n), r_{2i}(n)) t_{ij}(n), \tag{2.7}
\]

where each \( t_{ij}(n) \) is an outer subpolynomial of some \( q_i(n), 1 \leq l \leq L, \) so that \( \deg(t_{ij}) < \deg(q) \), and each \( r_{ji}(n) \) is a sum of induced inner subpolynomials of the \( q_i(n) \)'s and induced inner subpolynomials multiplied by indicator functions of the same form.

We will call the \( q_i(n) \)'s the **simple terms** of \( q(n) \).

**Proof:** We will use induction on \( B(q) \). Recall that \( B(q) \) is the length of the longest sequence of nested brackets in \( q(n) \), Definition 1.2.2. If \( B(q) = 0 \), then \( q(n) \) is a
polynomial, and the statement is trivially true. Suppose the lemma is true for all generalized polynomials \( q(n) \) with \( B(q) = K \) and let \( q(n) \) be such that \( B(q) = K + 1 \).

Without loss of generality we may assume \( q(n) \) has only one term, i.e., that

\[
q(n) = k \prod_{i=1}^{L} q_{1i}(n) p(n)
\]

with \( q_{1i}(n) \), \( t_{ij}(n) \), \( r_{ji}(n) \) as stated in the lemma. Now by Lemma 1.2.5, we have

\[
q(n) = \sum_{i=1}^{L} q_{1i}(n) p(n) + \sum_{i,j}^{L} 1_A(r_{1i}, r_{2i}) t_{ij}(n) p(n)
\]

which proves (2.7) in this case since \([q_{1i}(n)] p(n)\) is simple for any \( i \) and each \( r_{ji}(n) \) is an induced inner subpolynomial of some \( q_{ii}(n) \), hence of \([q_{ii}(n)] p(n)\). Since \( t_{ij}(n) \) is an outer subpolynomial of some \( q_{ii}(n) \), \( t_{ij}(n) \) is an induced inner subpolynomial of \([q_{ii}(n)] p(n)\) and \([t_{ij}(n)] p(n)\) is an outer subpolynomial of \([q_{ii}(n)] p(n)\).

Now, if \( q(n) = \prod_{i=1}^{k} [q_i(n)] p(n), k > 1 \), we get a product of expressions like (2.9), which when multiplied out, is of form (2.7).

\[\Box\]

Recall the definitions preceding Lemma 1.2.2 of \( \overline{p}(n), \overline{q}(n), q'(n) \) where \( q(n) \) is a generalized polynomial.
Corollary 2.1.3 Let $q(n), q_i(n), t_{ij}(n), r_{ji}(n)$ be as in Lemma 2.1.2 and let $v_1(n), \ldots, v_{m_1}(n)$ be all the distinct simple generalized polynomials appearing in the expressions of the $r_{ji}(n)$’s. Then $q(n)$ is uniformly distributed (mod 1) if either

(i) \( \left( \sum_i q_i(n) + \sum_{i,j} \varepsilon_i t_{ij}(n), v_1(n), \ldots, v_{m_1}(n) \right) \)

is uniformly distributed (mod 1) in $\mathbb{R}^{m_1+1}$ for all $\varepsilon_i \in \{0,1\}$, or

(ii) \( \left( \sum_i \bar{q}_i(n) + \sum_{i,j} \varepsilon_i \bar{t}_{ij}(n), \bar{w}_1(n), \ldots, \bar{w}_{m_2}(n) \right) \)

is uniformly distributed (mod 1) in $\mathbb{R}^{m_2+1}$ for all $\varepsilon_i \in \{0,1\}$, where $w_1(n), \ldots, w_{m_2}(n)$ are all the distinct simple generalized polynomials from the set $\{v_i(n), \bar{v}_i(n), q_i(n), t_{ij}(n) | i, j, l \}$.

Proof: (i) follows directly from Lemma 2.1.1.

(ii). By (i) and Theorem 1.2.1, $q(n)$ is uniformly distributed (mod 1) if for all $(m_1+1)$-tuples of integers $a = (a_0, \ldots, a_{m_1}) \neq (0, \ldots, 0),$

\[
    g_a(n) = a_0 \left( \sum_i q_i(n) + \sum_{i,j} \varepsilon_i t_{ij}(n) \right) + \sum_{i=1}^{m_1} a_i v_i(n) \tag{2.10}
\]

is uniformly distributed (mod 1). Let $u_1(n), \ldots, u_{m_3}(n)$ be all the distinct elements from the set $\{v_i(n), \bar{v}_i(n), t_{ij}(n) | i, j, l \}$. By Lemma 1.2.2,

\[
g_a(n) \equiv \left( a_0 \left( \sum_i \bar{q}_i(n) + \sum_{i,j} \varepsilon_i \bar{t}_{ij}(n) \right) + \sum_i a_i \bar{v}_i(n), \bar{w}_1(n), \ldots, \bar{w}_{m_3}(n) \right) \tag{2.11}
\]

So if $w_1(n), \ldots, w_{m_2}(n)$ are as stated in the lemma, $q(n)$ is uniformly distributed (mod 1) if

\[
    \left( \sum_i \bar{q}_i(n) + \sum_{i,j} \varepsilon_i \bar{t}_{ij}(n), \bar{w}_1(n), \ldots, \bar{w}_{m_2}(n) \right) \tag{2.12}
\]
is uniformly distributed (mod 1) in $\mathbb{R}^{m_2+1}$.

\[ \square \]

**Definition 2.1.1** Let $S$ be the space of all linear combinations over $\mathbb{Z}$ of simple generalized polynomials. For any $h \in \mathbb{N}$, define a generalized derivative $V_h : S \rightarrow S$ inductively by

(i) $V_h n^l = lhn^{l-1}$, $l = 0, 1, 2, \ldots$

(ii) $V_h \left( q_1(n)q_2(n) \right) = q_1(n)V_h \left( q_2(n) \right) + V_h \left( q_1(n) \right) q_2(n)$, when $q_1(n)$ and $q_2(n)$ are simple generalized polynomials.

(iii) If $q(n)$ is a simple generalized polynomial and $V_h q(n) = \sum_i q_i(n)$, $q_i(n)$ simple, then $V_h \left[ q(n) \right] = \sum_i \left[ q_i(n) \right]$.

Also, let $V_0(q(n)) = q(n)$.

Note that if $p(n)$ is a usual polynomial, then $V_1 p(n)$ is the usual derivative $p'(n)$, and that for polynomials, (ii) is just the product rule.

**Example 3** If

$$ q(n) = \left[ p_1(n)p_2(n) \right] \left[ p_3(n) \right]^2 p_4(n), \quad (2.13) $$

where the $p_i(n)$’s are monomials, then

$$ V_h q(n) = \left[ [V_hp_1(n)p_2(n)] \left[ p_3(n) \right] \right]^2 p_4(n) + \left[ [p_1(n)V_hp_2(n)] \left[ p_3(n) \right] \right]^2 p_4(n) + 2 \left[ [p_1(n)p_2(n)] \left[ p_3(n) \right] V_hp_3(n) \right] p_4(n) + \left[ [p_1(n)p_2(n)] \left[ p_3(n) \right] \right]^2 V_hp_4(n). \quad (2.14) $$
Note that if \( q(n) \) is a simple generalized polynomial having \( k \) non-constant polynomials as subpolynomials, then \( V_h q(n) \) is a sum of \( k \) simple generalized polynomials.

As in the case of usual derivatives of polynomials, the generalized derivative reduces the degree of a generalized polynomial (Definition 1.2.6).

**Lemma 2.1.4** Let \( q(n) \) be a non-constant generalized polynomial and let \( h \in \mathbb{N} \). Then \( \deg(V_h q) = \deg(q) - 1 \).

**Proof:** This is obvious if \( q(n) \) is a polynomial. We will show the general case by induction on \( \deg(q) \). Note first that if \( q(n) = [\cdots[q_1(n)]\lambda_1 \cdots]\lambda_k \) and \( V_h q_1(n) = \sum_i v_i(n) \), then \( V_h[q(n)] = \sum_i [\cdots[v_i(n)]\lambda_1 \cdots]\lambda_k \), so that \( \deg(V_h q) = \max_i \deg(v_i) \). It is therefore enough to consider polynomials and generalized polynomials of the form \( q(n) = [q_1(n)]q_2(n) \), where \( q_1(n) \) and \( q_2(n) \) are generalized polynomials of positive degrees. Hence if \( \deg(q) = 1 \), then we may assume that \( q(n) \) is a linear polynomial so that \( \deg(V_h q) = \deg(q) - 1 = 0 \).

Assume the lemma is true for \( q(n) \) where \( \deg(q) \leq K \), and let \( q(n) = [q_1(n)]q_2(n) \), where \( \deg(q_1) > 0 \) and \( \deg(q_2) > 0 \), be a generalized polynomial of degree \( K + 1 \).

Then

\[
V_h q(n) = [q_1(n)]V_h q_2(n) + V_h([q_1(n)])q_2(n). \tag{2.15}
\]

Since \( \deg(q_2) > 0 \), we are done by induction.

We will end this section by stating a version of van der Corput’s difference theorem. The proof can be found in [24, p.26, see also Cor.2.1, p.251].
Theorem 2.1.5 (Van der Corput’s difference theorem) Let \( x(n), n = 1, 2, \ldots, \) be a sequence of real numbers. If for all but finitely many integers \( h, \) the sequence \( x(n + h) - x(n), n = 1, 2, \ldots, \) is uniformly distributed (mod 1), then \( x(n) \) is uniformly distributed (mod 1).

Remark: There is a well-distribution version of van der Corput’s difference theorem ([24], p.240). Therefore any generalized polynomial that we prove is uniformly distributed (mod 1) by using van der Corput’s difference theorem will be well-distributed as well.

2.2 Generalized polynomials with independent coefficients

Recall the definition of the set \( S(q) \) of coefficients of a generalized polynomial \( q(n), \) Definition 1.2.3, and let

\[
R(q) = \bigcup S(v),
\]

where the union is taken over all generalized polynomials \( v(n) \) which are obtained from \( q(n) \) by removing any number of nested brackets in \( q(n) \). So, if for example

\[
q(n) = \left[\alpha n\right][\beta n]\lambda n + [\delta n]^{3}\sigma\right] \gamma n^2,
\]

then \( S(q) = \{\alpha, \beta, \lambda, \delta, \sigma, \gamma\} \) and \( R(q) = S(q) \cup \{\lambda \gamma, \sigma \gamma, \alpha \lambda \gamma, \beta \lambda \gamma, \delta \sigma \gamma, \alpha \lambda, \beta \lambda, \delta \sigma\}.\)

Definition 2.2.1 A (representation of a) generalized polynomial \( q(n) \) has independent coefficients if \( R(q) \cup \{1\} \) is rationally independent.
The goal of this section is to prove the following theorem.

**Theorem 2.2.1** If (a representation of) a generalized polynomial \( q(x) \) has independent coefficients, then \( q(n) \) is uniformly distributed \((\mod 1)\).

We will use induction and van der Corput’s difference theorem to prove this theorem, similarly to what is done for polynomials, but it is more complicated in this case. The complications arise because of the form that \( q^h(n) \) takes. If \( q(n) \) is a polynomial, \( q^h(n) \) is a new polynomial of degree \( \deg(q) - 1 \). However, if \( q(n) \) is a generalized polynomial which is not a usual polynomial, we cannot write \( q^h(n) \) as a generalized polynomial of degree \( \deg(q) - 1 \) without obtaining indicator functions that have to be taken care of. Furthermore, \( q^h(n) \) has no longer independent coefficients. Therefore, instead of dealing directly with \( q(n) \), we will operate with a class of generalized polynomials coming from \( q(n) \).

Let \( Q(n) = \sum_i b_i Q_i(n) \) be a generalized polynomial where each \( b_i \in \mathbb{Z} \) and each \( Q_i(n) \) is simple. Suppose also that \( R(Q) \subset \{a\alpha \mid a \in \mathbb{N}, \alpha \in R(q)\} \) for some generalized polynomial \( q(n) \). By Lemma 1.2.5, integers can be brought out of brackets so that for each \( Q_i(n) \) there exists a corresponding generalized polynomial \( u_i(n) \) with \( R(u_i) \subset R(q) \) and such that

\[
Q(n) = \sum_i b_i a_i u_i(n) + \sum_i 1_{C_i}(s_i) t_i(n)
\]  

(2.18)

for some \( a_i \in \mathbb{N} \), some generalized polynomials \( t_i(n) \) and \( s_i(n) \) and some subintervals \( C_i \subset [0,1) \). We will say that \( u_i(n) \) is the reduced term of \( Q_i(n) \).
**Definition 2.2.2** Let \( Q(n) \) be as above. We will say that a simple generalized polynomial \( r(n) \) is a reduced subpolynomial of \( Q(n) \) if there exists a subpolynomial \( u(n) \) of \( Q(n) \) such that \( r(n) \) is the reduced term (with respect to some fixed \( q(n) \)) of \( u(n) \).

**Lemma 2.2.2** Let \( q(n) \) be a finite sum of simple generalized polynomials and \( q^h(n) = q(n + h) - q(n) \). Then

\[
q^h(n) = V_h q(n) + \sum_i b_i(h) s_i(n) + \sum_{i,j} 1_{B_i(*)} d_{ij}(h) t_{ij}(n),
\]

and by possibly adding more terms involving indicator functions,

\[
q^h(n) = \sum_i a_i(h) q_i(n) + \sum_i b_i(h) s_i(n) + \sum_{i,j} 1_{B_i(*)} d_{ij}(h) t_{ij}(n),
\]

where \( a_i(h), b_i(h), d_{ij}(h) \in \mathbb{Z} \), each \( s_i(n), t_{ij}(n) \) is a simple, reduced subpolynomial of \( V_{h_1} \circ \cdots \circ V_{h_l}q(n) \) for some \( l \), the \( q_i(n) \)'s are the simple, reduced terms of \( V_h q(n) \), \( B_i \) is some admissible subset of either \([0, 1)\) or \([0, 1)^2\) and each argument of the indicator functions is of form

\[
\sum_i c_i(h) r_i(n) + \sum_i 1_{B_i(*)} c_{ij}(h) r_{ij}(n)
\]

where \( c_i(h), c_{ij}(h) \in \mathbb{Z} \) and \( r_i(n), r_{ij}(n) \) are induced inner subpolynomials of \( q(n) \), \( q_i(n), s_i(n) \) and \( t_{ij}(n) \).

Moreover, \( \deg(s_i) < \deg(V_h q) \), \( \deg(r_i), \deg(r_{ij}) \leq \deg(q) \) and \( \deg(t_{ij}) \leq \deg(V_h q) \). If \( \deg(t_{ij}) = \deg(V_h q) \), then \( d_{ij}(h) = 1 \) and \( t_{ij}(n) \) equals a reduced term of \( V_h q(n) \). If \( q(n) = q(n) \), then \( \deg(r_i), \deg(r_{ij}) \leq \deg(V_h q) \).
Proof: Since the operators \( q(n) \mapsto q^h(n) \) and \( V_h \) are linear, we may assume that \( q(n) \) is simple. We will again use induction on \( B(q) \).

If \( B(q) = 0 \), then \( q(n) = \alpha n^k \) for some \( k \in \mathbb{N} \) and \( \alpha \in \mathbb{R} \). So

\[
q^h(n) = \alpha(n+h)^k - \alpha n^k
= \alpha kn^{k-1} + \alpha \sum_{i=2}^{k} \binom{k}{i} h^i n^{k-i}
= V_h q(n) + \sum_{i=2}^{k} \binom{k}{i} h^i \alpha n^{k-i}.
\] (2.22)

This gives \( b_i(h) = \binom{k}{i} h^i \in \mathbb{Z} \) and for each \( i = 2, \ldots, k-1 \), \( s_i(n) = \alpha n^{k-i} \) is the simple, reduced term of \( V_h \circ \cdots \circ V_h q(n) \). Also, \( a(h) = kh \).

Suppose now that the lemma is true for all simple generalized polynomials \( q(n) \) with \( B(q) \leq K \), and let \( q(n) \) be a simple generalized polynomial with \( B(q) = K + 1 \). Then \( q(n) = \prod_{i=1}^{k} [q_i(n)] p(n) \), where \( B(q_i) = K \) for some \( i, 1 \leq i \leq k \). We will prove it in the case \( k = 1 \). The general case follows similarly. So let \( q(n) = [q_1(n)] p(n) \) where \( B(q_1) = K \) and \( p(n) \) is a polynomial. By the induction hypothesis,

\[
q^h_1(n) = V_h q_1(n) + \sum_i b_{1i}(h)s_{1i}(n) + \sum_{i,j} 1_{B_i(*)} d_{1ij}(h)t_{1ij}(n)
\] (2.23)

where

\[
V_h q_1(n) = \sum_i q_{1i}(n) = \sum_i a_{1i}(h) u_{1i}(n) + \sum_i 1_{B_i(*)} v_{1i}(n),
\] (2.24)

and

\[
p^h(n) = V_h p(n) + \sum_i b_{2i}(h)s_{2i}(n) = \sum_i a_{2i}(h) p_i(n) + \sum_i b_{2i}(h)s_{2i}(n)
\] (2.25)
where \( q_{1i}(n), s_{ji}(n), t_{ij}(n), u_{1i}(n), v_{1i}(n) \) and \( p_i(n) \) are all simple generalized polynomials. By using these identities and Lemma 1.2.5, we have

\[
q^h(n) = [q_1(n + h)] p(n + h) - [q_1(n)] p(n)
\]

\[
= [q_1(n) + \sum_i q_{1i}(n) + \sum_i b_{1i}(h) s_{1i}(n) + \sum_{i,j} 1_{B_i}(*d_{1ij}(h)t_{1ij}(n))] (p(n) + V_h p(n) + \sum_i b_{2i}(h) s_{2i}(n)) - [q_1(n)] p(n)
\]

\[
= [q_1(n)] V_h p(n) + \sum_i [q_{1i}(n)] p(n)
+ \sum_i b_{2i}(h)[q_1(n)] s_{2i}(n) + \sum_i b_{1i}(h)[s_{1i}(n)] p(n) \tag{2.26}
\]

\[
+ \sum_{i,j} (a_{1i}(h)[u_{1i}(n)] + b_{1i}(h)[s_{1i}(n)]) (a_{2j}(h)p_j(n) + b_{2j}(h)s_{2j}(n)) \tag{2.27}
\]

\[
+ \sum_{i,j} 1_{B_i}(*d_{ij}(h)t_{ij}(n)) \tag{2.28}
\]

where

\[
\{ t_{ij}(n) \mid i, j \} = \{ [t_{1ij}(n)] p(n), [t_{1ij}(n)] p_i(n), [t_{1ij}(n)] s_{2i}(n), [v_{1i}(n)] p_j(n), [v_{1i}(n)] s_{2j}(n), p(n), p_i(n), s_{2i}(n) \mid i, j \} \tag{2.29}
\]

and \( d_{ij}(h) \) are the corresponding integer coefficients.

Now, \( V_h q(n) = [q_1(n)] V_h p(n) + \sum_i [q_{1i}(n)] p(n) \), so we only need to check that the terms in (2.26), (2.27) and (2.28) have the properties stated in the lemma.

Since each \( q_{1i}(n), s_{1i}(n), u_{1i}(n) \) is a reduced outer subpolynomial of some \( V_{h_i} \circ \cdots \circ V_{h_1} q_1(n) \) and each \( p(n), p_i(n), s_{2i}(n) \) is a reduced term of some \( V_{h_i} \circ \cdots \circ V_{h_1} p(n) \), each \([q_1(n)] s_{2i}(n), [s_{1i}(n)] p(n), [u_{1i}(n)] p_j(n), [u_{1i}(n)] s_{2i}(n), [s_{1i}(n)] p_j(n), [s_{1i}(n)] s_{2j}(n) \) is a reduced outer subpolynomial of some \( V_{h_i} \circ \cdots \circ V_{h_1} q(n) \). The same is also true about the \( t_{ij}(n) \)'s.
By the induction hypothesis, \( \deg(s_{1i}) < \deg(V_hq_1) \) and \( \deg(s_{2i}) < \deg(V_hp) \).

Therefore

\[
\deg([s_{1i}]s_{2i}) < \deg([q_1]s_{2i}) < \deg([q_1]V_hp) = \deg(V_hq),
\]

(2.30)

\[
\deg([s_{1i}]p_j) < \deg([s_{1i}]p) < \deg([V_hq_1]p) = \deg(V_hq),
\]

(2.31)

\[
\deg([u_{1i}]s_{2j}) < \deg([u_{1i}]p_j) \leq \deg([V_hq_1]V_hp) < \deg(V_hq).
\]

(2.32)

Also, \( \deg(t_{1ij}) \leq \deg(V_hq_1) \) and \( \deg(v_{1i}) < \deg(V_hq_1) \), so it follows that \( \deg(t_{1ij}) \leq \deg(V_hq) \). Now if \( \deg(t_{1ij}) = \deg(V_hq) \), then \( t_{1ij}(n) = p(n) \) if \( \deg(q_1) = 1 \), and \( t_{1ij}(n) = [t_{1ij}(n)]p(n) \) otherwise, where \( \deg(t_{1ij}) = \deg(V_hq_1) \). Therefore, by the induction hypothesis, \( d_{1ij}(h) = 1 \) and \( t_{1ij}(n) \) equals a reduced term of \( V_hq_1(n) \). Hence \( d_{ij}(h) = 1 \) and \( t_{ij}(n) \) equals a reduced term of \( V_hq(n) \).

By repeated use of Lemma 1.2.5, the arguments of the indicator functions can be written in the form (2.21). Any term coming from the arguments of \( q_1^h(n) \) has the desired properties by the induction hypothesis. Also, any simple reduced term of an argument comes from \( q_1(n + h) \), so it is an induced inner subpolynomial of \( q(n), q_i(n), s_i(n) \) or \( t_{ij}(n) \). If \( q(n) = \overline{q}(n) \), then \( \deg(p) \geq 1 \), so \( \deg(r_i), \deg(r_{ij}) \leq \deg(q_1) < \deg(q) \).

\[\square\]

**Lemma 2.2.3** Let \( q(n) \) be a simple generalized polynomial and \( u(n) \) a simple reduced subpolynomial of \( V_{h'} \circ \cdots \circ V_{h_1}q(n) \). Then the following hold:

(i) If \( x(n) \) is a reduced subpolynomial of \( V_{h_1} \circ \cdots \circ V_{h_1}\overline{u}(n) \), then there exists a reduced subpolynomial \( \check{x}(n) \) of \( V_{h_0} \circ \cdots \circ V_{h_1}u(n) \) such that \( \check{x}(n) = \pi(n) \).
(ii) If \( x(n) \) is a reduced subpolynomial of \( V_{h_i} \circ \cdots \circ V_{h_1} u(n) \), then \( x(n) \) is a reduced subpolynomial of \( V_{h_{1+l'}} \circ \cdots \circ V_{h_1} q(n) \).

Proof: (i) Suppose \( u(n) = [\cdots [v_1(n)] \lambda_1 \cdots ] \lambda_k \). Let \( v(n) = \pi(n) = v_1(n) \lambda_1 \cdots \lambda_k \) and \( V_{h_i} \circ \cdots \circ V_{h_1} v_1(n) = \sum_i v_{1i}(n) \). Then

\[
V_{h_i} \circ \cdots \circ V_{h_1} u(n) = \sum_i [\cdots [v_{1i}(n)] \lambda_1 \cdots ] \lambda_k
\]

(2.33)

and

\[
V_{h_i} \circ \cdots \circ V_{h_1} v(n) = \sum_i v_{1i}(n) \lambda_1 \cdots \lambda_k.
\]

(2.34)

\( x(n) \) is either a subpolynomial of \( v_{1i}(n) \) for some \( i \), in which case \( x(n) \) is also a subpolynomial of \( [\cdots [v_{1i}(n)] \lambda_1 \cdots ] \lambda_k \), or \( x(n) \) is an outer subpolynomial of \( v_{1i}(n) \lambda_1 \cdots \lambda_k \) for some \( i \), say \( x(n) = y(n) \lambda_1 \cdots \lambda_k \). But then \( \tilde{x}(n) = [\cdots [y(n)] \lambda_1 \cdots ] \lambda_k \) is a subpolynomial of \( [\cdots [v_{1i}(n)] \lambda_1 \cdots ] \lambda_k \) and \( \tilde{\pi}(n) = \pi(n) \).

(ii) Bringing out integers from the brackets of a generalized polynomial, which we will refer to as reducing the generalized polynomial, changes only the coefficients and not the structure of the generalized polynomial. Therefore, if \( v(n) \) is a subpolynomial of \( V_{h_i} \circ \cdots \circ V_{h_1} q(n) \), we will obtain the same set of reduced terms if we reduce \( v(n) \) first and then take \( l \) generalized derivatives and reduce as if we just reduce the terms of \( V_{h_i} \circ \cdots \circ V_{h_1} v(n) \). So if \( v(n) \) is such that \( u(n) \) is the reduced term of \( v(n) \), then \( x(n) \) is a reduced term of \( V_{h_i} \circ \cdots \circ V_{h_1} v(n) \).

Let \( y(n) \) be a subpolynomial of \( V_{h_i} \circ \cdots \circ V_{h_1} v(n) \) such that \( x(n) \) is the reduced term of \( y(n) \). Let \( Q(n) = V_{h_{1+l'}} \circ \cdots \circ V_{h_1} q(n) \). Recall that \( v(n) \) is a subpolynomial of \( Q(n) \). We will prove by induction on \( l \) that any subpolynomial \( y(n) \) of \( V_{h_i} \circ \)
\[ \cdots \circ \dot{V}_{h_1} \circ \cdots \circ V_{h_1} Q(n) = V_{h_1} \circ \cdots \circ \dot{V}_{h_1} \circ \cdots \circ \dot{V}_{h_1} q(n) = \dot{V}_{h_i \circ \cdots \circ V_{h_1}} q(n). \]

First, let \( l = 1 \) and let \( a(n) \) be the term of \( Q(n) \) such that \( v(n) \) is a subpolynomial of \( a(n) \). If \( V_{h_1} v(n) = \sum_{i=1}^{k} v_i(n) \), where \( v_i(n) \) is a simple generalized polynomial, \( i = 1, \ldots, k \), then \( V_{h_1} a(n) = \sum_{i=1}^{k} a_i(n) + a_0(n) \), where \( v_i(n) \) is a subpolynomial of \( a_i(n) \), \( i = 1, \ldots, k \), and \( v(n) \) is a subpolynomial of each term of \( a_0(n) \). Hence, any subpolynomial of \( V_{h_1} v(n) \) is a subpolynomial of \( V_{h_1} Q(n) \).

Let \( q(n) \) be a generalized polynomial with independent coefficients. We will put a partial order on the set \( S(q) \) of coefficients of \( q(n) \) by

\[ \alpha_1 \preceq \alpha_2 \text{ if there exist subpolynomials } u_1(n) \text{ and } u_2(n), \quad u_2(n) \neq [u_2(n)], \]

such that

\[ \alpha_1 \text{ is a coefficient of } u_1(n), \quad \alpha_2 \text{ is an outer coefficient of } u_2(n) \text{ and such that } \]

\[ [u_1(n)]u_2(n) \text{ is either an inner subpolynomial of } q(n) \text{ or a term of } q(n). \]

Since \( q(n) \) has independent coefficients, the relation \( \preceq \) is a strict ordering. In order to check that \( \preceq \) is transitive, let \( \alpha_1 \preceq \alpha_2 \) and \( \alpha_2 \preceq \alpha_3 \). Then there exist generalized polynomials \( u(n) = [u_1(n)]u_2(n) \) and \( v(n) = [v_1(n)]v_2(n) \) where \( \alpha_1 \) is a coefficient of \( u_1(n) \), \( \alpha_2 \) is an outer coefficient of \( u_2(n) \) and also a coefficient of \( v_1(n) \), and \( \alpha_3 \) is an outer coefficient of \( v_2(n) \). Since \( \alpha_2 \) is a coefficient of \( v_1(n) \) in addition to being an outer coefficient of \( u(n) \), \( u(n) \) is necessarily an inner subpolynomial of \( q(n) \) and therefore is a subpolynomial of \( v_1(n) \). Hence, \( \alpha_1 \preceq \alpha_3 \).
Example 4  If
\[ q(n) = \left[ \alpha n \right] \left[ \beta n \right] \gamma n \lambda + \delta n^2 \sigma, \]  
(2.35)
then
\[ \alpha \prec \lambda \prec \sigma \]
\[ \beta \prec \gamma \prec \lambda \prec \sigma \]
\[ \delta \prec \sigma. \]  
(2.36)

Note that there are no relations between \( \alpha \) and \( \beta \) or between \( \alpha \) and \( \gamma \).

If \( q_i(n) \) is a simple term (see Lemma 2.1.2) of \( q(n) \), then each simple, reduced term of \( V_h q_i(n) \) has independent coefficients and the coefficients have the same order as in \( q(n) \). We will say that \( V_h \) preserves the ordering of the coefficients.

If \( q(n) \) is a generalized polynomial with independent coefficients and \( q_i(n), i = 1, \ldots, m, \) are its simple terms, then let \( U_i(q_i) \) be the set of all non-constant, simple reduced subpolynomials of \( V_h q_i(n) \) and
\[ U(q) = \bigcup_{i=1}^{m} \bigcup_{l=0}^{D_i-1} U_l(q_i) \]  
(2.37)
where \( D_i = \deg(q_i) \). Note that if \( u(n) \in U(q) \), then \( S(u) \subset S(q) \) and that if \( \alpha, \beta \in S(u) \) and \( \alpha \prec \beta \) as elements of \( S(q) \), then \( \alpha \prec \beta \) as elements of \( S(u) \). The following proposition is important for the proof of Theorem 2.2.1.

Proposition 2.2.4 Let \( q(n) \) be a generalized polynomial with independent coefficients. Then for any subset \( \{ u_1, \ldots, u_k \} \) of \( U(q) \), where \( u_i \neq u_j \) if \( i \neq j \), both
\[ u(n) = (u_1(n), \ldots, u_k(n)) \text{ and } \pi(n) = (\pi_1(n), \ldots, \pi_k(n)) \]  
(2.38)
are uniformly distributed (mod 1) in \( \mathbb{R}^k \).

Let us illustrate this with an example. Let

\[
q(n) = \left[ (\alpha n^2)^3 \beta n^2 + \lambda n^4 \right]^2 \gamma n,
\]

where \(1, \alpha, \beta, \lambda, \gamma, \alpha \beta, \alpha \beta \gamma, \lambda \gamma\) are rationally independent. The simple terms of \( q(n) \) are then

\[
\left[ (\alpha n^2)^3 \beta n^2 \right]^2 \gamma n,\ 2 \left[ (\alpha n^2)^3 \beta n^2 \right] \lambda n^4 \gamma n,\ \left[ \lambda n^4 \right]^2 \gamma n.
\]

There are too many elements of \( U(q) \) to write them all up. So consider the subpolynomial \( u(n) = (\alpha n^2)^3 \beta n^2 \in U(q) \) of \( q(n) \), and

\[
U_4(u) = \left\{ (\alpha n^2)^2 \beta,\ (\alpha n^2)^2 \beta n^2,\ \alpha n(\alpha n^2)^2 \beta n,\ (\alpha n^2)^2 \beta n^2,\ (\alpha n)^2 \beta n,\ (\alpha n^2)^2 \beta \right\} \subset U(q).
\]

Then, by Proposition 2.2.4,

\[
\left( (\alpha n^2)^2 \beta,\ (\alpha n^2)^2 \beta n^2,\ \alpha n(\alpha n^2)^2 \beta n,\ (\alpha n^2)^2 \beta n^2,\ (\alpha n)^2 \beta n,\ (\alpha n^2)^2 \beta \right)
\]

is uniformly distributed (mod 1) in \( \mathbb{R}^6 \).

We will need the following lemma in the proof of Proposition 2.2.4.

**Lemma 2.2.5** If \( q(n) \) is a generalized polynomial with independent coefficients, let \( u_1(n), \ldots, u_k(n) \in U(q) \) be \( k \) distinct generalized polynomials with \( \deg(u_i) = D, i = 1, \ldots, k \). Let \( \{ r_i(n) \mid i \} \) be the set of distinct inner subpolynomials of the \( \pi_i(n) \)'s of degree \( D - 1 \) and let \( c_i \in \mathbb{Z} \setminus \{0\}, i = 1, \ldots, k \). If \( u_{ij}(n) \) are the reduced generalized
polynomials such that

\[ V_h \overline{v}_i(n) = \sum_{j=1}^{k_i} a_{ij}(h) u_{ij}(n) + \sum_j C_{ij}(\ast) t_{ij}(n) \]  \hspace{1cm} (2.43)

for some \( a_{ij}(h) \in \mathbb{Z} \), let \( v_1(n), \ldots, v_l(n) \) be all the distinct elements from the set \( \{ \overline{v}_{ij}(n), \overline{r}_i(n) \mid i,j \} \) and \( d_i(h) \in \mathbb{Q} \). Then

\[ \sum_{i=1}^{k} \sum_{j=1}^{k_i} c_{i} a_{ij}(h) \overline{v}_{ij}(n) + \sum_i d_i(h) \overline{r}_i(n) = \sum_{i=1}^{l} b_i(h) v_i(n) \]  \hspace{1cm} (2.44)

for some \( b_i(h) \in \mathbb{Q} \) such that \( (b_1(h), \ldots, b_l(h)) \neq (0, \ldots, 0) \) for all but finitely many \( h \).

Note that if \( u_1(n), \ldots, u_6(n) \) are the elements of \( U_4(u) \) in the order which they are written in (2.41), and \( c_1, \ldots, c_6 \in \mathbb{Z} \), not all equal 0, then (2.44) becomes

\[ c_1 4h[\alpha n^2][\alpha n] \beta + c_2 \left( 2h[\alpha n] \beta n^2 + 2h[\alpha n^2] \beta n \right) + c_3 \left( [\alpha h][\alpha n^2] \beta n \right. \]
\[ + 2h[\alpha n]^2 \beta n + h[\alpha n][\alpha n^2] \beta \bigg) + c_4 \left( 2[\alpha h][\alpha n] \beta n^2 + 2h[\alpha n]^2 \beta n \right) \]
\[ + c_5 \left( 3[\alpha h][\alpha n] \beta n + h[\alpha n]^3 \beta \right) + c_6 \left( 2h[\alpha n]^3 \beta + 2[\alpha h][\alpha n^2][\alpha n] \beta \right) \]
\[ = \left( h(4c_1 + c_3) + 2[\alpha h]c_6 \right) [\alpha n^2][\alpha n] \beta + \left( 2hc_2 + 2[\alpha h]c_4 \right) [\alpha n]^2 \beta n \]
\[ + \left( 2hc_2 + [\alpha h]c_3 \right) [\alpha n^2] \beta n + \left( 2h(c_3 + c_4) + 3c_5[\alpha h] \right) [\alpha n]^2 \beta n \]
\[ + \left( h(c_5 + 2c_6) \right) [\alpha n]^3 \beta \]  \hspace{1cm} (2.45)

which is non-zero for all but at most finitely many \( h \) (see Lemma 2.3.3).

**Proof of Lemma 2.2.5:** Suppose first that \( \overline{v}_1(n), \ldots, \overline{v}_k(n) \) are polynomials. Then

\[ u_i(n) = \left[ \cdots [\alpha_i n^p] \lambda_{i_1} \cdots \right] \lambda_{i_t}, \quad i = 1, \ldots, k \]  \hspace{1cm} (2.46)
where $\alpha_i, \lambda_{ij} \in S(q)$ and $\pi_i(n) = \alpha_i \prod_{j=1}^{k_i} \lambda_{ij}n^D$. Note that $u_{i_1}(n) \neq u_{i_2}(n)$ for $i_1 \neq i_2$ implies that $\alpha_i \prod_{j=1}^{k_i} \lambda_{ij} \neq \alpha_{i_2} \prod_{j=1}^{k_{i_2}} \lambda_{i_2j}$. For $\alpha_{i_j} \prec \lambda_{i_1} \prec \cdots \prec \lambda_{i_l} \prec \lambda_{i_l} \in R(q)$, $j = 1, 2$, and $R(q)$ is rationally independent. So

$$V_h \left( \sum_{i=1}^{k} c_i \pi_i(n) \right) = \sum_{i=1}^{k} c_i \alpha_i \prod_{j=1}^{k_i} \lambda_{ij} h D^{n^D-1} = \sum_{i=1}^{k} b_i(h) \pi_i(n) \quad (2.47)$$

where $b_i(h) = c_i Dh \neq 0$, which was to be proved.

Next, suppose that there is at least one $i$ such that $\pi_i(n)$ is not a polynomial. We will first show that if $w_1(n), \ldots, w_m(n)$ are all the distinct elements from the set $\{\pi_{ij}(n) \mid i, j\}$, and

$$\sum_{i,j} c_{ij} a_{ij}(h) \pi_{ij}(n) = \sum_i B_i(h) w_i(n), \quad (2.48)$$

then $(B_1(h), \ldots, B_m(h)) \neq (0, \ldots, 0)$ for all but finitely many $h$.

Let $\alpha n^t$ be an inner subpolynomial of some $\pi_i(n)$, $t \geq 1$, and let

$$T = \min \{ t \geq 1 \mid \alpha n^t \text{ is an inner subpolynomial of } \pi_j(n) \text{ for some } j \in \{1, \ldots, k\} \},$$

$$R_\alpha = \{ \pi_j(n) \mid \alpha n^T \text{ is an inner subpolynomial of } \pi_j(n) \} \quad (2.49)$$

and

$$S_\alpha = \{ s_{ij}(n) = a_{ij}(h) \pi_{ij}(n) \mid \pi_i(n) \in R_\alpha \}. \quad (2.50)$$

For each $\pi_i(n) \in R_\alpha$, there exists exactly one corresponding $s_{ij}(n) \in S_\alpha$ with $\alpha n^{T-1}$ as an inner subpolynomial if $T > 1$ and with $a_{ij}(h) = [\alpha h]$ if $T = 1$. With this corresponding $s_{ij}(n) = a_{ij}(h) \pi_{ij}(n)$, let $w_i(n) = \pi_{ij}(n)$. It follows that if $i_1 \neq i_2$ and $\pi_{i_1}(n), \pi_{i_2}(n) \in R_\alpha$, then $w_{i_1}(n) \neq w_{i_2}(n)$. 

If $T > 1$, the only $\pi_{ij}(n)$’s having $\alpha n^{T-1}$ as an inner subpolynomial, arise from elements in $R_\alpha$. Since all these $w_i(n) = \pi_{ij}(n)$’s are different, there are no cancellations so that the corresponding coefficients $B_i(h)$’s have the property that $B_i(h) \neq 0$ for all $h$.

If $T = 1$, there may exist $i_1 \neq i_2$ and $j$ such that $\pi_{i_1}(n) \in R_\alpha$ and $w_{ij}(n) = \pi_{i_2j}(n)$. However, we have already seen that we then have $\pi_{i_2}(n) \not\in R_\alpha$. Therefore $a_{i_2j}(h) \neq [\alpha h]$ because the coefficient $[\alpha h]$ can only come from terms having $\alpha n$ as an inner subpolynomial. Hence $B_{i_1}(h) = c_{i_1}[\alpha h] + Bh + \sum e_i[\beta_i h]$ for some $B, e_i \in \mathbb{Z}$ and $\beta_i \in S(q), \beta_i \neq \alpha$. Since $c_{i_1} \neq 0$ and $S(q)$ is rationally independent, $B_{i_1}(h) \neq 0$ for all but finitely many $h$, see Lemma 2.3.3.

It remains to show that none of the subpolynomials $r_i(n)$ can cancel out all the $w_i(n)$’s in (2.48). We may assume that $d_i(h) \in \mathbb{Z}$. Suppose that

$$\sum_{i=1}^{m} B_i(h)w_i(n) + \sum_{i} d_i(h)\pi_i(n) = \sum_{i=1}^{l} b_i(h)v_i(n). \quad (2.51)$$

We need to show that $(b_1(h), \ldots, b_l(h)) \neq (0, \ldots, 0)$ for all but finitely many $h$. Since terms with different outer coefficients cannot cancel each other out, and since the above argument shows that for each distinct outer coefficient of the $u_i(n)$’s there is some $B_i(h) \neq 0$, all the outer coefficients of the $u_i(n)$’s are found in the $w_i(n)$’s but possibly multiplied by some inner coefficients. Let $\gamma$ be a maximal element in $\bigcup_{i=1}^{k} S(u_i) \subset S(q)$, with respect to the partial order of the set of coefficients of the $u_i(n)$’s, and let $w_j(n)$ be a generalized polynomial where $\gamma$ is a factor of the outer coefficient and for which the corresponding $B_j(h) \neq 0$. All the $r_i(n)$’s are inner subpolynomials of the $u_i(n)$’s. Therefore if $\lambda$ is an outer coefficient of $r_i(n)$, then
either $\lambda \prec \gamma$ or there is no relation between $\lambda$ and $\gamma$. That means that $\pi_i(n) \neq w_j(n)$ for all $i$. Hence there is at least one $j$ such that $b_j(h) \neq 0$.

\[ \square \]

**Proof of Proposition 2.2.4:** Since the partial ordering of the coefficients is preserved under $V_{h_1} \circ \cdots \circ V_{h_k}$, $\pi_i(n) \neq \pi_j(n)$ if $i \neq j$. There is also at least one $i_0$ such that $\pi_{i_0}(n) \neq \pi_i(n)$ for all $i, j$. We need to show that for any $a = (a_1, \ldots, a_k) \in \mathbb{Z}^k \setminus \{(0, \ldots, 0)\}$,

\[ u(n) = \sum_{i=1}^{k} a_i \pi_i(n) \]  \hspace{1cm} (2.52)

is uniformly distributed (mod 1). Without loss of generality we may assume that $a_i \neq 0$ for all $i$. Let $w_1(n), \ldots, w_{k_1}(n)$ be all the distinct, simple generalized polynomials from $\{\pi_i(n) \mid i, j\}$ and let $v_1(n), \ldots, v_{k_2}(n)$ be all the distinct elements from the set $\{\pi_i(n), w_j(n) \mid i, j\}$. Then by Lemma 1.2.2,

\[ u(n) \leftrightarrow \left( \sum_{i=1}^{k} a_i \pi_i(n), w_1(n), \ldots, w_{k_1}(n) \right), \]  \hspace{1cm} (2.53)

and since $\pi_{i_0}(n) \neq w_j(n)$ for all $j$,

\[ u(n) \leftrightarrow (v_1(n), \ldots, v_{k_2}(n)). \]  \hspace{1cm} (2.54)

It is therefore enough to show that $v(n) = (v_1(n), \ldots, v_k(n))$ is uniformly distributed (mod 1) for any subset

\[ \{v_1(n), \ldots, v_k(n)\} \subset \overline{U}(q) = \{\pi(n) \mid u(n) \in U(q)\} \]  \hspace{1cm} (2.55)

such that $v_i(n) \neq v_j(n)$ if $i \neq j$, and we will do this by induction on $\deg(v) \overset{\text{def}}{=} \max_i(\deg(v_i))$. 


If \( \deg(v) = 1 \), \( v(n) \) is uniformly distributed (mod 1) since \( v_i(n) = \beta_i n \) for some \( \beta_i \in R(q) \), and \( R(q) \cup \{1\} \) is rationally independent.

Assume now that \( v(n) \) is uniformly distributed (mod 1) if \( \deg(v) \leq K \), and let \( \deg(v) = K + 1 \). We will show that \( v_c(n) = \sum_{i=1}^{k} c_i v_i(n) \) is uniformly distributed (mod 1) for any \( k \)-tuple of integers \( c = (c_1, \ldots, c_k) \neq (0, \ldots, 0) \). If \( \deg(v_{i_1}), \ldots, \deg(v_{i_{m_1}}) < \deg(v) \), then by the induction hypothesis, \( (v_{i_1}(n), \ldots, v_{i_{m_1}}(n)) \) is uniformly distributed (mod 1). So we may assume without loss of generality that \( c_i \neq 0, i = 1, \ldots, k \). Let \( u_c(n) = \sum_{i=1}^{k} c_i u_i(n) \), where \( u_i(n) \in U \) and \( \overline{u}_i(n) = v_i(n) \).

By Lemma 2.2.2

\[
v^k_c(n) = \sum_{i,j} c_i a_{ij}(h)v_{ij}(n) + \sum_i b_i(h)s_i(n) + \sum_{i,j} 1_{B_i}(*)d_{ij}(h)t_{ij}(n) \tag{2.56}
\]

where each \( v_{ij}(n), s_i(n), t_{ij}(n) \) is a simple reduced subpolynomial of some \( V_{h_1} \circ \cdots \circ V_{h_1} v_c(n) \). Also, \( \deg(s_i) < \deg(V_h v_c) = K \) and if \( \deg(t_{ij}) = \deg(V_h v_c) \), then \( d_{ij}(h) = 1 \).

Let \( \{r_i(n) \mid i\} \) be the set of all distinct simple, reduced generalized polynomials appearing in the arguments of the indicator functions. By Lemma 2.2.2, each \( r_i(n) \) is an induced inner subpolynomial of one of the generalized polynomials \( v_c(n), v_{ij}(n), s_i(n) \) or \( t_{ij}(n) \). Therefore each \( r_i(n) \) is a reduced subpolynomial of some \( V_{h_i} \circ \cdots \circ V_{h_1} v_c(n) \). Let furthermore \( w_1(n), \ldots, w_{k_1}(n) \) be all the distinct terms from the set \( \{r_i(n), r_i^1(n), v_{ij}^1(n), s_i^1(n), t_{ij}^1(n) \mid i, j, l\} \). Then by Lemma 2.1.1 and Lemma 1.2.2,

\[
v^k_c(n) = \left( \sum_{i,j} c_i a_{ij}(h)\overline{v}_{ij}(n) + \sum_i b_i(h)\overline{s}_i(n) + \sum_i \epsilon_i d_{ij}(h)\overline{t}_{ij}(n), \overline{w}_1(n), \ldots, \overline{w}_{k_1}(n) \right) \tag{2.57}
\]

Note that each \( w_i(n) \) is also a reduced subpolynomial of some \( V_{h_i} \circ \cdots \circ V_{h_1} v_c(n) \) so that if \( Q_1(n), \ldots, Q_{k_1}(n) \) are all the distinct generalized polynomials from the set
\{v_{ij}(n), t_{ij}(n), s_i(n), w_i(n) \mid i, j\}, then each \( Q_i(n) \) is a reduced subpolynomial of some \( V_{h_r} \circ \cdots \circ V_{h_1} v_c(n) \). By Lemma 2.2.3(i) there exist a generalized polynomial \( \tilde{Q}_i(n) \) such that \( V_{h_l} \circ \cdots \circ V_{h_1} v_c(n) \) for some \( l, i = 1, \ldots, k_4 \). So by Lemma 2.2.3(ii), \( \tilde{Q}_i(n) \in U(q) \) and therefore \( \tilde{Q}_i(n) \in U(q) \). Since \( \deg(\tilde{Q}_i) \leq K \), it follows by the induction hypothesis that \( (\tilde{Q}_1(n), \ldots, \tilde{Q}_{k_4}(n)) \) is uniformly distributed (mod 1) in \( \mathbb{R}^{k_4} \).

Now, if
\[
g(h) = \sum_{i,j} c_{ij} a_{ij}(h)v_{ij}(n) + \sum_i b_i(h)s_i(n) + \sum_i \epsilon_i d_{ij}(h)t_{ij}(n) + \sum_i e_i w_i(n) \neq 0 \tag{2.58}
\]
for any \( k_4 \)-tuple of rational numbers \( (\epsilon_1, \ldots, \epsilon_{k_4}) \neq (0, \ldots, 0) \), then
\[
v_c(h) = \tilde{Q}_1(n), \ldots, \tilde{Q}_{k_4}(n) \tag{2.59}
\]
It is enough, by van der Corput’s difference theorem, to show (2.59) and hence (2.58) for all but finitely many \( h \). Furthermore, it suffices to consider the terms of degree \( K \).

Note that \( \deg(s_i) < K \) and \( \deg(t_{ij}) < K \) unless \( d_{ij}(h) = 1 \), i.e. \( d_{ij}(h) \) is independent of \( h \). Since each \( a_{ij}(h) \) is an increasing function of \( h \), and it is sufficient that \( g(h) \neq 0 \) for all but finitely many \( h \), we need only to consider
\[
D(h) = \sum_{i,j} c_{ij} a_{ij}(h)v_{ij}(n) + \sum_i e_i w_i(n). \tag{2.60}
\]
However, \( D(h) \neq 0 \) for all but finitely many \( h \), by Lemma 2.2.5. Therefore \( g(h) \neq 0 \) and \( v_c(h) = \tilde{Q}_1(n), \ldots, \tilde{Q}_{k_4}(n) \) for all but finitely many \( h \).

\[\square\]
Proof of Theorem 2.2.1: Let \( q(n) \) be a generalized polynomial with independent coefficients. By Corollary 2.1.3(i), \( q(n) \) is uniformly distributed (mod 1) if
\[
\left( \sum_{i=1}^{k} q_i(n) + \sum_{i,j} \epsilon_i t_{ij}(n), r_1(n), \ldots, r_m(n) \right)
\]  
(2.61)
is uniformly distributed (mod 1) for any \( \epsilon_i \in \{0, 1\} \), where \( q_i(n), i = 1, \ldots, k \), are the simple terms of \( q(n) \), the \( t_{ij}(n) \)'s are outer subpolynomials of \( \sum_{i=1}^{k} q_i(n) \) and the generalized polynomials \( r_1(n), \ldots, r_m(n) \) are distinct induced inner subpolynomials of \( \sum_{i=1}^{k} q_i(n) \). So if \( t_1(n), \ldots, t_{k_1}(n) \) are all the distinct terms among the \( t_{ij}(n) \)'s, then
\[
q(n) \leftrightarrow (q_1(n), \ldots, q_k(n), t_1(n), \ldots, t_{k_1}(n), r_1(n), \ldots, r_m(n)).
\]  
(2.62)
Since all \( q_i(n), t_i(n), r_i(n) \) are distinct reduced subpolynomials of \( \sum q_i(n) \), it follows from Proposition 2.2.4 \( (l = 0) \), that \( q(n) \) is uniformly distributed (mod 1).

\[ \square \]

2.3 Main Result

Recall that \( R(q) \) is the set of coefficients of the generalized polynomial \( q(n) \) and of all generalized polynomials obtained from \( q(n) \) by removing nested brackets. In our main theorem, which follows below, we require that the subpolynomials have independent coefficients except that their outer coefficient can take the value 1. In this way, the subpolynomials \( q_i(n) \) (see below) can take the form \( \prod_{j=1}^{l} [q_{ij}(n)] \). Define therefore a set \( R'(q) = R(q) \setminus \{1\} \).
**Theorem 2.3.1** Let

\[ q(x) = p_0(x) + \sum_{i=1}^{k} [q_i(x)]p_i(x) \quad (2.63) \]

be a generalized polynomial such that \( q_i(x), i = 1, \ldots, k, \) are generalized polynomials and \( p_i(x), i = 0, \ldots, k, \) are polynomials. Suppose that there exists \( i_0 \) such that \([q_{i_0}(x)]p_{i_0}(x)\) has a simple term \( a(x) \) with \( \deg(a) = \deg(q) \) and \( \gamma(n) \) is not a polynomial, and such that

\[ R \cup R([q_{i_0}]\gamma_{i_0}) \quad (2.64) \]

is rationally independent, where \( R = \bigcup_{j=1}^{k} R'(q_j) \cup \{1\} \) and \( \gamma_{i_0} \) is the leading coefficient of \( p_{i_0}(x) \). Then \( q(n) \) is uniformly distributed (mod 1).

This gives us immediately the following corollary.

**Corollary 2.3.2** Let \( q_i(x), i = 1, \ldots, k, \) be generalized polynomials such that \( R = \bigcup_{i=1}^{k} R(q_i) \cup \{1\} \) is rationally independent. Then

\[ q(n) = \sum_{i=1}^{k} [q_i(n)]p_i(n) + p_0(n) \quad (2.65) \]

is uniformly distributed (mod 1) for all but countably many \( k \)-tuples of monomials \((p_1(n), \ldots, p_k(n))\), where \( \deg(p_i) \geq 1 \) if \( \eta_i(n) \) is a polynomial.

Before we prove Theorem 2.3.1 we need some lemmas.

**Lemma 2.3.3** Let \( q_0(n), \ldots, q_k(n) \) be generalized polynomials of degree 1 (with possibly \( q_0(n) = n \)) such that the coefficients of \( \eta_0(n), \ldots, \eta_k(n) \) are rationally independent
and let $a_i \in \mathbb{Q}$. Then
\begin{equation}
\sum_{i=0}^{k} [q_i(h)]a_i \neq 0
\end{equation}
for all but finitely many $h$ unless $a_i = 0$ for all $i = 0, \ldots, k$.

**Proof:** Let $\beta_i$ be the coefficient of $q_i(n)$ so that $q_i(n) = \beta_i n$. Since $q_i(n) = q_i(n) - \sum_j \{q_j^i(n)\}\lambda_{ij}$ for some $\lambda_{ij} \in \mathbb{R}$ (see equation (1.37) in the proof of Lemma 1.2.2), we have
\begin{equation}
\frac{[q_i(h)]}{h} = \frac{q_i(h)}{h} - \frac{\{q_i(h)\}}{h}
= \frac{q_i(h)}{h} - \sum_j \frac{\{q_j^i(h)\}\lambda_{ij}}{h} - \frac{\{q_i(h)\}}{h} \to \beta_i
\end{equation}
as $h \to \infty$. Suppose that
\begin{equation}
\sum_{i=0}^{k} [q_i(h)]a_i = 0
\end{equation}
for $h \in H \subset \mathbb{N}$ where $\text{card}(H) = \infty$. Then we also have $\sum_{i=0}^{k} [q_i(h)]a_i = 0$ for all $h \in H$.

By taking the limit as $h \to \infty$, we get $\sum_{i=0}^{k} a_i\beta_i = 0$. Since $\beta_0, \ldots, \beta_k$ are rationally independent, it follows that $a_0 = a_1 = \cdots = a_k = 0$.

\[\square\]

**Lemma 2.3.4** Let $q_0(n), \ldots, q_k(n)$ be generalized polynomials of degree 1 (with possibly $q_0(n) = n$) such that the coefficients of $\overline{q}_0(n), \ldots, \overline{q}_k(n)$ are rationally independent and let $R \subset \mathbb{R}$. Then
\begin{equation}
\sum_{i=0}^{k} [q_i(h)]\theta_i
\end{equation}
is rationally independent of $R$ for all but finitely many $h$ iff there exists $j$, $0 \leq j \leq k$, such that $\theta_j$ is rationally independent of $R$. 

Proof: Denote by \( \langle R, t_1, \ldots, t_l \rangle \) the set of all linear combinations over \( \mathbb{Q} \) of the elements in \( R \cup \{t_1, \ldots, t_l\} \). Clearly, if \( \theta_i \in \langle R \rangle \) for all \( i \), then \( \sum_{i=0}^k [q_i(h)] \theta_i \in \langle R \rangle \). Therefore, assume without loss of generality that \( \theta_1 \not\in \langle R \rangle \). We can construct inductively rationally independent elements

\[
\gamma_1, \ldots, \gamma_l \in \{\theta_0, \ldots, \theta_k\}
\]

by letting \( \gamma_1 = \theta_1 \) and \( \gamma_i = \theta_{k_i} \) where \( k_i = \min\{s > k_{i-1} \mid \theta_s \not\in \langle R, \theta_1, \ldots, \theta_{k_{i-1}} \rangle \} \).

Hence there are \( r_i \in \langle R \rangle \cup \{0\} \) and \( a_{ij} \in \mathbb{Q} \) so that \( \theta_i = r_i + \sum_{j=1}^l a_{ij} \gamma_j \) and

\[
\sum_{i=0}^k [q_i(h)] \theta_i = \sum_{i=0}^k [q_i(h)] \left( r_i + \sum_{j=1}^l a_{ij} \gamma_j \right)
= r + \sum_{j=1}^l \left( \sum_{i=0}^k [q_i(h)] a_{ij} \right) \gamma_j
\]

where \( r \in \langle R \rangle \). Therefore \( \sum_{i=0}^k [q_i(h)] \theta_i \not\in \langle R \rangle \) if there is at least one \( j \) such that \( \sum_{i=0}^k [q_i(h)] a_{ij} \neq 0 \). However, this is true by Lemma 2.3.3, because there is at least one \( a_{ij} \neq 0 \).

\[\square\]

Lemma 2.3.5 If \( q(n) \) is a simple generalized polynomial such that \( \pi(n) \) is not a polynomial and \( f_l(n) = V_{h_l} \circ \cdots \circ V_{h_1} q(n) \), \( l \in \{0, \ldots, \deg(q) - 1\} \), then \( \overline{f}_l(n) \) is not a polynomial either unless \( \deg(f_l) = 1 \) in which case \( \overline{f}_l(n) \) has an integer coefficient \( [u(h_l)] \), where \( u(n) \) is a subpolynomial of \( f_l(n) \) of degree 1.

Proof: The proof goes by induction on \( l \). The case \( l = 0 \) is trivial. Assume the statement is true for \( l - 1 \). We will show that \( f_l(n) \), where \( \deg(f_l) \geq 1 \), has the
desired properties. Let \( r(n) \) be a simple term of \( f_{l-1}(n) \) such that \( \tau(n) \) is not a polynomial. Since \( \deg(f_{l-1}) \geq 2 \), \( r(n) = [u(n)]v(n) \) for some generalized polynomials \( u(n) \) and \( v(n) \) of positive degrees. Let \( V_h u(n) = \sum_i u_i(n) \). Then

\[
V_h r(n) = [u(n)]v(n) + [u(n)]V_h v(n)
\]

which is a sum of simple terms of \( f_l(n) \). If \( \deg(r) \geq 3 \), then either \( \deg(u) > 1 \), in which case \( \deg(u_i) \geq 1 \), or \( \deg(v) > 1 \), in which case \( \deg(V_h v) \geq 1 \). Hence \( V_h r(n) \) is not a polynomial if \( \deg(r) \geq 3 \). If \( \deg(r) = 2 \), then \( \deg(u) = \deg(v) = 1 \) and \( u(n) = [\alpha] \cdot [\lambda_1 \cdot \lambda_k] \) for some \( \alpha, \lambda_1, \ldots, \lambda_k \in \mathbb{R} \). Therefore \( [V_h u(n)] = [\alpha] \cdot [\lambda_1 \cdot \lambda_k] = [u(h)] \) is an integer coefficient of \( V_h r(n) \).

Proof of Theorem 2.3.1:

In order to prove that

\[
q(n) = p_0(n) + \sum_{i=1}^{k} [g_i(n)]p_i(n)
\]

is uniformly distributed (mod 1), we write, by Lemma 2.1.2,

\[
q(n) = \sum_{i=1}^{k} Q_i(n) + \sum_{i,j} 1_A(r_{1i}, r_{2i})t_{ij}(n)
\]

where the \( Q_i(n) \)'s are the simple terms of \( q(n) \), each \( t_{ij}(n) \) is an outer subpolynomial of \( \sum_{i=1}^{k} Q_i(n) \) and each \( r_{ji}(n) \) is a sum of simple induced inner subpolynomials of \( \sum_{i=1}^{k} Q_i(n) \), some which may be multiplied by indicator functions. Note that \( \deg(t_{ij}) < \deg(q) \).

Let \( v_1(n), \ldots, v_{m_1}(n) \) be all the distinct simple generalized polynomials in the expressions for the \( r_{ji}(n) \)'s, and let \( w_1(n), \ldots, w_{m_2}(n) \) be all the distinct simple gen-
eralized polynomials from the set \( \{v_i(n), v_j(n), Q_i^l(n), t_{ij}^l(n) \mid i, j, l \} \). Then by Corollary 2.1.3, \( q(n) \) is uniformly distributed (mod 1) if for all \( \epsilon_i \in \{0, 1\} \),

\[
\left( \sum_{i=1}^{k_1} Q_i(n) + \sum_{i,j} \epsilon_i t_{ij}(n), w_1(n), \ldots, w_{m_2}(n) \right)
\]

is uniformly distributed (mod 1) in \( \mathbb{R}^{m_2+1} \). Note that all the \( w_i(n) \)'s are induced inner subpolynomials of \( \sum_{i=1}^{k} [q_i(n)] \), and that \( \sum_{i=1}^{k} [q_i(n)] \) has independent coefficients except that the outer coefficients equal 1. Proposition 2.2.4 can be applied to these induced inner subpolynomials, because if we let \( \lambda_i \in \mathbb{R} \) so that \( R(Q) \cup \{1\} \) is rationally independent, where \( Q(n) = \sum_{i=1}^{k} [q_i(n)] \lambda_i \), then \( w_i(n) \in U(Q) \) for each \( i \). Hence, \( (w_1(n), \ldots, w_{m_2}(n)) \) is uniformly distributed (mod 1). So by Theorem 1.2.1, (2.75) is uniform distribution (mod 1) if for all \( a_i \in \mathbb{Z}, a_0 \neq 0 \),

\[
a_0 \left( \sum_{i} Q_i(n) + \sum_{i,j} \epsilon_i t_{ij}(n) \right) + \sum_{i \geq 1} a_i w_i(n)
\]

is uniformly distributed (mod 1). It is therefore enough to show that for any \( a_i \in \mathbb{Z} \), where \( a_0 \neq 0 \), and for any subpolynomial \( u_i(n) \) of \( \sum_{j=1}^{k_1} Q_j(n), i = 1, \ldots, k_2 \), such that \( u_i(n) \neq Q_j(n) \) for all \( i, j \), the sequence

\[
a_0^{k_3} \sum_{i=1}^{k_1} Q_i(n) + \sum_{i=1}^{k_2} a_i w_i(n)
\]

is uniformly distributed (mod 1).

We will prove a more general statement:

Let \( A_l(q) \) be the set of \( u(n) \in \bigcup_{j=1}^{k_1} U(Q_j) \) such that \( \deg(u) \leq \deg(q) - l \) and if \( \deg(u) = \deg(q) - l \) then \( u(n) \in \bigcup_{j=1}^{k} U'(q_j) \), where \( U'(q_j) \) is the subset of \( U(q_j) \) which excludes all generalized polynomials having outer coefficient 1. For each \( l \in \{0, 1, \ldots, \deg(q) - 2\} \),
let $F_l$ be the set of generalized polynomials

$$f(n) = a_0 \left( \sum_{i=1}^{k_1} V_{h_i} \circ \cdots \circ V_{h_1} Q_i(n) + \sum_i V_{h_i} \circ \cdots \circ V_{h_{i+1}} t_i(n) \right) + \sum_{i \geq 1} a_i \overline{u}_i(n) \quad (2.78)$$

where $h_i \in \mathbb{N}, i = 1, \ldots, l$, $a_i \in \mathbb{Z}, a_0 \neq 0$, $t_i(n)$ is a reduced term of $V_{h_i} \circ \cdots \circ V_{h_1} Q_j(n)$ for some $j$, $1 \leq l_i \leq l$, and $u_i(n) \in A_l(q)$. Note that $F_0$ contains all generalized polynomials of form (2.77). Observe also that each $f(n) \in F_l$ has degree $\text{deg}(q) - l$, which we will denote by $\text{deg}(F_l)$. We will use induction on $\text{deg}(F_l)$ to show that any $f(n) \in F_l$ is uniformly distributed (mod 1).

First, let $f(n) \in F_l$, where $\text{deg}(F_l) = 2$, i.e., $l = \text{deg}(q) - 2$. By bringing out integer coefficients from the brackets,

$$a_0 \sum_{i=1}^{k_2} V_{h_i} \circ \cdots \circ V_{h_1} Q_i(n) + a_0 \sum_i V_{h_i} \circ \cdots \circ V_{h_{i+1}} t_i(n) \quad (2.79)$$

can be written as a linear combination, $\sum_{i=1}^{k_2} d_i v_i(n)$, over $\mathbb{Z}$ of distinct simple reduced terms $v_1(n), \ldots, v_{k_2}(n)$, and a sum of indicator functions multiplied by outer subpolynomials of the generalized polynomial (2.79). When reduced, the outer subpolynomials of (2.79) are members of $A_l(q)$. The simple reduced terms of the arguments of the indicator functions are also in $A_l(q)$. Hence, $f(n)$ is uniformly distributed (mod 1) if $g(n) = c_0 \sum_{i=1}^{k_2} d_i v_i(n) + \sum_i c_i w_i(n) \quad (2.80)$
is uniformly distributed (mod 1) for any $w_i(n) \in A_l(q)$ and $c_i \in \mathbb{Z}, c_0 \neq 0$. We will use van der Corput’s difference theorem to show that $g(n)$ is uniformly distributed (mod 1). By Lemma 2.2.2,

$$g^h(n) = g(n + h) - g(n) = c_0 \sum_{i=1}^{k_2} d_i V_{h_i} v_i(n) + \sum_{i \geq 1} c_i V_{h_i} w_i(n) + \sum_i 1_{A(*)} s_i(n). \quad (2.81)$$
Note that if \( V_h w_i(n) \) is not a constant, then \( R(w_i) \subset R \). Therefore, since the coefficients of the reduced simple terms of the \( s_i(n) \)'s are independent of \( h \), and the coefficients of the simple reduced terms of the arguments of the indicator functions are in \( R \), it is enough to prove that the coefficient \( \sigma(h) \) of \( \sum_{i=1}^{k_2} d_i V_h v_i(n) \) is rationally independent of \( R \) for all but finitely many \( h \). We can write

\[
\sigma(h) = h\theta_0 + \sum_{i=1}^{m} [\phi_i(h)]\theta_i, \tag{2.82}
\]

where \( \phi_1(n), \ldots, \phi_m(n) \) are distinct generalized polynomials of degree 1, and the coefficients of \( \overline{\phi_1(n)}, \ldots, \overline{\phi_m(n)} \) are rationally independent and contained in \( R \). Furthermore, \( \theta_i \in R([q_{j_i}\gamma_{j_i}]) \) for some \( j_i \) such that \( \gamma_{j_i} \) is a factor of \( \theta_i \), \( i = 1, \ldots, m \), where \( \gamma_i \) is the leading coefficient of \( p_i(n) \). Since \([q_{j_0}(n)]p_{j_0}(n)\) has a term \( a(n) \) with \( \deg(a) = \deg(q) \) and so that \( \pi(n) \) is not a polynomial, it follows from Lemma 2.3.5 that \( \gamma_{j_0} \) is factor of some \( \theta_i \), say \( \theta_{j_0} \). So by Lemma 2.3.4, \( \sigma(h) \) is rationally independent of \( R \) for all but finitely many \( h \), which was to be proved.

Assume \( f(n) \in F_l \) is uniformly distributed \((\mod 1)\) if \( \deg(F_l) < K \), for some \( K \geq 3 \), and let \( f(n) \in F_l \), where \( l = \deg(q) - K \), i.e. \( \deg(f) = K \). Let \( f^h(n) = f(n + h) - f(n) \). Then by Lemma 2.2.2,

\[
f^h(n) = a_0 \sum_{i=1}^{k_1} V_h \circ V_{h_i} \circ \cdots \circ V_{h_i} Q_i(n) + a_0 \sum_{i} V_h \circ V_{h_i} \circ \cdots \circ V_{h_i} t_i(n) + \sum_{i \geq 1} a_i e_{ij}(h)u_{ij}(n) + \sum_{i} b_i(h)s_i(n) + \sum_{i,j} 1_{B_i}(*)d_{ij}(h)t_{ij}(n), \tag{2.83}
\]

where \( b_i(h), d_{ij}(h), e_{ij}(h) \in \mathbb{Z} \), each \( u_{ij}(n) \) is a reduced term of \( V_h \pi_i(n) \), and \( s_i(n), t_{ij}(n) \) are simple, reduced subpolynomials of some \( V_{h_i'} \circ \cdots \circ V_{h_i} \circ \cdots \circ V_{h_i} Q_i(n) \) or \( V_{h_i'} \circ \cdots \circ V_{h_i} \pi_i(n) \). Here we have used that any simple, reduced subpolyno-
mial of \( V_{h_i'} \circ \cdots \circ V_{h_1' \circ \cdots \circ V_{h_1} \ell_i(n)} \) is a simple, reduced subpolynomial of \( V_{h_i'} \circ \cdots \circ V_{h_1' \circ \cdots \circ V_{h_1} Q_i(n)} \), by Lemma 2.2.3. Each simple, reduced generalized polynomial \( r_i(n) \) appearing in the argument of an indicator function is a reduced inner subpolynomial of either \( V_h \circ V_{h_i'} \circ \cdots \circ V_{h_1} Q_i(n) \), \( u_{ij}(n) \), \( s_i(n) \) or \( t_{ij}(n) \). Also, all the generalized polynomials involved have degree less than \( \deg(f) \) since \( f(n) = f(n) \).

Let \( w_1(n), \ldots, w_{m_1}(n) \) be all the distinct non-constant terms from the set of all simple, reduced terms of \( (V_h \circ V_{h_i'} \circ \cdots \circ V_{h_1} Q_i(n))^j(n) \) and of \( (V_h \circ V_{h_i'} \circ \cdots \circ V_{h_1} \ell_i(n))^j(n) \), all \( j \), and the terms \( u_{ij}^1(n), s_i^1(n), t_{ij}^1(n), r_i(n) \) and \( r_i^1(n) \). By Proposition 2.2.4,

\[ (w_1(n), \ldots, w_{m_1}(n)) \] is uniformly distributed (mod 1). So by Corollary 2.1.3(ii) and Theorem 1.2.1, since \( \overline{V_h q} = \overline{V_h q} \) for any generalized polynomial \( q(n) \), \( f^h(n) \) is uniformly distributed (mod 1) if

\[
c_0 \left( a_0 \sum_{i=1}^{k_1} V_h \circ V_{h_i} \circ \cdots \circ V_{h_1} Q_i(n) + a_0 \sum_{i} V_h \circ V_{h_i} \circ \cdots \circ V_{h_1} \ell_i(n) \right) + \sum_{i,j} a_i \epsilon_{ij}(h) \overline{u_{ij}(n)} + \sum_i b_i(h) \overline{s_i(n)} + \sum_{i,j} \epsilon_i d_{ij}(h) \overline{t_{ij}(n)} + \sum_{i \geq 1} c_i \overline{w_i(n)} \tag{2.84} \]

is uniformly distributed (mod 1) for all \( \epsilon_i \in \{0,1\} \) and any \( c_i \in \mathbb{Z} \), \( c_0 \neq 0 \). Now, each \( u_{ij}(n), s_i(n), t_{ij}(n), w_i(n) \) is a reduced subpolynomial of some \( V_{h_i'} \circ \cdots \circ V_{h_1} \circ V_{h_i} Q_i(n) \). By Lemma 2.2.3, there exist corresponding \( \overline{u_{ij}(n)}, \overline{s_i(n)}, \overline{t_{ij}(n)} \), \( \overline{w_i(n)} \) which are reduced subpolynomials of the \( V_{h_{i+1}'} \circ \cdots \circ V_{h_1} Q_i(n) \)’s such that

\[
\overline{u_{ij}(n)} = \overline{u_{ij}(n)}, \overline{s_i(n)} = \overline{s_i(n)}, \overline{t_{ij}(n)} = \overline{t_{ij}(n)}, \overline{w_i(n)} = \overline{w_i(n)} \tag{2.85}
\]

and let \( V_1 \) be the set of all generalized polynomials in \( V \) which is either an element of \( \bigcup_{i=1}^{k} U(q_i) \) or has degree strictly less than \( \deg(f) - 1 \), i.e., \( V_1 = V \cap A_{i+1}(q) \). Note that
all $\tilde{u}_{ij}(n), \tilde{s}_i(n), \tilde{w}_i(n) \in V_1$. Denote by $v_1(n), \ldots, v_{k_2}(n)$ all the distinct elements of $V_1$.

If $\tilde{t}_{ij}(n) \in V \setminus V_1$, then $t_{ij}(n)$ is a reduced subpolynomial of some $V_h \circ V_{h_1} \circ \cdots \circ V_{h_i} Q_{i_1}(n)$ and since $\deg(t_{ij}) = \deg(f) - 1$, it follows from Lemma 2.2.2 that $t_{ij}(n)$ equals a reduced term of $V_h \circ V_{h_1} \circ \cdots \circ V_{h_i} Q_{i_1}(n)$ and that $d_{ij}(h) = 1$.

Hence we can write (2.84) as

$$g_{d,h}(n) = a_0 c_0 \left( \sum_{i=1}^{k_1} V_h \circ V_{h_1} \circ \cdots \circ V_{h_i} Q_{i_1}(n) + \sum_{i} V_h \circ V_{h_i} \circ \cdots \circ V_{h_i} t_{ij}(n) \right)$$

$$+ \sum_{t_{ij} \notin V_1} \epsilon_i \tilde{t}_{ij}(n) \right) + \sum_{i=1}^{k_2} d_i \pi_i(n)$$

(2.86)

which for each $h \in \mathbb{N}$, each $d = (d_1, \ldots, d_{k_2}) \in \mathbb{Z}^{k_2}$ and any $\epsilon_i \in \{0, 1\}$ lies in $F_{l+1}$, and hence is uniformly distributed (mod 1) by the induction hypothesis. Therefore $f(n)$ is uniformly distributed (mod 1) by the van der Corput’s difference theorem.
CHAPTER III

Some special results

The results in the previous chapter concern only generalized polynomials having independent inner coefficients. In this chapter we classify all simple generalized polynomials $q(n)$ of degree two and three for which $q(n) = \overline{q}(n)$, by using van der Corput’s difference theorem and treating each dependence relation separately. We also show in Section 3.3 that for any $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$, $k \geq 3$, the generalized polynomial $\lfloor \alpha_1n \rfloor \cdots \lfloor \alpha_kn \rfloor \gamma$ is uniformly distributed (mod 1) if $\gamma$ is irrational.

3.1 Some classes of generalized polynomials of degree two

In this section we confine ourselves to (sums of) generalized polynomials of degree two and we show that many generalized polynomials having dependent coefficients are uniformly distributed (mod 1), but there are also some which are not.

**Lemma 3.1.1** Let $1, \alpha_1, \ldots, \alpha_k$ be rationally independent and

$$q(n) = \sum_{i=1}^{k} [\alpha_i n] \gamma_i n + \alpha_0 n^2 + \beta n.$$  \hfill (3.1)
Then \( q(n) \) is uniformly distributed (mod 1) if either \( 2\alpha_0 + \sum_{i=1}^k \alpha_i \gamma_i \) or some \( \gamma_i \), is rationally independent of \( 1, \alpha_1, \ldots, \alpha_k \).

**Proof:** Since

\[
q^h(n) \leftrightarrow \left( \sum_{i=1}^k \lfloor \alpha_i n \rfloor \gamma_i h + \lfloor \alpha_i h \rfloor \gamma_i n + 2\alpha_0 h n + \beta h, \alpha_1 n, \ldots, \alpha_k n \right)
\]

it follows from the van der Corput’s difference theorem that \( q(n) \) is uniformly distributed (mod 1) if \( \sum_{i=1}^k \lfloor \alpha_i h \rfloor \gamma_i n + \left( \sum_{i=1}^k \alpha_i \gamma_i + 2\alpha_0 \right) h n, \alpha_1 n, \ldots, \alpha_k n \). By Lemma 2.3.4, this is so if either \( 2\alpha_0 + \sum_{i=1}^k \alpha_i \gamma_i \) or some \( \gamma_i \), is rationally independent of \( 1, \alpha_1, \ldots, \alpha_k \).

\[\square\]

The identity

\[
[a]b + [b]a = ab + [a][b] - \{a\}\{b\},
\]

where \( a, b \) are real numbers, will be used to prove the following.

**Proposition 3.1.2** Let \( 1, \alpha_1, \ldots, \alpha_k \) be rationally independent. Then

\[
q(n) = \sum_{i=1}^k [\alpha_i n] \beta_i n + \alpha_0 n^2
\]

is uniformly distributed (mod 1) if and only if one of the following conditions holds:

(i) There exists \( i \) such that \( \beta_i \) is rationally independent of \( 1, \alpha_1, \ldots, \alpha_k \).

(ii) \( \beta_i = a_{i0} + \sum_{j=1}^k a_{ij} \alpha_j \), \( a_{ij} \in \mathbb{Q} \), \( i = 1, \ldots, k \), and there exist \( i, j \) such that \( a_{ij} \neq a_{ji} \).
(iii) \( \beta_i = a_{i0} + \sum_{j=1}^{k} a_{ij} \alpha_j, \ a_{ij} \in \mathbb{Q}, \ i = 1, \ldots, k, \ a_{ij} = a_{ji} \) for all \( i, j \) and
\[
\sum_{i=1}^{k} \sum_{j=1}^{i-1} a_{ij} \alpha_i \alpha_j + \frac{1}{2} \sum_{i=1}^{k} a_{ii} \alpha_i^2 + \alpha_0 \not\in \mathbb{Q}.
\]

**Proof:** It follows from Lemma 3.1.1 that \( q(n) \) is uniformly distributed (mod 1) if (i) holds. Suppose that \( \beta_i = \sum_{j=1}^{k} a_{ij} \alpha_j, \ i = 1, \ldots, k. \) Let \( b \in \mathbb{N} \) be such that \( ba_{ij} \in \mathbb{Z} \) and \( ba_{ii} \in 2\mathbb{Z} \) for all \( i, j. \) Then by using identity (3.3) we have
\[
bq(n) = \sum_{i=1}^{k} \sum_{j=1}^{k} ba_{ij} [\alpha_i n] \alpha_j n + b\alpha_0 n^2
\]
\[
\equiv \sum_{i<j} ba_{ij} [\alpha_i n] \alpha_j n - \sum_{i>j} ba_{ij} [\alpha_i n] \alpha_i n + \sum_{i=1}^{k} ba_{ii} [\alpha_i n] \alpha_i n
\]
\[
+ \sum_{i>j} ba_{ij} \alpha_i \alpha_j n^2 - \sum_{i>j} ba_{ij} \{\alpha_i n\} \{\alpha_j n\} + b\alpha_0 n^2 \pmod{1}
\]
\[
= \sum_{i<j} b(a_{ij} - a_{ji})[\alpha_i n] \alpha_j n + \sum_{i>j} ba_{ij} \alpha_i \alpha_j n^2 + \frac{1}{2} b \sum_{i=1}^{k} a_{ii} \alpha_i^2 n^2 + b\alpha_0 n^2
\]
\[
- \sum_{i>j} ba_{ij} \{\alpha_i n\} \{\alpha_j n\} - \frac{1}{2} b \sum_{i=1}^{k} a_{ii} \{\alpha_i n\}^2 \pmod{1}.
\]

The generalized polynomial \( bq(n) \) is therefore uniformly distributed (mod 1) by Lemma 3.1.1 if there exist \( i, j \) such that \( a_{ij} \neq a_{ji}. \) If that is not the case, then \( bq(n) \) behaves like the polynomial \( b\left(\sum_{i>j} a_{ij} \alpha_i \alpha_j + \frac{1}{2} \sum_{i=1}^{k} a_{ii} \alpha_i^2 + \alpha_0\right)n^2 \) which is uniformly distributed (mod 1) if its coefficient is irrational. Note that \( b^2q(n) \) has these same properties. In order to show that \( q(n) \) is uniformly distributed (mod 1) if (ii) or (iii) is satisfied, write for each \( n, \ n = mb + r \) where \( 0 \leq r < b. \) By Lemma 1.2.5 we have
\[
[\alpha_i(bm + r)] = b[\alpha_i m] + [\alpha_i r] + 1_A(b\alpha_i m, \alpha_i r) + \sum_{l=0}^{b-1} l I_{\left[\frac{1}{b}, \frac{b+1}{b}\right]}(\alpha_i m)
\]
so that
\[
q_r(m) = q(bm + r) = \sum_{i=1}^{k} \sum_{j=1}^{k} a_{ij} b^2[\alpha_i m] \alpha_j m + \alpha_0 b^2 m^2 + \phi_r(m) + I_r(m)
\]
where $\phi_r(m)$ is a generalized polynomial of degree 1 and $I_r(m)$ is a sum of generalized polynomials multiplied by indicator functions whose arguments are constants and generalized polynomials of degree 1. It follows from Corollary 2.1.3 that $q_r(m)$ is uniformly distributed (mod 1) if for certain linear polynomials $p_r(m)$, $(b^2q(m) + p_r(m), \alpha_1m, \ldots, \alpha_km)$ is uniformly distributed (mod 1). Since $(\alpha_1m, \ldots, \alpha_km)$ is uniformly distributed (mod 1) and we have already shown by using Lemma 3.1.1 that $b^2q(m)$ is uniformly distributed (mod 1), it follows from the same lemma that $q_r(m)$ is uniformly distributed (mod 1) for each $r$. Hence, by letting $M = \lfloor N/b \rfloor$, we have for each Riemann-integrable function $f(x)$ on $[0,1]$,

$$
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(q(n)) = \lim_{N \to \infty} \frac{1}{N} \sum_{r=0}^{b-1} \frac{\lfloor N/b \rfloor}{N} \lim_{M \to \infty} \frac{1}{M} \sum_{m=0}^{M-1} f(q_r(m)) = \sum_{r=0}^{b-1} \frac{1}{b} \int_0^1 f(x)dx = \int_0^1 f(x)dx
$$

(3.8)

which shows that $q(n)$ is uniformly distributed (mod 1).

If neither (i), (ii) nor (iii) holds, then $\beta_i = \sum_{j=1}^{k} a_{ij} \alpha_j, a_{ij} \in \mathbb{Q}, i = 1, \ldots, k, a_{ij} = a_{ji}$ for all $i, j$ and $\sigma = \sum_{i=1}^{k} \sum_{j=1}^{i-1} a_{ij} \alpha_i \alpha_j + \frac{1}{2} \sum_{i=1}^{k} a_{ii} \alpha_i^2 + \alpha_0 \in \mathbb{Q}$. Let $b \in \mathbb{N}$ be such that $ba_{ij} \in \mathbb{Z}$ and $b\sigma \in \mathbb{Z}$. Then it follows from (3.5) that

$$
2bq(n) \equiv -\left(2 \sum_{i>j} ba_{ij} \{\alpha_in\} \{\alpha_jn\} + \sum_{i=1}^{k} ba_{ii} \{\alpha_in\}^2 \right) \pmod{1}
$$

(3.9)

which is not uniformly distributed (mod 1). By Weyl’s criterion for uniform distribution (see Theorem 1.2.1), $2bq(n)$ is uniformly distributed (mod 1) if $q(n)$ is uniformly distributed (mod 1). Hence, $q(n)$ is not uniformly distributed (mod 1).
This gives us for example that

\[ [\alpha n] \alpha n - \alpha^2 n^2 \] (3.10)

is uniformly distributed (mod 1) if and only if \( \alpha^2 \not\in \mathbb{Q} \). Note however, that it follows from Proposition 3.1.2 that \(2[\alpha n] \alpha n - \alpha^2 n^2\) is not uniformly distributed (mod 1) for any \( \alpha \).

**Remark:**

We showed in the proof of Proposition 3.1.2 that the generalized polynomial \( q(n) \) of degree two is uniformly distributed (mod 1) if and only if \( bq(n) \) is uniformly distributed (mod 1), \( b \in \mathbb{N} \). The same technique can be used for any generalized polynomial \( q(n) \). Therefore, it is enough to prove that \( bq(n) \) is uniformly distributed (mod 1) for some \( b \in \mathbb{N} \) in order to prove that \( q(n) \) is uniformly distributed (mod 1). Furthermore, by the identity (ii) of Lemma 1.2.5 we may assume without loss of generality that all the rational numbers appearing in relations between the coefficients of \( q(n) \) are integers. Therefore, in the rest of this chapter, we will use the following simplifications frequently.

1. We will treat rational numbers as integers. The letters \( a, b, c, d, e \) (with indices) will always denote rational numbers.

2. Rational numbers can be moved between brackets, i.e., \([aq(n)] \leftrightarrow a[q(n)]\), \( a \in \mathbb{Q} \), \( q(n) \) a generalized polynomial.

3. If \( \gamma \) is an outer coefficient of \( q(n) \) and \( \gamma = a + b\alpha \) for some irrational number \( \alpha \) and \( a, b \in \mathbb{Q} \), then we can assume \( \gamma = \alpha \).
If by some application of (2) and (3) above, a generalized polynomial $q_1(n)$ is converted into a generalized polynomial $q_2(n)$, then we sometimes will say that $q_1(n)$ reduces to $q_2(n)$.

**Proposition 3.1.3** The generalized polynomial $[\alpha n][\beta n]\gamma$ is uniformly distributed (mod 1) if and only if one of the following conditions hold.

(i) $\frac{\alpha}{\beta} \neq \sqrt{c}$ for all $c \in \mathbb{Q}^+$ and $\gamma$ is irrational.

(ii) $\frac{\alpha}{\beta} = \sqrt{c}$ for some $c \in \mathbb{Q}^+$ and $\gamma$ is rationally independent of $1, \sqrt{c}$.

By choosing special values of $\alpha, \beta$ we have

**Corollary 3.1.4** $\gamma n^2$ is uniformly distributed (mod 1) if and only if $\gamma$ is irrational.

$[\alpha n] \gamma n$ is uniformly distributed (mod 1) if and only if $\alpha^2 \notin \mathbb{Q}$ and $\gamma$ is irrational, or $\alpha^2 \in \mathbb{Q}$ and $\gamma$ is rationally independent of $1, \alpha$.

**Proof of Proposition 3.1.3:** By (3.3) we have

$$[\alpha n][\beta n] \gamma = [\alpha n] \beta \gamma n + [\beta n] \alpha \gamma n - \alpha \beta \gamma n^2 + \{\alpha n\}{\beta n}\gamma. \quad (3.11)$$

Therefore, by Proposition 3.1.2, if $1, \alpha, \beta$ are rationally independent, then $[\alpha n][\beta n] \gamma$ is uniformly distributed (mod 1) if either $\alpha \gamma$ or $\beta \gamma$ is rationally independent of $1, \alpha, \beta$, or if

$$\alpha \gamma = a_0 + a_1 \beta + a_2 \alpha \quad (3.12)$$

$$\beta \gamma = b_0 + b_1 \alpha + b_2 \beta \quad (3.13)$$
where \(a_i, b_i \in \mathbb{Q}\) and \(b_2 \neq a_2\). Suppose \(b_2 = a_2\). Then \(\alpha \beta \gamma = b_0 \alpha + b_1 \alpha^2 + b_2 \alpha \beta\) and by the identity (3.3) we have,

\[
\begin{align*}
[\alpha n][\beta n] & \equiv b_1[\alpha n] \alpha n + b_2[\alpha n] \beta n + a_1[\beta n] \beta n + a_2[\beta n] \alpha n - \alpha \beta \gamma n^2 + \{\alpha n\} \{\beta n\} \\
& \equiv \left(\frac{1}{2}b_1 \alpha^2 + \frac{1}{2}a_1 \beta^2 + a_2 \alpha \beta - (b_0 \alpha + b_1 \alpha^2 + b_2 \alpha \beta)\right)n^2 \\
& \quad - \frac{1}{2}b_1 \{\alpha n\}^2 - \frac{1}{2}a_1 \{\beta n\}^2 - \{\alpha n\} \{\beta n\} a_2 + \{\alpha n\} \{\beta n\} \gamma \\
& = \frac{1}{2}(a_1 \beta^2 - b_1 \alpha^2 - 2b_0 \alpha)n^2 \\
& \quad - \frac{1}{2}b_1 \{\alpha n\}^2 - \frac{1}{2}a_1 \{\beta n\}^2 + \{\alpha n\} \{\beta n\} (\gamma - a_2).
\end{align*}
\]

(3.14)

Since \(a_2 = b_2\) it follows from (3.12) and (3.13) that

\[
a_0 \beta + a_1 \beta^2 = b_0 \alpha + b_1 \alpha^2.
\]

(3.15)

Therefore

\[
[\alpha n][\beta n] \gamma \equiv - \frac{1}{2}(b_0 \alpha + a_0 \beta)n^2 - \frac{1}{2}b_1 \{\alpha n\}^2 - \frac{1}{2}a_1 \{\beta n\}^2 + \{\alpha n\} \{\beta n\} (\gamma - a_2),
\]

(3.16)

which is uniformly distributed (mod 1) if and only if \(a_0 \neq 0\) or \(b_0 \neq 0\). For if \(a_0 = b_0 = 0\), then

\[
[\alpha n][\beta n](\gamma - a_2) \equiv \{\alpha n\} \{\beta n\} \gamma - b_1 \{\alpha n\}^2 - a_1 \{\beta n\}^2 \pmod{1}
\]

(3.17)

which is not uniformly distributed (mod 1). It follows from (3.15) that \(a_1 \beta^2 = b_1 \alpha^2\) so that \(\frac{a}{\beta} = \sqrt{c}\) and \(\gamma = b_1 \sqrt{c} + b_2\), where \(c = \frac{a_2}{b_1}\).

Suppose \(\beta = a + b \alpha\). Then by (3.3) and Proposition 3.1.2,

\[
[\alpha n][\beta n] \gamma = a[\alpha n] \gamma n + b[\alpha n]^2 \gamma
\]

(3.18)

\[
\quad \equiv a[\alpha n] \gamma n + b(2[\alpha n] \alpha n - \alpha^2 n^2) \gamma
\]

(3.19)

\[
\quad = [\alpha n](a \gamma + 2b \alpha \gamma)n - b \alpha^2 \gamma n^2
\]

(3.20)
is uniformly distributed (mod 1) if \( a\gamma + 2b\alpha \gamma \) is rationally independent of 1, \( \alpha \) or \( a\gamma + 2b\alpha \gamma = c + d\alpha \) for some \( c, d \in \mathbb{Q} \) and \( \frac{1}{2}d\alpha^2 - b\alpha^2 \gamma \) is irrational. If that fails, then

\[
\gamma = \frac{c + d\alpha}{a + 2b\alpha} = \frac{e}{\alpha^2} + \frac{d}{2b} \tag{3.21}
\]

for some \( e \in \mathbb{Q} \). We can therefore assume \( d = 0 \). Then

\[
\frac{c}{a + 2b\alpha} = \frac{e}{\alpha^2} \quad \text{so that} \quad 1, \alpha, \alpha^2 \quad \text{are rationally dependent.}
\]

We need therefore to check the cases when

\[
\begin{align*}
\alpha &= r_1 + s_1\sqrt{c} \\
\beta &= r_2 + s_2\sqrt{c} \\
\gamma &= r_3 + s_3\sqrt{c}
\end{align*} \tag{3.22}
\]

where \( r_i, s_i \in \mathbb{Q}, \ c \in \mathbb{Q}^+ \). Without loss of generality, we may assume \( r_3 = 0 \) and \( s_3 = 1 \). Since by the identity (3.3),

\[
[\sqrt{cn}]^2\sqrt{c} = 2[\sqrt{cn}]cn - c\sqrt{cn}^2 + \{\sqrt{cn}\}^2\sqrt{c} \tag{3.23}
\]

and

\[
[\sqrt{cn}]\sqrt{cn} \equiv \frac{1}{2}cn^2 - \frac{1}{2}\{\sqrt{cn}\}^2 \pmod{1}, \tag{3.24}
\]

we have

\[
[\alpha n][\beta n]\gamma = [(r_1 + s_1\sqrt{c})n][(r_2 + s_2\sqrt{c})n]\sqrt{c} \\
= r_1r_2\sqrt{cn}^2 + (r_1s_2 + s_1r_2)[\sqrt{cn}]\sqrt{cn} + s_1s_2[\sqrt{cn}]^2\sqrt{c} \\
\equiv (r_1r_2 - s_1s_2c)\sqrt{cn}^2 + (s_1s_2\sqrt{c} - \frac{1}{2})\{\sqrt{cn}\}^2. \tag{3.25}
\]

Hence, \( q(n) \) is uniformly distributed (mod 1) if and only if \( r_1r_2 \neq s_1s_2c \). Since

\[
\frac{\alpha}{\beta} = \frac{r_1 + s_1\sqrt{c}}{r_2 + s_2\sqrt{c}} = \frac{r_1r_2 - s_1s_2c - (r_1s_2 + s_1r_2)\sqrt{c}}{r_2^2 - s_2^2c}, \tag{3.26}
\]

is uniformly distributed (mod 1) if \( a\gamma + 2b\alpha \gamma \) is rationally independent of 1, \( \alpha \) or \( a\gamma + 2b\alpha \gamma = c + d\alpha \) for some \( c, d \in \mathbb{Q} \) and \( \frac{1}{2}d\alpha^2 - b\alpha^2 \gamma \) is irrational. If that fails, then

\[
\gamma = \frac{c + d\alpha}{a + 2b\alpha} = \frac{e}{\alpha^2} + \frac{d}{2b} \tag{3.21}
\]

for some \( e \in \mathbb{Q} \). We can therefore assume \( d = 0 \). Then

\[
\frac{c}{a + 2b\alpha} = \frac{e}{\alpha^2} \quad \text{so that} \quad 1, \alpha, \alpha^2 \quad \text{are rationally dependent.}
\]

We need therefore to check the cases when

\[
\begin{align*}
\alpha &= r_1 + s_1\sqrt{c} \\
\beta &= r_2 + s_2\sqrt{c} \\
\gamma &= r_3 + s_3\sqrt{c}
\end{align*} \tag{3.22}
\]

where \( r_i, s_i \in \mathbb{Q}, \ c \in \mathbb{Q}^+ \). Without loss of generality, we may assume \( r_3 = 0 \) and \( s_3 = 1 \). Since by the identity (3.3),

\[
[\sqrt{cn}]^2\sqrt{c} = 2[\sqrt{cn}]cn - c\sqrt{cn}^2 + \{\sqrt{cn}\}^2\sqrt{c} \tag{3.23}
\]

and

\[
[\sqrt{cn}]\sqrt{cn} \equiv \frac{1}{2}cn^2 - \frac{1}{2}\{\sqrt{cn}\}^2 \pmod{1}, \tag{3.24}
\]

we have

\[
[\alpha n][\beta n]\gamma = [(r_1 + s_1\sqrt{c})n][(r_2 + s_2\sqrt{c})n]\sqrt{c} \\
= r_1r_2\sqrt{cn}^2 + (r_1s_2 + s_1r_2)[\sqrt{cn}]\sqrt{cn} + s_1s_2[\sqrt{cn}]^2\sqrt{c} \\
\equiv (r_1r_2 - s_1s_2c)\sqrt{cn}^2 + (s_1s_2\sqrt{c} - \frac{1}{2})\{\sqrt{cn}\}^2. \tag{3.25}
\]

Hence, \( q(n) \) is uniformly distributed (mod 1) if and only if \( r_1r_2 \neq s_1s_2c \). Since

\[
\frac{\alpha}{\beta} = \frac{r_1 + s_1\sqrt{c}}{r_2 + s_2\sqrt{c}} = \frac{r_1r_2 - s_1s_2c - (r_1s_2 + s_1r_2)\sqrt{c}}{r_2^2 - s_2^2c}, \tag{3.26}
\]
we have shown that \([\alpha n][\beta n][\gamma] \) fails to be uniformly distributed (mod 1) only if \(\gamma\) is rational or \(\frac{\alpha}{\beta} = \sqrt{c}\) for some \(c \in \mathbb{Q}^+\) and \(\gamma\) is rationally dependent of 1, \(\sqrt{c}\).

\[\Box\]

**Examples:**

The following generalized polynomials are uniformly distributed (mod 1).

- \([\sqrt{2}n]\sqrt{3}n\]
- \([\sqrt{2}n]\sqrt{2}n\]
- \([\sqrt{2}n][\sqrt{2}n]\sqrt{2}\]

The following generalized polynomials are *not* uniformly distributed (mod 1).

- \([\sqrt{2}n]\sqrt{2}n\]
- \([\sqrt{2}n][\sqrt{3}n]\sqrt{6}\]
- \([\sqrt{2}\pi n][\pi n]\sqrt{2}\]
- \([(1 + \sqrt{2})n][(2 + \sqrt{2})n]\sqrt{2}\]

### 3.2 A class of generalized polynomials of degree three

Any simple generalized polynomial of degree three is either of form

\[q_1(n) = [\alpha n][\beta n][\lambda n]\gamma, \quad q_2(n) = \left[[\alpha n][\beta n][\lambda n]\gamma \text{ or } \left[\cdots \left[q_i(n)\sigma_i\right]\cdots\right]\sigma_k\right],\]

where \(i = 1\) or \(i = 2\). By Theorem 3.3.1 of the next section, \([\alpha n][\beta n][\lambda n]\gamma\) is uniformly distributed
(mod 1) if and only if $\gamma$ is irrational. We will here give necessary and sufficient conditions for $\lfloor \alpha n \rfloor \beta n \lfloor \lambda n \rfloor \gamma$ to be uniformly distributed (mod 1).

The method we used to prove uniform distribution of a generalized polynomial $q(n)$ of degree two was to write first the generalized polynomial $q(n)$ in an appropriate form and then compute $q^h(n)$ in order to apply the van der Corput’s difference theorem. We will use this method for higher degree generalized polynomials as well, and we will refer to it as van der Corput’s method.

Van der Corput’s method can be used to show uniform distribution of $q(n)$ only if we can find generalized polynomials $v_1(n), \ldots, v_l(n)$ which will depend on the sub-polynomials of $q(n)$ such that $(v_1(n), \ldots, v_l(n))$ is uniformly distributed (mod 1) and $q^h(n) = V_h q(n) + \phi(v_1(n), \ldots, v_l(n))$ for some Riemann-integrable periodic mod 1 function $\phi$. Problems arise when $q(n)$ has subpolynomials which we do not know how to deal with. However, we have the following lemma.

**Lemma 3.2.1** Let $q_1(n), \ldots, q_k(n)$ be generalized polynomials of degree one and two. If $(q_1(n), \ldots, q_k(n))$ is not uniformly distributed (mod 1) then there exists generalized polynomials $v_1(n), \ldots, v_l(n)$ such that $(v_1(n), \ldots, v_l(n))$ is uniformly distributed (mod 1) and $q_i(n) = \phi_i(v_1(n), \ldots, v_l(n))$, where $\phi_i, i = 1, \ldots, k$, are Riemann-integrable periodic mod 1 functions.

**Proof:** This follows from Lemma 1.2.2, Lemma 1.2.5, the identity $\lfloor \alpha n \rfloor \lfloor \beta n \rfloor \gamma = \lfloor \alpha n \rfloor \beta \gamma n + [\beta n] n^{\alpha} - \alpha \beta n^2 + \{\alpha n\} \{\beta n\} \gamma$ and Proposition 3.1.2. \qed
Lemma 3.2.2 Let $q_1(n), q_0(n)$ be generalized polynomials such that $\deg(q_0) < \deg(q_1)$ and so that any inner subpolynomial of

$$q(n) = q_1(n) + q_0(n)$$

is of degree one or two. Then $q(n)$ is uniformly distributed (mod 1) by van der Corput’s method if $q(n)$ is uniformly distributed (mod 1) by van der Corput’s method.

Proof: The proof goes by induction on $\deg(q)$. By the previous section the lemma is true when $\deg(q) = 2$. Let $q(n)$ be a generalized polynomial of degree bigger than 2 and which we will assume is a sum of simple generalized polynomials. Assume the lemma is true for generalized polynomials of degree less than $\deg(q)$. If $q_0(n) \neq \overline{\pi}(n)$, then $q_0(n)$ has terms of degree one or two. By Lemma 1.2.2 and Lemma 3.2.1, there exist $v_1(n), \ldots, v_k(n)$ such that $v_i(n) = \overline{v}_i(n)$, $(v_1(n), \ldots, v_k(n))$ is uniformly distributed (mod 1) and $q(n) \Rightarrow (\overline{\pi}(n), v_1(n), \ldots, v_k(n))$. We will use the identity $[u(n)]\gamma n = u(n)\gamma n - [\gamma n]u(n) - \{u(n)\}\{\gamma n\}$ (mod 1) to get rid of terms of form $[u(n)]\gamma n$. Let $\tilde{q}(n)$ be the generalized polynomial obtained from $q(n)$ by replacing any term of form $[u(n)]\gamma n$, where $u(n) \neq \overline{u}(n)$ and $\deg(u) = \deg(q) - 1$, by $u(n)\gamma n - [\gamma n]u(n)$. Suppose $[u_i(n)]\gamma i n, i = 1, \ldots, l$, are all such terms. Let $w_1(n), \ldots, w_r(n)$ be generalized polynomials such that $(w_1(n), \ldots, w_r(n))$ is uniformly distributed (mod 1) and for any $u(n) \in \{v_j(n), \gamma_i n, u_i(n) \mid j = 1, \ldots, k, i = 1, \ldots, l\}$ there exists a Riemann-integrable function $\phi$ such that $u(n) = \phi(w_1(n), \ldots, w_r(n))$. Then $q(n) \Rightarrow (\tilde{q}(n), w_1(n), \ldots, w_r(n))$ and for any $b_i \in \mathbb{Q}$, $b_0\tilde{q}(n) + \sum_{i=1}^r b_iw_i(n)$ has
only inner subpolynomials of degree one and two. We may therefore assume that 
\( q(n) = q_1(n) + q_0(n) \) has the property that \( \tilde{q}_1(n) = q_1(n) \) and \( \tilde{q}_0(n) = q_0(n) \). Then by Lemma 2.2.2,

\[
q^h(n) = q^h_1(n) + q^h_0(n) = V_hq_1(n) + V_hq_0(n) + s(n) + \sum_i 1_{B_i}(*)t_i(n),
\]

(3.28)

where \( \deg(s) < \deg(V_hq_1) \), any term of \( t_i(n) \) has degree less than \( \deg(V_hq_1) \) unless it equals a reduced term of \( V_hq_1(n) \) and the arguments of the indicator functions are generalized polynomials of degree one and two. Let \( w_1(n), \ldots, w_k(n) \) be generalized polynomials such that \( (w_1(n), \ldots, w_k(n)) \) is uniformly distributed (mod 1) and such that if \( v(n) \) is a term of an argument \( (*) \), then there exist a Riemann-integrable periodic mod 1 function \( \phi \) such that \( v(n) = \phi(w_1(n), \ldots, w_k(n)) \). Since \( q(n) = \tilde{q}(n) \) and \( q_0(n) = \tilde{q}_0(n) \), we have \( \deg(w_i) < \deg(V_hq) \). So since \( \deg(V_hq) = \deg(q) - 1 \) and \( \deg(V_hq_0) = \deg(q_0) - 1 \), it follows by the induction hypothesis that \( q^h(n) \) is uniformly distributed (mod 1) if \( V_hq_1(n) + t(n) \) is uniformly distributed (mod 1) for any \( t(n) \) which is a sum of reduced terms of \( V_hq_1(n) \). However, any such \( V_hq_1(n) + t(n) \) is uniformly distributed (mod 1) since \( q_1(n) \) is uniformly distributed (mod 1) by van der Corput’s method. Hence, by van der Corput’s theorem, \( q(n) \) is uniformly distributed (mod 1).

\( \square \)

**Lemma 3.2.3** If \( \lambda_1, \ldots, \lambda_l \) are rationally independent and \( q_h(n) = \sum_{i=1}^{l} [\lambda_i h]u_i(n) \),
where $u_i(n) = \pi_i(n)$, $\deg(u_i) = \deg(q_h)$ and any inner subpolynomial of $\sum_{i=1}^{t} u_i(n)$ is of degree one or two, then $q_h(n)$ is uniformly distributed (mod 1) for all but finitely many $h$ if some $u_i(n)$ is uniformly distributed (mod 1) by van der Corput’s method.

**Terminology:** We will refer to $u_i(n)$ as the $[\lambda_i h]$-term(s) of $q_h(n)$.

**Remark:** This may fail if $u_i(n) \neq \pi_i(n)$. For let $1$, $\lambda_1$, $\lambda_2$ and $1$, $\alpha_1$, $\alpha_2$ be rationally independent, and let

$$q_h(n) = [\lambda_1 h][\alpha_1 n^2]\beta_1 + [\lambda_2 h][\alpha_2 n^2]\beta_2 = [\lambda_1 h]\alpha_1 \beta_1 n^2 + [\lambda_2 h]\alpha_2 \beta_2 n^2 - [\lambda_1 h]\{\alpha_1 n^2\}\beta_1 - [\lambda_2 h]\{\alpha_2 n^2\}\beta_2. \quad (3.29)$$

If $\alpha_2 = \alpha_1$, $\beta_1$ and $\alpha_1 = \alpha_2$, then both $[\alpha_1 n^2]\beta_1$ and $[\alpha_2 n^2]\beta_2$ are uniformly distributed (mod 1), but

$$q_h(n) = ([\lambda_1 h] - [\lambda_2 h]\beta_2)\{\alpha_2 n^2\} + ([\lambda_2 h] - [\lambda_1 h]\beta_1)\{\alpha_1 n^2\} \quad (3.30)$$

is not uniformly distributed (mod 1).

**Proof:** The case when $\deg(q_h) = 1$ follows from Lemma 2.3.3 since each $u_i(n)$ is a linear polynomial. Suppose $\deg(q_h) = 2$. Since

$$[\alpha_1 n][\alpha_2 n]\gamma\varphi[\alpha_1 n]\alpha_2 \gamma n + [\alpha_2 n]\alpha_1 \gamma n - \alpha_1 \alpha_2 \gamma n^2 \quad (3.31)$$

we may assume each $u_i(n)$ is of the form $\sum_j [\alpha_j n] \beta_j n + \alpha_0 n^2$. Furthermore, by using the relations $[an]\beta n \leftrightarrow \alpha \beta n^2 - \beta n \alpha n$ and $[an]\alpha n \leftrightarrow \frac{1}{2}\alpha^2 n^2$ we can reduce any $u_i(n)$ which is not uniformly distributed (mod 1) to a polynomial $an^2, a \in \mathbb{Q}$. Therefore we may assume that each $u_i(n)$ is uniformly distributed (mod 1). Also, if not all the
$u_i(n)$’s are polynomials, let $k \geq 1$ be the least integer such that there exist $\alpha_1, \ldots, \alpha_k$ with $1, \alpha_1, \ldots, \alpha_k$ rationally independent and

$$u_i(n) = \sum_{j=1}^{k} [\alpha_j n] \beta_{ij} + \beta_{i0} n^2,$$

(3.32)

for some $\beta_{ij} \in \mathbb{R}$. Then there exists some $\beta_{ij}$ rationally independent of $1, \alpha_1, \ldots, \alpha_k$. For otherwise $\beta_{ij} = \sum_{r=1}^{k} a_{ijr} \alpha_r + a_{ij0}$ for all $i, j$, where $a_{ijr} \in \mathbb{Q}$, so that

$$u_i(n) = \sum_{j,r} c_{ijr} [\alpha_j n] \beta_{ij} \alpha_r n + \beta_{i0} n^2$$

(3.33)

for all $i$ and some $c_{ijr} \in \mathbb{Q}$. If we rewrite all the terms containing $[\alpha_k n]$ in the way, $[\alpha_k n] \alpha_j n \leftrightarrow \alpha_j \alpha_k n^2 - [\alpha_j n] \alpha_k n$, then $\alpha_k$ is not needed any more except among the $\beta_{ij}$’s. This contradicts the minimality of $k$.

Now,

$$q_h(n) = \sum_{i=1}^{l} [\lambda_i h] \left( \sum_{j=1}^{k} [\alpha_j n] \beta_{ij} n + \beta_{i0} n^2 \right)$$

$$= \sum_{j=1}^{k} [\alpha_j n] \left( \sum_{i=1}^{l} [\lambda_i h] \beta_{ij} \right) n + \sum_{i=1}^{l} [\lambda_i h] \beta_{i0} n^2.$$  

(3.34)

So by Proposition 3.1.2 $q_h(n)$ is uniformly distributed (mod 1) if there exists $j$ such that $\sum_{i=1}^{l} [\lambda_i h] \beta_{ij}$ is rationally independent of $1, \alpha_1, \ldots, \alpha_k$. We showed above that if $q_h(n)$ is not a polynomial, there exists some $\beta_{ij}$ rationally independent of $1, \alpha_1, \ldots, \alpha_k$. So by Lemma 2.3.4, $\sum_{i=1}^{l} [\lambda_i h] \beta_{ij}$ is rationally independent of $1, \alpha_1, \ldots, \alpha_k$. If $q_h(n)$ is a polynomial, then the coefficient $\sum_{i=1}^{l} [\lambda_i h] \beta_{i0}$ is irrational by Lemma 2.3.4. This proves the degree two case.

We will prove the general statement by induction on $\text{deg}(q_h)$. Assume it is true if $\text{deg}(q_h) < d$ and let $\text{deg}(q_h) = d > 2$. If $u_i(n)$ has a term $[u_{i1}(n)] \gamma n$, where
\[ u_{11}(n) \neq u_{11}(n), \text{ then by the identity } [u_{11}(n)] \gamma n \equiv u_{11}(n) \gamma n - [\gamma n] u_{11}(n) - \{u_{11}(n)\} \{\gamma n\} \] (mod 1) and Lemma 3.2.2 we can use \([u_{11}(n)] \gamma n \equiv u_{11}(n) \gamma n - [\gamma n] u_{11}(n).\) So we may assume \(q_h(n)\) has no term of form \([u_{11}(n)]\gamma n.\) Therefore, we can find subpolynomials or generalized polynomials derived from subpolynomials of \(q_h(n), v_1(n), \ldots, v_s(n),\) of degree less than \(d - 1 = \deg(V_k u_i)\) so that by Lemma 3.2.2
\[
q_h^k(n) \equiv \left( \sum_{i=1}^{l} [\lambda_i h] V_k u_i(n), v_1(n), \ldots, v_s(n) \right) \equiv \sum_{i=1}^{l} [\lambda_i h] V_k u_i(n). \quad (3.35)
\]
Since \(q_h(n)\) has no term of form \([u_{11}(n)]\gamma n\) we have \(V_k u_i(n) = V_k u_i(n)\) and \(\deg(V_k u_i) = d - 1\) for all \(i.\) So by the induction hypothesis, \(q_h^k(n)\) is uniformly distributed (mod 1). Hence by van der Corput’s difference theorem, \(q_h(n)\) is uniformly distributed (mod 1).

\[\square\]

**Remark:** Instead of requiring that any subpolynomial of \(q(n)\) in Lemma 3.2.2 and Lemma 3.2.3 has inner subpolynomials of degree one and two only, one could require that the generalized polynomial has only *manageable* subpolynomials, i.e., whenever \(v_1(n), \ldots, v_k(n)\) are subpolynomials of \(q(n)\) and \(a_i \in \mathbb{Q},\) then there exists a Riemann-integrable periodic mod1 function \(\phi\) on \(\mathbb{R}^l\) and generalized polynomials \(u_1(n), \ldots, u_l(n)\) such that \(\left( u_1(n), \ldots, u_l(n) \right)\) is uniformly distributed (mod 1) and \(\sum_{i=1}^{k} a_i v_i(n) = \phi \left( u_1(n), \ldots, u_l(n) \right).\) Note that all subpolynomials of degree one and two are manageable, and that the subpolynomials of generalized polynomials with independent coefficients are manageable.

**Lemma 3.2.4** The generalized polynomial \(q(n) = [\alpha n] \beta n \gamma \) is not uniformly distributed (mod 1) if either (1) or (2) below take place.
\(1\) \(\alpha^3 \in \mathbb{Q}, \frac{\beta}{\alpha} = a, \ a \in \mathbb{Q} \setminus \{0\}, \ \lambda \in \mathbb{Q} \) and \(\gamma\) is rationally dependent of \(1, \alpha\).

\(2\) \(\alpha^3 \in \mathbb{Q}, \frac{\beta}{\alpha} = a\lambda, \ a \in \mathbb{Q} \setminus \{0\}, \ \lambda = b_0 + b_1\alpha, \ b_0, b_1 \in \mathbb{Q}, \ b_1 \neq 0 \) and \(\gamma\) is rationally dependent of \(1, \frac{1}{\lambda}\).

**Proof:** In case (1) we may assume that \(q(n) = \lfloor \alpha n \rfloor \alpha n\). By using the identity (3.3), we have for any real number \(a\) that

\[
3\lfloor a \rfloor a \equiv 3[a]a^2 - 3[a]^2a - 3\{[a]a\}a \pmod{1}
\]

\[
\equiv a^3 + ([a] - a)^3 - 3\{[a]a\}a \pmod{1}
\]

\[
\equiv a^3 - \{a\}^3 - 3\{[a]a\}a \pmod{1}. \quad (3.36)
\]

Therefore

\[
3\lfloor \alpha n \rfloor \alpha n \equiv \alpha^3 n^3 - \{\alpha n\}^3 - 3\{\lfloor \alpha n \rfloor \alpha n\} \{\alpha n\} \pmod{1} \quad (3.37)
\]

which is uniformly distributed \((\text{mod } 1)\) if and only if \(\alpha^3 \not\in \mathbb{Q}\).

In case (2) we will assume \(\gamma = \frac{1}{\lambda}\) and that \(\alpha^3, b_0, b_1, a \in \mathbb{Z}^+\). We will use that

\[
\lfloor \lambda n \rfloor \frac{1}{\lambda} = n - \{\lambda n\} \frac{1}{\lambda} = n - 1 - \{\lambda n\} \frac{1}{\lambda} = n - 1 - \sum_{i=0}^{\lfloor 1/\lambda \rfloor} i 1_{\lfloor \lambda i, (i+1) \lambda \rfloor} \{\lambda n\}. \quad (3.38)
\]

In order to simplify the writing, we will assume \(\lambda > 1\) so that \(\lfloor \lambda n \rfloor \frac{1}{\lambda} = n - 1\). By (3.3) we have

\[
q(n) = \lfloor \alpha n \rfloor a\alpha \lambda n \lfloor \lambda n \rfloor \frac{1}{\lambda}
\]

\[
\equiv [\alpha n][\lambda n]a \alpha n - [\lambda n] \frac{1}{\lambda} [\alpha n]a \alpha \lambda n - \{[\alpha n]a \alpha \lambda n\} \{\lambda n\} \frac{1}{\lambda} \pmod{1}
\]

\[
\equiv [\alpha n][\lfloor (b_0 + b_1\alpha) n \rfloor]a \alpha n - [\alpha n]a \alpha (b_0 + b_1\alpha) (n^2 - n)
\]

\[
- \{[\alpha n]a \alpha (b_0 + b_1\alpha) n\} \{[(b_0 + b_1\alpha) n] \frac{1}{(b_0 + b_1\alpha)} \} \pmod{1} \quad (3.39)
\]
which is not uniformly distributed \((\text{mod } 1)\).

\[ = \lfloor an \rfloor b_1 a n + \lfloor an \rfloor a n^2 - \lfloor an \rfloor b_0 a n^2 - \lfloor an \rfloor b_1 a^2 n^2 \]

\[+ \lfloor an \rfloor a \alpha (b_1 \alpha + b_0) n - \left\{ \lfloor an \rfloor a \alpha (b_0 + b_1 \alpha) n \right\} \left\{ ((b_0 + b_1 \alpha) n) \frac{1}{(b_0 + b_1 \alpha)} \right\} \]

\[= b_1 a \left( \lfloor an \rfloor^2 a n - \lfloor an \rfloor (an)^2 \right) + \sum_{i=0}^{b_1-1} i 1_{\frac{i}{b_1}, \frac{i+1}{b_1}} (an) a [an] a n \]

\[+ \lfloor an \rfloor a \alpha (b_1 \alpha + b_0) n - \left\{ \lfloor an \rfloor a \alpha (b_0 + b_1 \alpha) n \right\} \left\{ ((b_0 + b_1 \alpha) n) \frac{1}{(b_0 + b_1 \alpha)} \right\} \]

\[\equiv \frac{1}{3} b_1 a \{an\}^3 + \sum_{i=0}^{b_1-1} i 1_{\frac{i}{b_1}, \frac{i+1}{b_1}} (an) \frac{a}{2} (\alpha^2 n^2 - \{an\}^2) \]

\[+ \left\{ \lfloor an \rfloor a \alpha (b_0 + b_1 \alpha) n \right\} \left( 1 - \left\{ ((b_0 + b_1 \alpha) n) \frac{1}{(b_0 + b_1 \alpha)} \right\} \right) \pmod{1}, \quad (3.40) \]

Theorem 3.2.5  The generalized polynomial \( q(n) = \lfloor an \rfloor \beta \lfloor bn \rfloor \gamma \) is uniformly distributed \((\text{mod } 1)\) for any irrational \( \gamma \) unless (1) or (2) below take place.

(1) \( \alpha^3 \in \mathbb{Q}, \frac{\beta}{\alpha} = a, \ a \in \mathbb{Q} \setminus \{0\} \) and \( \lambda \in \mathbb{Q} \).

In this case \( q(n) \) is uniformly distributed \((\text{mod } 1)\) if and only if \( \gamma \) is rationally independent of \( 1, \alpha \).

(2) \( \alpha^3 \in \mathbb{Q}, \frac{\beta}{\alpha} = a \lambda, \ a \in \mathbb{Q} \setminus \{0\} \) and \( \lambda = b_0 + b_1 \alpha, b_0, b_1 \in \mathbb{Q}, b_1 \neq 0 \).

In this case \( q(n) \) is uniformly distributed \((\text{mod } 1)\) if and only if \( \gamma \) is rationally independent of \( 1, \frac{1}{\lambda} \).

Proof: By Lemma 3.2.4 , we need only prove uniform distribution of the generalized polynomials.
Note that if \( \beta \in \mathbb{Q} \), then \( q(n) \rightarrow [\alpha n][\lambda n] \gamma n \) which is a special case of Theorem 3.3.1. We will therefore assume \( \beta \) is irrational.

By using the identity (3.3), we have

\[
q(n) = \left[ [\alpha n] \beta n \right] \gamma n \tag{3.41}
\]

\[
\rightarrow \left[ [\alpha n] \beta n \right] \lambda \gamma n + [\lambda n] [\alpha n] \beta \gamma n - [\alpha n] \beta \lambda \gamma n^2 \tag{3.42}
\]

\[
\rightarrow [\alpha n] [\lambda n] \beta \gamma n - [\lambda \gamma n] [\alpha n] \beta n. \tag{3.43}
\]

**Case 1:** \( \lambda \in \mathbb{Q} \).

We will show that \( q(n) = \left[ [\alpha n] \beta n \right] \gamma n \) is uniformly distributed (mod 1) for any irrational number \( \gamma \) if either \( \alpha^3 \not\in \mathbb{Q} \) or if \( \alpha^3 \in \mathbb{Q} \) and \( \frac{\alpha}{\beta} \not\in \mathbb{Q} \), and that \( q(n) \) is uniformly distributed (mod 1) for \( \gamma \) rationally independent of 1, \( \alpha \) if \( \alpha^3 \in \mathbb{Q} \) and \( \alpha \beta \not\in \mathbb{Q} \).

We have

\[
q(n) = \left[ [\alpha n] \beta n \right] \gamma n + [\alpha n] \beta \gamma n^2 - [\gamma n] [\alpha n] \beta n. \tag{3.44}
\]

So

\[
q^h(n) \rightarrow [\alpha h] \left( \beta n^2 - [\gamma n] \beta n \right) - [\gamma h] [\alpha n] \beta n + h \left( 2[\alpha n] \beta n - [\gamma n] [\alpha n] \beta \right). \tag{3.45}
\]

By van der Corput’s difference theorem it is enough to show that \( q^h(n) \) is uniformly distributed (mod 1).

If \( \alpha \in \mathbb{Q}, \ [\alpha n] \beta n \) is uniformly distributed (mod 1). So since \( \gamma \) is irrational, it follows from Lemma 3.2.3 and (3.45) that \( q^h(n) \) is uniformly distributed (mod 1).

Suppose now that 1, \( \alpha, \gamma \) are rationally independent. By Lemma 3.2.3, \( q^h(n) \) is uniformly distributed (mod 1) by Proposition 3.1.3 if \( [\alpha n] \beta n \) is uniformly distributed (mod 1). Hence \( q^h(n) \) is uniformly distributed (mod 1) if either \( \alpha^2 \not\in \mathbb{Q} \) and \( \beta \) is
irrational or $\alpha^2 \in \mathbb{Q}$ and $\beta$ is rationally independent of 1, $\alpha$. If $\alpha^2 \in \mathbb{Q}$ and $\beta = a + b\alpha$, $a, b \in \mathbb{Q}$, $b \neq 0$, then the $[\alpha h]$-term of $q^h(n)$ is uniformly distributed (mod 1).

If $\gamma = a + b\alpha$ for some $a, b \in \mathbb{Q}$, where $\alpha$ is irrational, we may assume that $\gamma = \alpha$. Then

$$q^h(n) \leftrightarrow [\alpha h] \left( \alpha \beta n^2 - 2[\alpha n] \beta n \right) + h \left( 2[\alpha n] \beta \alpha n - [\alpha n]^2 \beta \right)$$

$$\leftrightarrow [\alpha h] \left( \alpha \beta n^2 - 2[\alpha n] \beta n \right) + h \alpha^2 \beta n^2, \quad (3.46)$$

since $[\alpha n]^2 \beta - 2[\alpha n] \alpha \beta n - \alpha^2 \beta n^2$. So $q^h(n)$ is uniformly distributed (mod 1) if $\beta$ is rationally independent of 1, $\alpha$. If $\beta = c + d\alpha$, $c, d \in \mathbb{Q}$, then

$$q^h(n) \leftrightarrow [\alpha h] \left( (c\alpha + d\alpha)^2 n^2 - 2d[\alpha n] \alpha n \right) + h(c\alpha^2 + d\alpha^3)n^2$$

$$\leftrightarrow [\alpha h] c\alpha n^2 + h(c\alpha^2 + d\alpha^3)n^2. \quad (3.47)$$

If $c \neq 0$ then $c\alpha n^2$ and hence $q^h(n)$ is uniformly distributed (mod 1). However, if $c = 0$, then $q^h(n)$ is uniformly distributed (mod 1) only if $\alpha^3 \notin \mathbb{Q}$.

**Case 2:** $\alpha \in \mathbb{Q}$ and $\lambda$ is irrational.

We will show that $q(n) = [\beta n^2][\lambda n]\gamma$ is uniformly distributed (mod 1) for any $\beta, \lambda \in \mathbb{R}$ and any irrational $\gamma$.

We have

$$q^h(n) \leftrightarrow [\lambda h] \left( [\beta n^2][\beta n] \gamma + 2h[\beta n][\lambda n] \gamma, \beta n^2 \right) \quad (3.48)$$

which is uniformly distributed (mod 1) by a modification of Lemma 3.2.3 if $([\beta n^2] \gamma, \beta n^2)$ is uniformly distributed (mod 1), hence if $\beta \gamma$ is rationally independent.
of 1, β. If \( \beta \gamma = a + b\beta \) for some \( a, b \in \mathbb{Q} \), then we may assume \( \gamma = \frac{1}{\beta} \), \( \beta \) irrational. Then

\[
q(n) = [\beta n^2][\lambda n] \left( \frac{1}{\beta^2} + \frac{\lambda n}{\beta} \right)n - \lambda n^3 - \left[ \frac{\lambda}{\beta^2} \right] \beta n^2
\]

(3.49)

so that \( q^h(n) - [\frac{\lambda}{\beta} h] \beta n^2 - 2h[\frac{\lambda}{\beta^2}] \beta n \). Hence \( q(n) \) is uniformly distributed (mod 1) if \( \frac{\lambda}{\beta} \) is irrational since \( \beta \) is irrational. If \( \frac{\lambda}{\beta} \in \mathbb{Q} \), then \( q(n) \) is uniformly distributed (mod 1).

\( \Box \)

**Case 3:** 1, \( \alpha, \lambda \) are rationally independent.

Since

\[
q^\lambda h(n) - [\frac{\lambda}{\beta} h] \beta n^2 - 2h[\frac{\lambda}{\beta^2}] \beta n \].

(3.50)

\( q(n) \) is uniformly distributed (mod 1) by Lemma 3.2.3 and van der Corput’s difference theorem if \( ([\alpha n] \beta \gamma n, [\alpha n] \beta n) \) is uniformly distributed (mod 1), hence if either

(i) \( \alpha^2 \not\in \mathbb{Q} \) and 1, \( \beta, \beta \gamma \) are rationally independent, i.e., \( \gamma \neq \frac{1}{\beta}(a + b\beta) \) for all \( a, b \in \mathbb{Q} \),

or

(ii) \( \alpha^2 \in \mathbb{Q} \) and \( \beta \gamma \) is rationally independent of 1, \( \alpha, \beta \), i.e., \( \gamma \neq \frac{1}{\beta}(a + b\alpha + c\beta) \) for all \( a, b, c \in \mathbb{Q} \).

So we need to check the following two cases.

(i) \( \alpha^2 \not\in \mathbb{Q} \), \( \gamma = \frac{1}{\beta} \).

(ii) \( \alpha^2 \in \mathbb{Q} \), \( \gamma = \frac{1}{\beta}(a + b\alpha), a, b \in \mathbb{Q} \).
We will use the form (3.43) of \( q(n) \). In case (i), \( q(n) \mapsto [\frac{1}{\beta} n][\alpha n] \beta n \), so by Theorem 3.3.1 \( q(n) \) is uniformly distributed (mod 1) since \( \beta \) is irrational.

In case (ii), we have

\[
q(n) \mapsto b[an][\lambda n] an - a[\frac{\lambda}{\beta} n][\alpha n] \beta n - b[\frac{\alpha \lambda}{\beta} n][\alpha n] \beta n
\]  

so that

\[
q^h(n) \mapsto [\alpha h](b[an][\lambda n] an - a[\frac{1}{\beta} n][\alpha n] \beta n - b[\frac{\alpha \lambda}{\beta} n][\alpha n] \beta n) -\frac{\lambda}{\beta} h][\alpha n] a\beta n - \frac{\alpha \lambda}{\beta} h][\alpha n] b\beta n + h(b[an][\lambda n] an - a[\frac{\lambda}{\beta} n][\alpha n] \beta - b[\frac{\alpha \lambda}{\beta} n][\alpha n] \beta).
\]  

Here we used that \([\alpha n] an \equiv -\frac{1}{2} \{\alpha n\}^2 \mapsto 0. \) Let \( \alpha = \sqrt{c} \).

If \( \beta = d + e \alpha = d + e \sqrt{c} \), then we may assume \( \gamma = \sqrt{c} \), and we have

\[
q(n) \mapsto [\sqrt{cn}][\lambda n](d + e \sqrt{c}) \sqrt{cn} - [\lambda \sqrt{cn}][\sqrt{cn}] e \sqrt{cn}
\]

\[
\Rightarrow d[\sqrt{cn}][\lambda n] \sqrt{cn} - e[\lambda \sqrt{cn}][\sqrt{cn}] \sqrt{cn}.
\]

So

\[
q^h(n) \mapsto d[\lambda h][\sqrt{cn}] \sqrt{cn} - e[\lambda \sqrt{cn}][\sqrt{cn}] \sqrt{cn} + [\sqrt{ch}](d[\lambda n] \sqrt{cn} - e[\lambda \sqrt{cn}][\sqrt{cn}])
\]

\[
+ h(d[\sqrt{cn}][\lambda n] \sqrt{c} - e[\lambda \sqrt{cn}][\sqrt{cn}] \sqrt{c})
\]

\[
\Rightarrow [\sqrt{ch}](d[\lambda n] \sqrt{cn} - e[\lambda \sqrt{cn}][\sqrt{cn}]) + h(d[\sqrt{cn}][\lambda n] \sqrt{c} - e[\lambda \sqrt{cn}][\sqrt{cn}] \sqrt{c})
\]

which is uniformly distributed (mod 1) by its \([\sqrt{ch}]\)-terms, since \( \sqrt{c} \) is rationally independent of 1, \( \lambda, \lambda \sqrt{c} \).
Now let 1, \( \sqrt{c}, \beta \) be rationally independent. Then \( \sqrt{cn} \beta n \) is uniformly distributed (mod 1). So if 1, \( \sqrt{c}, \frac{\lambda}{\beta}, \frac{\sqrt{c}\lambda}{\beta} \) are rationally independent, it follows from (3.52) that \( q(n) \) is uniformly distributed (mod 1). Suppose 1, \( \sqrt{c}, \lambda \beta, \sqrt{c}\lambda \beta \) are rationally independent and that \( \frac{\sqrt{c}\lambda}{\beta} = r + d\sqrt{c} + e\frac{\lambda}{\beta}, r, d, e \in \mathbb{Q} \). Then the \([\frac{\lambda}{\beta}h] \)-term of \( q^h(n) \) is uniformly distributed (mod 1) if \( a + be \neq 0 \). If \( a + be = 0 \), then the \([\alpha h] \)-terms of \( q^h(n) \) are

\[
\left( b[\lambda n] \sqrt{cn} - a\left( d + e\sqrt{c}\right) n \right) \beta n - b\left( r + d\sqrt{c} + e\frac{\lambda}{\beta} \right) n - bd[\sqrt{cn}] \beta n \equiv -b[\sqrt{cn}] \left( \lambda + 2d\beta \right) n + b(\sqrt{c}\lambda - r\beta)n^2 \tag{3.55} \]

which is uniformly distributed (mod 1) unless possibly when \( \lambda + 2d\beta \) is rationally dependent of 1, \( \sqrt{c} \) and \( \sqrt{c}\lambda - r\beta \) is rational. Hence \( q(n) \) is uniformly distributed (mod 1) since 1, \( \sqrt{c}, \beta \) are rationally independent. Similarly, we find that \( q(n) \) is uniformly distributed (mod 1) when 1, \( \sqrt{c}, \frac{\sqrt{c}\lambda}{\beta} \) are rationally independent and \( \frac{\lambda}{\beta} = r + d\sqrt{c} + e\frac{\sqrt{c}\lambda}{\beta} \).

If \( \frac{\lambda}{\beta} = d + e\sqrt{c} \), then \( \frac{\sqrt{c}\lambda}{\beta} = d\sqrt{c} + ec \) and the \([\alpha h] \)-terms of \( q^h(n) \) are

\[
b[\lambda n] \sqrt{cn} - a\left( d + e\sqrt{c} \right)n \beta n - b\left( (d\sqrt{c} + ec) n \right) \beta n - [\sqrt{cn}](ae + bd)\beta n \equiv -[\sqrt{cn}] \left( b\lambda + (2ae + 2bd)\beta \right)n + \left( b\lambda\sqrt{c} - (ad + bec)\beta \right)n^2 \tag{3.56} \]

which is uniformly distributed (mod 1) unless \( b\lambda + (2ae + 2bd)\beta \) is rationally dependent of 1, \( \sqrt{c} \) and \( b\lambda\sqrt{c} - (ad + bec)\beta \in \mathbb{Q} \), i.e., unless there exist \( a_i \in \mathbb{Q} \) such that \( \lambda = a_0 + a_1\sqrt{c} + a_2\beta = a_3\sqrt{c}\beta + a_4\sqrt{c} \), in which case 1, \( \alpha, \beta \) are rationally dependent, contrary to our consumption.

\[ \square \]

**Case 4:** \( \lambda = s_0 + s_1\alpha, s_0, s_1 \in \mathbb{Q}, s_1 \neq 0. \)
By using the form (3.43) of \( q(n) \), we have
\[
q(n) = r s_0[an] \beta \gamma n^2 + s_1[an]^2 \beta \gamma n - s_0[\gamma n][an] \beta n - s_1[\gamma an][an] \beta n
\] (3.57)
so that
\[
q(n) \mapsto s_0[\alpha n][\beta n^2] + s_1[\alpha n] \gamma n - s_0[\gamma n][\alpha n] \beta n - s_1[\gamma \alpha n][\alpha n] \beta n
\] (3.57)
\[
\begin{align*}
q_h(n) & \mapsto [\alpha h][2\beta \gamma n - [\gamma n]s_0 \beta n - [\alpha \gamma n]s_1 \beta n + s_0 \beta \gamma n^2) \\
& - [\gamma h][an]s_0 \beta n - [\alpha \gamma h][an]s_1 \beta n \\
& + h([an](s_0 \beta \gamma + s_1 \alpha \beta \gamma) n - [\gamma n]s_0 \alpha \beta n - [\alpha \gamma n]s_1 \alpha \beta n + s_0 \alpha \beta \gamma n^2).
\end{align*}
\] (3.58)
If \( 1, \alpha, \gamma, \alpha \gamma \) are rationally independent, then because of the \([\gamma h]-\) and \([\alpha \gamma h]-\)terms of \( q_h(n) \), \( q_h(n) \) and hence \( q(n) \) by van der Corput’s difference theorem, is uniformly distributed (mod 1) unless possibly when \( \alpha^2 \in \mathbb{Q} \) and \( \beta \) is rationally dependent of \( 1, \alpha \). We need to consider the following two cases:

(i) \( 1, \alpha, \alpha^2 \) rationally independent.

(ii) \( 1, \alpha, \alpha^2 \) rationally dependent.

(i) Since \( 1, \alpha, \alpha^2 \) are rationally independent, both \( \alpha \gamma \) and \( \gamma \) cannot be rationally dependent of \( 1, \alpha \). If \( \gamma \) is rationally dependent of \( 1, \alpha \), say \( \gamma = \alpha \), then \( q(n) \) is uniformly distributed (mod 1) because of the \([\alpha \gamma h]-\)term \([an]s_1 \beta n \) of \( q_h(n) \). So we will assume \( 1, \alpha, \gamma \) are rationally independent and that \( \alpha \gamma = a_0 + a_1 \alpha + a_2 \gamma \). We may assume \( a_1 = 0 \), so that
\[
\alpha \gamma = a_0 + a_2 \gamma.
\] (3.59)
If \( a_2 = 0 \) then by the \([\gamma h]-\)term \([an]s_0 \beta n \) of \( q_h(n) \), \( q(n) \) is uniformly distributed (mod 1) if \( s_0 \neq 0 \). If \( a_2 = 0 = s_0 \), let \( \gamma = \frac{1}{\alpha} \). Then \( q_h(n) \mapsto [an][2s_1 \frac{\beta n}{\gamma} - s_1 \beta n^2] - \)
$h s_1 \alpha \beta n^2$, which is uniformly distributed (mod 1) unless $\frac{\beta}{\alpha} = a \alpha = \frac{\alpha}{s_1} \lambda$ for some $a \in \mathbb{Q}$ and $\alpha^3 \in \mathbb{Q}$.

Now, let $a_2 \neq 0$. Then the $[\gamma h]$-term of $q^h(n)$ is $[\alpha n](a_2 s_1 + s_0) \beta n$. Therefore $q(n)$ is uniformly distributed (mod 1) if $a_2 s_1 + s_0 \neq 0$. Suppose

$$a_2 s_1 = -s_0.$$  \hfill (3.60)

Then

$$q^h(n) \Longleftrightarrow [ah][\alpha n]s_1 2 \beta \gamma n - [\gamma n]s_0 \beta n - [(a_0 + a_2 \alpha)n] s_1 \beta n + s_0 \beta \gamma n^2 \quad + h(\alpha n)(s_0 \beta \gamma + s_1 \beta (a_0 + a_2 \gamma) - a_0 s_1 \beta) n - [\gamma n] s_0 \alpha \beta n$$

$$- [(a_0 + a_2 \alpha)n] s_1 \alpha \beta n + s_0 \alpha \beta \gamma n^2 \quad \Longleftrightarrow [ah][\alpha n]2 s_1 \gamma \beta n + (s_0 \gamma - a_0 s_1) \beta n^2 \quad + h(s_0 \gamma - a_0 s_1) \alpha \beta n^2$$  \hfill (3.61)

which is uniformly distributed (mod 1) by Lemma 3.2.3 and Proposition 3.1.2 if at least one of the following conditions holds:

(a) $\beta \gamma$ is rationally independent of $1, \alpha$

(b) $\beta \gamma = b_0 + b_1 \alpha$ and $s_1 b_1 \alpha^2 + (s_0 \gamma - a_0 s_1) \beta \not\in \mathbb{Q}$.

(c) $(s_0 \gamma - a_0 s_1) \alpha \beta \not\in \mathbb{Q}$.

Suppose all three fails to hold. Then

$$\gamma = \frac{1}{\beta}(b_0 + b_1 \alpha).$$  \hfill (3.62)

By (3.59) and (3.62) we have $\gamma = \frac{1}{\beta}(b_0 + b_1 \alpha) = \frac{a_0}{\alpha - a_2}$ which implies

$$b_0 a_2 - (b_0 - b_1 a_2) \alpha + a_0 \beta = b_1 \alpha^2.$$  \hfill (3.63)
So if condition (b) is not true, then
\[ s_1(b_0a_2 - (b_0 - b_1a_2)\alpha + a_0\beta) + s_0b_1\alpha - a_0s_1\beta \in \mathbb{Q} \]
which implies, by using (3.60), that \( s_1b_0\alpha \in \mathbb{Q} \), hence that \( b_0 = 0 \). So \( \gamma = b_1\frac{\alpha}{\beta} \). Let \( b_1 = 1 \). Since we assume (c) fails to hold, we have \( s_0\alpha^2 - a_0s_1\alpha \beta \in \mathbb{Q} \), which implies that \( 1, \alpha, \beta, \alpha \beta \) are rationally dependent, say
\[ \alpha\beta = r_0 + r_1\alpha + r_2\beta. \tag{3.64} \]
Since \( 1, \alpha, \alpha^2 \) are rationally independent and \( \alpha \beta \) is rationally dependent of \( 1, \alpha, \beta \), the numbers \( 1, \alpha, \beta \) must be rationally independent. Therefore
\[ s_0(a_2\alpha + a_0\beta) - a_0s_1(r_0 + r_1\alpha + r_2\beta) = (s_0a_2 - a_0s_1r_1)\alpha + a_0(s_0 - s_1r_2)\beta \in \mathbb{Q} \]
implies that \( s_0a_2 = a_0s_1r_1 \) and \( s_0 = s_1r_2 \). Hence
\[ \frac{s_0}{s_1} = \frac{a_0r_1}{a_2} = r_2. \tag{3.65} \]
From (3.60) we have \( \frac{s_0}{s_1} = -a_2 \). So \( r_2 = -a_2 \) and \( s_0 = s_1r_2 \) so that \( \lambda = s_1(r_2 + \alpha) \) and
\[ \gamma = \frac{a_0}{\alpha + r_2} = \frac{a_0s_1}{\lambda}. \tag{3.66} \]
Since \( \gamma = \frac{\alpha}{\beta} \), we have \( \frac{\beta}{\alpha} = (a_0s_1)^{-1}\lambda \). Also, from (3.64) and (3.66), we have \( \alpha = \beta\gamma = \frac{r_0 + r_1\alpha}{\alpha + r_2} \frac{a_0}{\alpha + r_2} \), which together with (3.65) and \( r_2 = -a_2 \) gives \( \alpha^3 = r_0a_0 \in \mathbb{Q} \). Hence when \( 1, \alpha, \alpha^2 \) are rationally independent, \( q(n) \) can fail to be uniformly distributed (mod 1) only if \( \alpha^3 \in \mathbb{Q} \) and there exist \( b_0, b_1, a \in \mathbb{Q} \), \( b_1 \neq 0 \) such that \( \lambda = b_0 + b_1\alpha \), \( \frac{\beta}{\alpha} = a\lambda \) and \( \gamma \) is rationally dependent of \( 1, \frac{1}{\lambda} \).

(ii) \( 1, \alpha, \alpha^2 \) rationally dependent.

We have already seen from (3.58) that \( q(n) \) is uniformly distributed (mod 1) when \( 1, \alpha, \gamma, \alpha\gamma \) are rationally independent and either \( \alpha^2 \notin \mathbb{Q} \) or \( \alpha^2 \in \mathbb{Q} \) and \( \beta \) are rationally independent of \( 1, \alpha \). The following cases are left to check:
(a) \( \alpha = a + b\sqrt{c}, a \neq 0, \gamma = \sqrt{c} \) and \( \beta \) is rationally independent of \( 1, \sqrt{c} \).

(b) \( \alpha = \sqrt{c}, \beta = a + b\sqrt{c} \) and \( \gamma \) is rationally independent of \( 1, \sqrt{c} \).

(c) \( \alpha, \beta, \gamma \) are rationally dependent of \( 1, \sqrt{c} \).

(a) In this case both \( \gamma \) and \( \alpha\gamma \) are rationally dependent of \( 1, \sqrt{c} \). Therefore, by (3.58) the \( \sqrt{ch} \)-terms of \( q^h(n) \) are

\[ [\sqrt{cn}]b(2bs_1\beta\sqrt{c}-2(s_0+as_1)\beta)\gamma n + (2abs_1+s_0b)\beta\sqrt{c}-(b^2cs_1+as_0+a^2s_1)\beta)n^2. \quad (3.67) \]

Hence since \( 1, \sqrt{c}, \beta, \sqrt{e} \) are rationally independent and \( 2bs_1 \neq 0, q^h(n) \) is uniformly distributed (mod 1).

(b) By (3.58) the \( \sqrt{ch} \)-terms of \( q^h(n) \) are

\[ [\sqrt{cn}]2s_1(a+b\sqrt{c})\gamma n - [\gamma n]s_0b\sqrt{cn} - [\sqrt{c}\gamma n]s_1b\sqrt{c} + s_0(a+b\sqrt{c})\gamma n^2 \]

\[ + \gamma [\sqrt{cn}]((2s_1a+s_0b)\gamma + 3s_1b\sqrt{c}\gamma)n + (s_0a\gamma - s_1b\gamma)n^2, \quad (3.68) \]

which is uniformly distributed (mod 1) since \( 1, \sqrt{c}, \gamma, \sqrt{e} \) are rationally independent and \( 3s_1b \neq 0 \).

(c) Let

\[ \alpha = a_1 + b_1\sqrt{c} \]
\[ \beta = a_2 + b_2\sqrt{c} \]
\[ \lambda = a_3 + b_3\sqrt{c} \]
\[ \gamma = \sqrt{c}, \]

where \( a_i, b_i \in \mathbb{Q}, i = 1, 2, 3 \) and \( b_1, b_2, b_3 \neq 0 \). By using (3.43) we have

\[ q(n) \cdot [((a_1+b_1\sqrt{c})n][(a_3+b_3\sqrt{c})n]a_2\sqrt{cn} - [(a_3\sqrt{c}+b_3c)n][(a_1+b_1\sqrt{c})n]b_2\sqrt{cn} \]

\[ + a_1a_2a_3 - a_1b_2b_3c)\sqrt{cn}^3 + (a_1a_2b_3 + a_2a_3b_1 - a_1a_3b_2 - b_1b_2b_3c)[\sqrt{cn}]\sqrt{cn}^2 \]

\[ +(a_2b_1b_3 - a_3b_1b_2)[\sqrt{cn}]^2\sqrt{cn} \quad (3.70) \]
so that

\[
q^h(n) \leftrightarrow [\sqrt{ch}](a_1a_2b_3 + a_2a_3b_1 - a_1a_3b_2 - b_1b_2b_3c)\sqrt{cn}^2 \\
+ 2(a_2b_1b_3 - a_3b_1b_2)[\sqrt{cn}]\sqrt{cn} \\
+ h\left(3(a_1a_2a_3 - a_1b_2b_3c)\sqrt{cn}^2 \\
+ 2(a_1a_2b_3 + a_2a_3b_1 - a_1a_3b_2 - b_1b_2b_3c)[\sqrt{cn}]\sqrt{cn} \\
+ (a_2b_1b_3 - a_3b_1b_2)[\sqrt{cn}]^2 \sqrt{c}\right)
\]

\[
\leftrightarrow [\sqrt{ch}](a_1a_2b_3 + a_2a_3b_1 - a_1a_3b_2 - b_1b_2b_3c)\sqrt{cn}^2 \\
+ h\left(3(a_1a_2a_3 - a_1b_2b_3c) - c(a_2b_1b_3 - a_3b_1b_2)\right)\sqrt{cn}^2.
\] (3.71)

So by Lemma 3.2.3, \(q(n)\) is uniformly distributed (mod 1) unless

\[
3(a_1a_2a_3 - a_1b_2b_3c) - c(a_2b_1b_3 - a_3b_1b_2) = 0 \\
a_1a_2b_3 + a_2a_3b_1 - a_1a_3b_2 - b_1b_2b_3c = 0,
\] (3.72)

which is the same as

\[
3a_1(a_2a_3 - b_2b_3c) = b_1c(a_2b_3 - a_3b_2) \\
a_1(a_2b_3 - a_3b_2) = b_1(b_2b_3c - a_2a_3).
\] (3.73)

If \(a_1 = 0\), then \(a_2b_3 = b_2a_3\) and \(b_2b_3c = a_2a_3\), which implies that \(c = \frac{a_2a_3}{b_2b_3} = \left(\frac{a_2}{b_2}\right)^2\) so that \(\sqrt{c} \in \mathbb{Q}\), a contradiction. So we get that

\[
c = \frac{3a_1}{b_1} \frac{a_2a_3 - b_2b_3c}{a_2b_3 - a_3b_2} = -3\left(\frac{a_2a_3 - b_2b_3c}{a_2b_3 - a_3b_2}\right)^2 < 0
\] (3.74)

which contradicts that \(\sqrt{c} \in \mathbb{R}\).
3.3 A class of generalized polynomials which are products of brackets

We show in this section the following theorem.

**Theorem 3.3.1** Let $\alpha_1, \ldots, \alpha_k \in \mathbb{R} \setminus \{0\}$, $k \geq 3$. Then

$$q(n) = [\alpha_1 n][\alpha_2 n] \cdots [\alpha_k n] \gamma$$  \hspace{1cm} (3.75)

is uniformly distributed (mod 1) if and only if $\gamma$ is irrational.

**Remark:** By Proposition 3.1.3, $[\alpha_1 n][\alpha_2 n] \gamma$ is not uniformly distributed (mod 1) if

$$\frac{\alpha_1}{\alpha_2} = \sqrt{c}$$

for some $c \in \mathbb{Q}^+$ and $\gamma$ is rationally dependent of $1, \sqrt{c}$.

**Proposition 3.3.2** A generalized polynomial

$$Q(n) = b_0[\lambda_1 n]^2 \gamma + b_1[\lambda_1 n][\lambda_2 n] \gamma + b_2[\lambda_2 n]^2 \gamma$$  \hspace{1cm} (3.76)

is uniformly distributed (mod 1) unless there exist $a, c, k_1, k_0 \in \mathbb{Q}$, $c > 0$, and $b \in \{+1, -1\}$ such that

$$\frac{\lambda_2}{\lambda_1} = a + b \sqrt{c}, \gamma = k_0 + k_1 \sqrt{c}$$

and

$$b_0 + b_1 a + b_2 (a^2 - c) = 0.$$  \hspace{1cm} (3.77)

**Proof:** By the identity (3.3), we have

$$Q(n) \leftrightarrow 2b_0[\lambda_1 n][\lambda_1 \gamma n] - b_0 \lambda_1^2 \gamma n^2$$

$$+b_1[\lambda_1 n][\lambda_2 \gamma n] + b_1[\lambda_2 n][\lambda_1 \gamma n] - b_1 \lambda_1 \lambda_2 \gamma n^2$$

$$+2b_2[\lambda_2 n][\lambda_2 \gamma n] - b_2 \lambda_2^2 \gamma n^2$$
\begin{align*}
&= [\lambda_1 n] \left( 2b_0 \lambda_1 \gamma + b_1 \lambda_2 \gamma \right) n + [\lambda_2 n] \left( b_1 \lambda_1 \gamma + 2b_2 \lambda_2 \gamma \right) n \\
&\quad - \left( b_0 \lambda_1^2 \gamma + b_1 \lambda_1 \lambda_2 \gamma + b_2 \lambda_2^2 \gamma \right) n^2 \\
&= [\lambda_1 n] An + [\lambda_2 n] Bn - \frac{1}{2} (\lambda_1 A + \lambda_2 B) n^2.
\end{align*}

(3.78)

where

\begin{align*}
A &= 2b_0 \lambda_1 \gamma + b_1 \lambda_2 \gamma \\
B &= b_1 \lambda_1 \gamma + 2b_2 \lambda_2 \gamma.
\end{align*}

(3.79)

Suppose first that $1, \lambda_1, \lambda_2$ are rationally independent. Then by Proposition 3.1.2, $Q(n)$ is fails to be uniformly distributed (mod 1) if and only if there exist $a_i \in \mathbb{Q}$ such that $A = a_0 + a_1 \lambda_1 + a_2 \lambda_2$, $B = a_4 + a_2 \lambda_1 + a_3 \lambda_2$ and

\begin{align*}
\frac{1}{2} \lambda_1 A + \lambda_2 B &= -\frac{1}{2} (a_0 \lambda_1 + a_4 \lambda_2) \in \mathbb{Q},
\end{align*}

in which case $a_0 = a_4 = 0$. So $Q(n)$ is not uniformly distributed (mod 1) if and only if

\begin{align*}
A &= a_1 \lambda_1 + a_2 \lambda_2 \\
B &= a_2 \lambda_1 + a_3 \lambda_2 
\end{align*}

for some $a_1, a_2, a_3 \in \mathbb{Q}$. (3.80)

Now, if $\lambda_1 = 1$, then $Q(n) \equiv [\lambda_2 n] B - \frac{1}{2} (\lambda_2 B - A) n^2$. Therefore, $Q(n)$ is not uniformly distributed (mod 1) if and only if there exist $a_i \in \mathbb{Q}$ such that $B = a_2 + a_3 \lambda_2$ and $a_3 \lambda_2^2 - (\lambda_2 (a_2 + a_3 \lambda_2) - A) = A - a_2 \lambda_2 \in \mathbb{Q}$. So also in this case $Q(n)$ is not uniformly distributed (mod 1) if and only if $A$ and $B$ satisfy the condition (3.80).

Suppose (3.80) is true. Then by (3.79) and (3.80) we have

\begin{align*}
2b_0 \lambda_1 \gamma + b_1 \lambda_2 \gamma &= a_1 \lambda_1 + a_2 \lambda_2 \\
b_1 \lambda_1 \gamma + 2b_2 \lambda_2 \gamma &= a_2 \lambda_1 + a_3 \lambda_2
\end{align*}

(3.81)

which implies

\begin{align*}
(b_1^2 - 4b_0 b_2) \lambda_2 \gamma &= (b_1 a_1 - 2b_0 a_2) \lambda_1 + (b_1 a_2 - 2b_0 a_3) \lambda_2.
\end{align*}

(3.82)
If $b_1^2 = 4b_0b_2$ then $\frac{2b_0}{b_1} = \frac{a_1}{a_2} = \frac{a_2}{a_3} = b$ and $b_1(b_\lambda_1 + \lambda_2)\gamma = a_2(b_\lambda_1 + \lambda_2)$ which implies that $\gamma \in \mathbb{Q}$, a contradiction.

If $b_1^2 \neq 4b_0b_2$, then both $\lambda_1\gamma$ and $\lambda_2\gamma$ are rationally dependent of $\lambda_1, \lambda_2$, say

$$\lambda_1\gamma = c_1\lambda_1 + c_2\lambda_2$$
$$\lambda_2\gamma = d_1\lambda_1 + d_2\lambda_2,$$

(3.83)

where $c_1, c_2, d_1, d_2$ are given by (3.81). This gives us the equation

$$2b_0c_2 + b_1(d_2 - c_1) - 2b_2d_1 = 0.$$  

(3.84)

It follows from (3.83) that

$$c_2\lambda_2^2 + (c_1 - d_2)\lambda_1\lambda_2 - d_1\lambda_1^2 = 0.$$  

(3.85)

This gives $\lambda_2 = (a + b\sqrt{c})\lambda_1$ where $a = \frac{d_2 - c_1}{2c_2}$, $c = a^2 + \frac{d_1}{c_2}$ and $b \in \{+1, -1\}$, if $a^2 + \frac{d_1}{c_2} > 0$. Otherwise, (3.85) is not possible. Note that $c_2 \neq 0$ unless $\lambda_1, \lambda_2$ are rationally dependent. By dividing (3.84) by $2c_2$ we have

$$b_0 + ab_1 + (a^2 - c)b_2 = 0.$$  

(3.86)

So if $1, \lambda_1, \lambda_2$ are rationally independent or $\lambda_1 = 1$ and $1, \lambda_2$ are rationally independent, then $Q(n)$ fails to be uniformly distributed (mod 1) if and only there exist $a, c \in \mathbb{Q}$ satisfying (3.86) and such that for some $b \in \{+1, -1\}$, we have $\lambda_2 = (a + b\sqrt{c})\lambda_1$ and $\gamma = k_0 + k_1\sqrt{c}$ for some $k_0, k_1 \in \mathbb{Q}$.

If $1, \lambda_1, \lambda_2$ are rationally dependent, then because of symmetry with respect to $\lambda_1$ and $\alpha_2$, we may assume that $\lambda_1 = 1 + d\lambda_2$, $d \in \mathbb{Q}$ such that

$$Q(n) \leftrightarrow b_0(n + d[\lambda_2n])^2\gamma + b_1(n + d[\lambda_2n])[\lambda_2n]\gamma + b_2[\lambda_2n]^2\gamma$$
$$= b_0n^2\gamma + (b_1 + 2b_0d)[\lambda_2n]n\gamma + (b_2 + b_1d + b_0d^2)[\lambda_2n]^2\gamma$$  

(3.87)
which by the above result is not uniformly distributed \( \mod 1 \) if and only if there exist \( a, c, b, k_0, k_1 \in \mathbb{Q} \) such that 
\[
\lambda_2 = a + b \sqrt{c}, \quad \gamma = k_0 + k_1 \sqrt{c} \quad \text{and} \quad 0 = b_0 + (b_1 + 2b_0d)a + (b_2 + b_1d + b_0d^2)(a^2 - c)
\]

\[= b_0(1 + 2ad + (a^2 - c)d^2) + b_1(a + (a^2 - c)d) + b_2(a^2 - c). \tag{3.88} \]

So
\[
b_0 + b_1 \frac{a + (a^2 - c)d}{1 + 2ad + (a^2 - c)d^2} + b_2 \frac{a^2 - c}{1 + 2ad + (a^2 - c)d^2} = 0. \tag{3.89} \]

Now,
\[
\frac{\lambda_2}{\lambda_1} = \frac{a + b \sqrt{c}}{1 + d(a + b \sqrt{c})} = \frac{(a + b \sqrt{c})(1 + ad - bd \sqrt{c})}{(1 + ad)^2 - cd^2} = \frac{a + (a^2 - c)d + b \sqrt{c}}{1 + 2ad + (a^2 - c)d^2} \equiv a' + b \sqrt{c'}. \tag{3.90} \]

Also,
\[
a'^2 - c' = \frac{(a + (a^2 - c)d)^2 - c}{(1 + 2ad + (a^2 - c)d^2)^2} = \frac{a^2 - c}{1 + 2ad + (a^2 - c)d^2}. \tag{3.91} \]

Hence, equation (3.89) is \( b_0 + b_1 a' + b_2(a'^2 - c') = 0 \), where \( \lambda_2 = (a' + b \sqrt{c'}) \lambda_1 \), 
\[ \gamma = k_0' + k_1' \sqrt{c'}. \]

\[
\square
\]

**Lemma 3.3.3** Let
\[
M = \begin{pmatrix}
A_1 & B_2 & 0 & 0 & \cdots & 0 \\
C_1 & A_2 & B_3 & 0 & \cdots & 0 \\
0 & C_2 & A_3 & B_4 & \cdots & 0 \\
& & & \ddots & & \vdots \\
0 & \cdots & 0 & C_{k-3} & A_{k-2} & B_{k-1} \\
0 & \cdots & 0 & 0 & C_{k-2} & A_{k-1}
\end{pmatrix}. \tag{3.92}
\]
where $A_j, B_j, C_j \in \mathbb{Q}$ such that $C_jB_{j+1} = dA_jA_{j+1}$, $d = \frac{1}{4}(1 - \frac{c}{a^2})$, $c, a \in \mathbb{Q}$, $c > 0$, $c \neq a^2$. Then $\det(M) \neq 0$.

**Proof:** Define a sequence $c_j$ inductively by $c_0 = c_1 = 1$ and $c_j = c_{j-1} - dc_{j-2}$ for $j \geq 2$.

We will show that $c_j > 0$ for all $j = 1, \ldots, k - 1$, and that $\det(M) = A_1 \cdots A_{k-1}c_{k-1}$.

Let

$$M_j = \begin{pmatrix}
A_j & B_{j+1} & 0 & \cdots & 0 \\
C_j & A_{j+1} & 0 & \cdots & 0 \\
0 & \cdots & 0 & A_{k-2} & B_{k-1} \\
0 & \cdots & 0 & C_{k-2} & A_{k-1}
\end{pmatrix}, \quad (3.93)$$

$j = 1, \ldots, k - 1$, and let $M_k = (1)$ and $M_{k+1} = (0)$. We will show by induction on $j$ that

$$\det(M) = A_1 \cdots A_j \left( c_j \det(M_{j+1}) - dc_{j-1}A_{j+1}\det(M_{j+2}) \right) \quad (3.94)$$

for $j = 1, \ldots, k - 1$. Since

$$\det(M_j) = A_j \det(M_{j+1}) - B_{j+1}C_j \det(M_{j+2}) = A_j \left( \det(M_{j+1}) - dA_{j+1} \det(M_{j+2}) \right), \quad (3.95)$$

$j = 1, \ldots, k - 1$, and $c_0 = c_1 = 1$, (3.94) is true for $j = 1$. If (3.94) is true for some $l$, $l < k - 1$, then

$$\det(M) = A_1 \cdots A_j \left( c_j \det(M_{j+1}) - dc_{j-1}A_{j+1}\det(M_{j+2}) \right)$$

$$= A_1 \cdots A_j \left( c_jA_{j+1} \left( \det(M_{j+2}) - dA_{j+2} \det(M_{j+3}) \right) - dc_{j-1}A_{j+1} \det(M_{j+2}) \right)$$

$$= A_1 \cdots A_{j+1} \left( (c_j - dc_{j-1}) \det(M_{j+2}) - c_j dA_{j+2} \det(M_{j+3}) \right) \quad (3.96)$$

which gives (3.94) for $j + 1$ since $c_{j+1} = c_j - dc_{j-1}$. 


By letting \( j = k - 1 \) in (3.94), we have \( \det(M) = A_1 \cdots A_{k-1} c_{k-1} \).

Since \( d = \frac{1}{4}(1 - \frac{c}{a^2}) \) and \( \frac{c}{a^2} > 0 \), either \( d < 0 \) or \( 0 < d < \frac{1}{4} \). If \( d < 0 \) then \( c_j > 0 \) for all \( j \) by the definition of the sequence \( c_j \). If \( 0 < d < \frac{1}{4} \), let \( b_j = \frac{c_j}{c_{j-1}} \) for all \( j \) such that \( c_{j-1} \neq 0 \). We will show by induction on \( j \) that \( \frac{1}{2} < b_j < 1 \). We have \( b_2 = \frac{c_2}{c_1} = 1 - d \) so that \( \frac{1}{2} < \frac{3}{4} < b_2 < 1 \). If \( \frac{1}{2} < b_j < 1 \) for some \( j \), then \( c_j \neq 0 \) and \( b_{j+1} = \frac{c_{j+1}}{c_j} = \frac{c_j - dc_{j-1}}{c_j} = 1 - \frac{d}{b_j} \). So since \( 0 < \frac{d}{b_j} < \frac{1}{2} \), we have \( \frac{1}{2} < b_{j+1} < 1 \). Hence \( c_j > 0 \) for all \( j \).

\[ \square \]

**Lemma 3.3.4** Let \( k \) be even and let \( b_3, \ldots, b_k \in \mathbb{Q} \). Define \( \sigma_1 = 1 \) and

\[
\sigma_j = \sum_{3 \leq i_1 < \cdots < i_{j-1} \leq k} b_{i_1} \cdots b_{i_{j-1}}, \quad j = 2, \ldots, k. \tag{3.97}
\]

Then it is impossible to have

\[
\sigma_{2j} = 0 \quad \text{and} \quad \sigma_{2j+1} > 0 \quad \text{for all} \quad j = 1, \ldots, \frac{k-2}{2}. \tag{3.98}
\]

**Proof:** Suppose (3.98) is true. We will first show by induction on \( j \) that

\[
\sigma_{k-3} \leq -b_3^j \sum_{4 \leq i_1 < \cdots < i_{k-4(j+4)} \leq k} b_{i_1} \cdots b_{i_{k-4(j+4)}} - b_3^{-1}b_4 \cdots b_k, \tag{3.99}
\]

\( j = 2, 4, \ldots, k - 4 \) and where we treat a sum over the empty set as 1. Since

\[
0 = \sigma_{2j} = \sum_{3 \leq i_1 < \cdots < i_{2j-1} \leq k} b_{i_1} \cdots b_{i_{2j-1}}
= b_3 \sum_{4 \leq i_1 < \cdots < i_{2j-2} \leq k} b_{i_1} \cdots b_{i_{2j-2}} + \sum_{4 \leq i_1 < \cdots < i_{2j-1} \leq k} b_{i_1} \cdots b_{i_{2j-1}} \tag{3.100}
\]
we have

\[ \sum_{4 \leq i_1 < \cdots < i_{2j-1} \leq k} b_{i_1} \cdots b_{i_{2j-1}} = -b_3 \sum_{4 \leq i_1 < \cdots < i_{2j-2} \leq k} b_{i_1} \cdots b_{i_{2j-2}} \]  

(3.101)

and especially for \( j = \frac{k-2}{2} \), we have

\[ \sum_{4 \leq i_1 < \cdots < i_{k-4} \leq k} b_{i_1} \cdots b_{i_{k-4}} = -b_3^{-1}b_4 \cdots b_k. \]

(3.102)

So by using (3.101) for \( j = \frac{k-4}{2} \) and (3.102) we get

\[ \sigma_{k-3} = \sum_{3 \leq i_1 < \cdots < i_{k-4} \leq k} b_{i_1} \cdots b_{i_{k-4}} 
\]

\[ = b_3 \sum_{4 \leq i_1 < \cdots < i_{k-5} \leq k} b_{i_1} \cdots b_{i_{k-5}} + \sum_{4 \leq i_1 < \cdots < i_{k-4} \leq k} b_{i_1} \cdots b_{i_{k-4}} 
\]

\[ = -b_3^{-1} \sum_{4 \leq i_1 < \cdots < i_{k-6} \leq k} b_{i_1} \cdots b_{i_{k-6}} - b_3^{-1}b_4 \cdots b_k, \]

(3.103)

which shows (3.99) for \( j = 2 \).

Suppose now that (3.99) is true for \( j \). From

\[ 0 < \sigma_{2j+1} = \sum_{3 \leq i_1 < \cdots < i_{2j} \leq k} b_{i_1} \cdots b_{i_{2j}} 
\]

\[ = b_3 \sum_{4 \leq i_1 < \cdots < i_{2j-1} \leq k} b_{i_1} \cdots b_{i_{2j-1}} + \sum_{4 \leq i_1 < \cdots < i_{2j} \leq k} b_{i_1} \cdots b_{i_{2j}} \]

(3.104)

it follows that

\[ - \sum_{4 \leq i_1 < \cdots < i_{2j} \leq k} b_{i_1} \cdots b_{i_{2j}} < b_3 \sum_{4 \leq i_1 < \cdots < i_{2j-1} \leq k} b_{i_1} \cdots b_{i_{2j-1}}, \]

(3.105)

\( j = 1, \ldots, \frac{k-2}{2} \). By combining (3.101) and (3.105), we have

\[ - \sum_{4 \leq i_1 < \cdots < i_{2j} \leq k} b_{i_1} \cdots b_{i_{2j}} < -b_3^2 \sum_{4 \leq i_1 < \cdots < i_{2j-2} \leq k} b_{i_1} \cdots b_{i_{2j-2}}. \]

(3.106)
So by the induction hypothesis and (3.106),

\[
\sigma_{k-3} \leq -b_3^j \sum_{4 \leq i_1 < \cdots < i_{k-(j+4)} \leq k} b_{i_1} \cdots b_{i_{k-(j+4)}} - b_3^{-1} b_4 \cdots b_k \\
< -b_3^{j+2} \sum_{4 \leq i_1 < \cdots < i_{k-(j+6)} \leq k} b_{i_1} \cdots b_{i_{k-(j+6)}} - b_3^{-1} b_4 \cdots b_k
\]

(3.107)

which shows (3.99) for \( j + 2 \). Hence (3.99) is proved.

Let \( j = k - 4 \). Then

\[
\sigma_{k-3} \leq -(b_3^{k-4} + b_3^{-1} b_4 \cdots b_k) < 0
\]

(3.108)

since \( k - 4 \) is even and \( b_3 \cdots b_k = \sigma_{k-1} > 0 \). However, this contradicts that \( \sigma_{k-3} > 0 \). Therefore (3.98) is impossible.

\[\square\]

**Proof of Theorem 3.3.1:**

Let \( l \) be the dimension of the vector space over \( \mathbb{Q} \) spanned by \( \alpha_1, \ldots, \alpha_k \). By possibly reordering the \( \alpha_i \)'s, we may take \( \alpha_1, \ldots, \alpha_l \) as basis of this vector space so that

\[
\alpha_i = \sum_{j=1}^{l} a_{ij} \alpha_j, \quad a_{ij} \in \mathbb{Q}, \quad i = l + 1, \ldots, k.
\]

(3.109)

If \( l = 1 \), then \( q(n) \mapsto c[\alpha n]^k \gamma \), \( c \in \mathbb{Q} \). Since \( c[\alpha n]^2 \gamma \) is uniformly distributed (mod 1) by Proposition 3.1.3 and \( q^h(n) \mapsto c \alpha^{k-1} \gamma \), it follows by induction and van der Corput’s Theorem that \( q(n) = c[\alpha n]^k \gamma \) is uniformly distributed (mod 1).

If \( l = 2 \), then

\[
q(n) \mapsto [\alpha_1 n][\alpha_2 n] \prod_{i=3}^{k} (a_{i1}[\alpha_1 n] + a_{i2}[\alpha_2 n]) \gamma \\
= \sum_{i=1}^{k-1} a_i [\alpha_1 n]^{k-i} [\alpha_2 n]^i \gamma
\]

(3.110)
where

$$a_1 = \prod_{i=3}^{k} a_i$$
$$a_2 = \sum_{i=3}^{k} \left( \prod_{j \neq i} a_j \right) a_{i2}$$
$$\vdots$$
$$a_{k-1} = \prod_{i=3}^{k} a_{i2}.$$  \hfill (3.111)

Note that since \(q(n) \neq 0\) there exists \(i, 1 \leq i \leq k - 1\) so that \(a_i \neq 0\).

Let \(a_0 = a_k = 0\) so that we can write

$$q(n) = \sum_{i=0}^{k} a_i [\alpha_1 n]^{k-i} [\alpha_2 n]^i \gamma.$$  \hfill (3.112)

By van der Corput's theorem, \(q(n)\) is uniformly distributed (mod 1) if

$$q^h(n) = [\alpha_1 h] \left( \sum_{i=0}^{k} a_i (k-i) [\alpha_1 n]^{k-i-1} [\alpha_2 n]^i \gamma \right) + [\alpha_2 h] \left( \sum_{i=0}^{k} a_i i [\alpha_1 n]^{k-i} [\alpha_2 n]^{i-1} \gamma \right)$$  \hfill (3.113)

is uniformly distributed (mod 1), and by Lemma 3.2.3, \(q^h(n)\) is uniformly distributed (mod 1) if either

$$v_1(n) = \sum_{i=0}^{k-1} a_i (k-i) [\alpha_1 n]^{k-i-1} [\alpha_2 n]^i \gamma$$  \hfill (3.114)

or

$$v_2(n) = \sum_{i=1}^{k} a_i i [\alpha_1 n]^{k-i} [\alpha_2 n]^{i-1} \gamma$$  \hfill (3.115)

is uniformly distributed (mod 1). The same argument can be repeated for \(v_1(n)\) and \(v_2(n)\). Note that \(v_1(n)\) and \(v_2(n)\) can be seen as the partial derivatives of the polynomial function

$$f(x, y) = \sum_{i=0}^{k} a_i x^{k-i} y^i \gamma$$  \hfill (3.116)

on \(\mathbb{R}^2\), evaluated at \(([\alpha_1 n], [\alpha_2 n])\). So by using induction it follows from van der Corput's theorem and Lemma 3.2.3 that \(q(n)\) is uniformly distributed (mod 1) if at
least one of the \((k - 2)’th\) partial derivatives evaluated at \(([\alpha_1 n], [\alpha_2 n])\),

\[
Q_j(n) = \frac{\partial^{k-2} f}{\partial x^{k-2-j} \partial y^j} ([\alpha_1 n], [\alpha_2 n]), \quad j = 0, 1, \ldots, k - 2
\]  

(3.117)
is uniformly distributed \((\text{mod } 1)\). Now,

\[
\frac{\partial^{k-2} f}{\partial x^{k-2-j} \partial y^j}(x, y) = a_j \frac{(k-j)!}{2!} j! x^2 \gamma + a_{j+1} (k-j-1)! (j+1)! xy \gamma + a_{j+2} (k-j-2)! \frac{(j+2)!}{2!} y^2 \gamma,
\]

(3.118)
so that

\[
Q_j(n) = \frac{(k-j)!}{2!} j! a_j [\alpha_1 n]^2 \gamma + (k-j-1)! (j+1)! a_{j+1} [\alpha_1 n] [\alpha_2 n] \gamma + (k-j-2)! \frac{(j+2)!}{2!} a_{j+2} [\alpha_2 n]^2 \gamma,
\]

(3.119)
\(j = 0, \ldots, k - 2\). If none of the \(Q_j(n)’s\) is uniformly distributed \((\text{mod } 1)\), there exist by Proposition 3.3.2, \(a, c \in \mathbb{Q}, c > 0, c \neq a^2\) such that

\[
\frac{(k-j)!}{2!} j! a_j + a(k-j-1)! (j+1)! a_{j+1} + (a^2 - c)(k-j-2)! \frac{(j+2)!}{2!} a_{j+2} = 0,
\]

(3.120)
\(j = 0, \ldots, k - 2\).

We will show that this leads to a contradiction. Now, use the fact that \(a_0 = a_k = 0\), and let \(a_1, \ldots, a_{k-1}\) be the unknowns in the system (3.120) of \(k - 1\) equations.

If \(a \neq 0\), the system (3.120) has the unique solution \((0, \ldots, 0)\) if the matrix

\[
M = \begin{pmatrix}
A_1 & B_2 & 0 & 0 & \cdots & 0 \\
C_1 & A_2 & B_3 & 0 & \cdots & 0 \\
0 & C_2 & A_3 & B_4 & \cdots & 0 \\
& & & & & \ddots \n0 & \cdots & 0 & C_{k-3} & A_{k-2} & B_{k-1} \\
0 & \cdots & 0 & 0 & C_{k-2} & A_{k-1}
\end{pmatrix}
\]  

(3.121)
where $A_j = (k-j)!j!a$, $B_j = (k-j)!\frac{j!}{2!} (a^2 - c)$, $C_j = \frac{(k-j)!}{j!} a$, is non-singular. Since $B_j = \frac{a^2 - c}{2a} A_j$, $C_j = \frac{1}{2a} A_j$, it follows that $C_j B_{j+1} = d A_j A_{j+1}$, where $d = \frac{1}{4} (1 - \frac{c}{a^2})$. So by Lemma 3.3.3, $\det(M) \neq 0$. Therefore $q(n) = 0$, a contradiction.

If $a = 0$, then (3.120) gives

$$\frac{(k-j)!}{2!} j! a_j = c(k-j-2)! \frac{(j+2)!}{2!} a_{j+2}, \quad j = 0, \ldots, k-2. \quad (3.122)$$

Since $a_0 = a_k = 0$, we have $A_{2j} = 0$ for all $j$. If $k$ is odd, then we also have $a_{2j+1} = 0$ for all $j$ such that $q(n) = 0$, a contradiction.

Let $k$ be even. If $a_1 = 0$ then $a_i = 0$ for all $i$. So we may assume that $a_1 \neq 0$. It follows from (3.122) that $\frac{a_{2j+1}}{a_1} > 0$ for all $j$. Recall that the $a_i$’s satisfy the equations (3.111). Let $b_i = \frac{a_i}{a_1}$. Then it follows from (3.111) and (3.122) that

$$\frac{a_{2j}}{a_1} = \sum_{3 \leq i_1 < \cdots < i_{2j-1} \leq k} b_{i_1} \cdots b_{i_{2j-1}} = 0, \quad j = 1, \ldots, \frac{k-2}{2} \quad (3.123)$$

and

$$\frac{a_{2j+1}}{a_1} = \sum_{3 \leq i_1 < \cdots < i_{2j} \leq k} b_{i_1} \cdots b_{i_{2j}} > 0, \quad j = 0, \ldots, \frac{k-2}{2}, \quad (3.124)$$

which by Lemma 3.3.4 is impossible. This ends the proof for the case $l = 2$.

Let $l > 2$. We will show by induction on $k$ that

$$q(n) \equiv \prod_{i=1}^l [\alpha_i n] \prod_{i=l+1}^k \left( \sum_{j=1}^l a_{ij} [\alpha_j n] \right) \gamma \quad (3.125)$$

is uniformly distributed (mod 1) for any $l \leq k$.

If $k = 3$ and $l = 3$ then $\alpha_1, \alpha_2, \alpha_3$ are rationally independent and

$$q^h(n) \equiv [\alpha_1 h] [\alpha_2 n] [\alpha_3 n] \gamma + [\alpha_2 h] [\alpha_1 n] [\alpha_3 n] \gamma + [\alpha_3 h] [\alpha_1 n] [\alpha_2 n] \gamma$$

which by Lemma 3.3.4 is also uniformly distributed (mod 1) for any $l \leq k$. 

If $k = 3$ and $l = 2$ then $\alpha_1, \alpha_2$ are rationally independent and

$$q^h(n) \equiv [\alpha_1 h] [\alpha_2 n] \gamma + [\alpha_1 h] [\alpha_2 n] \gamma$$

and by Lemma 3.3.4 is also uniformly distributed (mod 1) for any $l \leq k$. 

If $k = 3$ and $l = 1$ then $\alpha_1$ is rationally independent and

$$q^h(n) \equiv \alpha_1 h \cdot \gamma$$

which by Lemma 3.3.4 is also uniformly distributed (mod 1) for any $l \leq k$. 

If $k = 3$ and $l = 0$ then $\alpha_1$ is rationally independent and

$$q^h(n) \equiv 1$$

which by Lemma 3.3.4 is also uniformly distributed (mod 1) for any $l \leq k$. 

If $k = 2$ and $l = 2$ then $\alpha_1, \alpha_2$ are rationally independent and

$$q^h(n) \equiv [\alpha_1 h] [\alpha_2 n] \gamma + [\alpha_2 h] [\alpha_1 n] \gamma$$

and by Lemma 3.3.4 is also uniformly distributed (mod 1) for any $l \leq k$. 

If $k = 2$ and $l = 1$ then $\alpha_1$ is rationally independent and

$$q^h(n) \equiv \alpha_1 h \cdot \gamma$$

which by Lemma 3.3.4 is also uniformly distributed (mod 1) for any $l \leq k$. 

If $k = 2$ and $l = 0$ then $\alpha_1$ is rationally independent and

$$q^h(n) \equiv 1$$

which by Lemma 3.3.4 is also uniformly distributed (mod 1) for any $l \leq k$. 

If $k = 1$ then $\alpha_1$ is rationally independent and

$$q^h(n) \equiv \alpha_1 h \cdot \gamma$$

which by Lemma 3.3.4 is also uniformly distributed (mod 1) for any $l \leq k$. 

If $k = 0$ then $\alpha_1$ is rationally independent and

$$q^h(n) \equiv \alpha_1 h \cdot \gamma$$

which by Lemma 3.3.4 is also uniformly distributed (mod 1) for any $l \leq k$. 

Therefore, $q(n) = 0$, a contradiction.
By Lemma 3.2.3, it is enough that some \([\alpha_i n][\alpha_j n]\gamma\) is uniformly distributed (mod 1), \(i \neq j, i, j \in \{1, 2, 3\}\). If all of them fail to be uniformly distributed (mod 1), then by Proposition 3.1.3,

\[
\frac{\alpha_i}{\alpha_j} = a_{ij}\sqrt{c} \quad \text{and} \quad \gamma = k_0\sqrt{c}, \quad i, j = 1, 2, 3, \quad a_{ij}, k_0 \in \mathbb{Q} \setminus \{0\}. \tag{3.127}
\]

If this is the case then \(\alpha_1 = a_{12}\sqrt{c}\alpha_2 = a_{13}\sqrt{c}\alpha_3\) which contradicts that \(\alpha_1, \alpha_2, \alpha_3\) are rationally independent. So \(q(n)\) is uniformly distributed (mod 1) if \(k = 3\).

Let \(k > 3\) and suppose \(q(n)\) is uniformly distributed (mod 1) if \(\deg(q) < k\). By rewriting the expressions (3.125) for \(q(n)\), we have

\[
q(n) = \prod_{r=0}^{k-1} [\alpha_1 n]^{k-l+1-r} \prod_{i=2}^{l} [\alpha_i n] \sum_{l<i_1<\cdots<i_r<k} \left( \prod_{j \neq i_1, \ldots, i_r} a_{ij} \right) \prod_{j=1}^{r} \left( \sum_{s=2}^{l} a_{ij_s}[\alpha_s n] \right). \tag{3.128}
\]

Let \(l_1, \ldots, l_m, 0 \leq m \leq k - 1\), be all the indices \(i\) for which \(a_{i1} = 0\). Then

\[
q(n) = \prod_{r=m}^{k-1} [\alpha_1 n]^{k-l+1-r} \prod_{i=2}^{l} [\alpha_i n] \sum_{l<i_1<\cdots<i_r<k} \left( \prod_{j \neq i_1, \ldots, i_r} a_{ij} \right) \prod_{j=1}^{r} \left( \sum_{s=2}^{l} a_{ij_s}[\alpha_s n] \right), \tag{3.129}
\]

and if

\[
q_1(n) = \prod_{i=2}^{l} [\alpha_i n] \sum_{l<i_1<\cdots<i_m<k} \left( \prod_{j \neq i_1, \ldots, i_m} a_{ij} \right) \prod_{j=1}^{m} \left( \sum_{s=2}^{l} a_{ij_s}[\alpha_s n] \right), \tag{3.130}
\]

then

\[
q_1(n) = \prod_{i=2}^{l} [\alpha_i n] \left( \prod_{j \neq i_1, \ldots, i_m} a_{ij} \right) \prod_{j=1}^{m} \left( \sum_{s=2}^{l} a_{ij_s}[\alpha_s n] \right), \tag{3.131}
\]

which is a non-zero generalized polynomial of form \(\prod_{i=1}^{k_1} [\lambda_i n] \gamma\) where \(k_1 = m + l - 1 \leq k - l + l - 1 = k - 1\) and \(l_1 = l - 1\). By the induction hypothesis, \(q_1(n)\) is uniformly distributed (mod 1) if \(m \neq 0\) or \(m = 0\) and \(l > 3\).
If we see $q(n)$ as a polynomial function $f$ on $\mathbb{R}^l$ evaluated at $\left([\alpha_1 n], [\alpha_2 n], \ldots, [\alpha_l n]\right)$, we have similarly as in the case $l = 2$ that $q(n)$ is uniformly distributed (mod 1) if one of the generalized polynomials

$$\frac{\partial^i f}{\partial x_1^{i_1} \cdots \partial x_l^{i_l}} \left([\alpha_1 n], [\alpha_2 n], \ldots, [\alpha_l n]\right), j = 1, \ldots, k - 1, i \leq j$$

(3.132)

is uniformly distributed (mod 1), and especially if $\frac{\partial^{k-l+1-m} f}{\partial x_1^{k-l+1-m}} \left([\alpha_1 n], [\alpha_2 n], \ldots, [\alpha_l n]\right)$ is uniformly distributed (mod 1). From (3.129) we see that

$$\frac{\partial^{k-l+1-m} f}{\partial x_1^{k-l+1-m}} \left([\alpha_1 n], [\alpha_2 n], \ldots, [\alpha_l n]\right) = (k - l + 1 - m)!q_1(n)$$

(3.133)

which we have shown is uniformly distributed (mod 1) if $m \neq 0$ or $m = 0$ and $l > 3$.

Note that if $m = 0$ and $l = 3$, then $(k - 2)!q_1(n) = a[\alpha_1 n][\alpha_2 n] \gamma$, $a \in \mathbb{Q}$, which may fail to be uniformly distributed (mod 1).

Since there was nothing special with $\alpha_1$, we could use $\alpha_2$ or $\alpha_3$ instead of $\alpha_1$. If the corresponding $m_2 \neq 0$ or $m_3 \neq 0$, then $q(n)$ is uniformly distributed (mod 1) as above. However, if $m_1 = m_2 = m_3 = 0$, then the problem is reduced to show that one of $[\alpha_1 n][\alpha_2 n] \gamma$, $[\alpha_2 n][\alpha_3 n] \gamma$ or $[\alpha_3 n][\alpha_3 n] \gamma$ is uniformly distributed (mod 1). See the case $k = l = 3$ treated above for this situation. This completes the proof that $q(n)$ is uniformly distributed (mod 1).
CHAPTER IV

Dynamical approach to uniform distribution of generalized polynomials

4.1 Introduction

The sequence $\alpha_n \pmod{1}$ can be seen as the orbit of the point 0 under the rotation $T$ on the torus $K = \mathbb{R}/\mathbb{Z}$, $Tx = x + \alpha \pmod{1}$, $x \in K$. If $\alpha$ is irrational, then for any $k \in \mathbb{Z} \setminus \{0\}$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi ikT^n x} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi ik(x+n\alpha)} = \lim_{N \to \infty} e^{2\pi ikx} \frac{1}{N} \frac{1-e^{2\pi ik\alpha}}{1-e^{2\pi ik}} = 0 \quad (4.1)$$

uniformly in $x$. It follows therefore that for any $f \in C(K)$, $\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = f \, f d\mu$ uniformly in $x$, where $\mu$ is the normalized Haar measure on $K$. Note that Haar measure is the only $T$-invariant measure on $K$. This example suggests a dynamical way of proving uniform distribution of sequences. By a (topological) dynamical system $(X, T)$ we shall mean a compact metric space $X$ and a homeomorphism $T$ on $X$. Denote by $\mathcal{B}(X)$ the $\sigma$-algebra of Borel subsets of $X$. A measure $\mu$ on $\mathcal{B}(X)$ is $T$-invariant if $\mu(T^{-1}A) = \mu(A)$ for all $A \in \mathcal{B}(X)$. By a theorem of Krylov and
Bogolioubov, there always exists a $T$-invariant probability measure $\mu$ on $B(X)$, see for example [39, p. 152].

**Definition 4.1.1** A dynamical system $(X, T)$ is uniquely ergodic if there exists a unique $T$-invariant probability measure on the Borel subsets of $X$.

If $\mu$ is unique then $\mu$ is automatically ergodic, i.e., if $A \in B(X)$, $\mu(A) > 0$ and $T^{-1}A = A$, then $\mu(A) = 0$ or $\mu(A) = 1$. The ergodic theorem [39] says that if $(X, B, \mu, T)$ is an ergodic dynamical system and $f \in L^1(X)$ then

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = \int f \, d\mu, \quad a.e. \ x. \quad (4.2)$$

If $\mu$ is a unique $T$-invariant measure then one can say somewhat more.

**Theorem 4.1.1** ([28]) Let $(X, T)$ be a dynamical system. The following are equivalent.

(i) $T$ is uniquely ergodic.

(ii) There exists a $T$-invariant measure $\mu$ such that for all $f \in C(X)$ and all $x \in X$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = \int f \, d\mu. \quad (4.3)$$

(iii) For every $f \in C(X)$, $\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x)$ converges pointwise to a constant.

(iv) For every $f \in C(X)$, $\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x)$ converges uniformly to a constant.

Remarks:
(1) Note that if \( T \) is uniquely ergodic then by Theorem 4.1.1, the averages 
\[
\frac{1}{N} \sum_{n=M}^{M+N-1} f(T^nx_0)
\]
converge to \( \int fd\mu \) uniformly in \( M \) for any \( x_0 \in X \), where \( f \in C(X) \). This implies that any sequence \( x(n), n = 1, 2, \ldots \), which we will prove in the sequel to be uniformly distributed (mod 1) by dynamical methods, is well-distributed (see Definition 1.2.8) as well.

(2) The results of Theorem 4.1.1 are still true if we allow the functions \( f \) to be discontinuous at a set of measure 0, see Lemma 4.3.7.

Let \((X, T)\) and \((Y, S)\) be two dynamical systems. A *homomorphism* from \((X, T)\) to \((Y, S)\) is given by a continuous map \( \phi : X \to Y \) satisfying
\[
\phi(Tx) = S\phi(x) \quad \text{for all } x \in X.
\]

**Definition 4.1.2** A dynamical system \((X, T)\) is an extension of a dynamical system \((Y, S)\) if there exists a homomorphism from \((X, T)\) to \((Y, S)\) given by a map \( \phi \) of \( X \) onto \( Y \). In this case we also say that \((Y, S)\) is a factor of \((X, T)\).

Especially important for us are group extensions.

**Definition 4.1.3 ([14])** Let \((X, T)\) be a dynamical system and suppose that a group \( G \) acts on \( X \) in such a way that \((x, g) \mapsto xg\) defines a continuous map from \( X \times G \to X \) with the property that if \( xg = x \) for some \( x \in X \), then \( g \) is the identity. Assume in addition that the action of \( G \) commutes with that of \( T \), \( T(xg) = (Tx)g \).
Then the orbit space \( X/G \) determines a dynamical system \((Y, T) = (X/G, T)\), where \( T(xG) = (Tx)G \). \((Y, T)\) is then a factor of \((X, T)\) and we say that \((X, T)\) is a group extension of \((Y, T)\). If the group \( G \) is a torus, \((X, T)\) will sometimes be called a torus extension of \((Y, S)\).

Let \((Y, S)\) be a dynamical system, \( G \) a compact group and \( \psi : Y \to G \) a continuous mapping. Form \( X = Y \times G \) and define \( T : X \to X \) by \( T(y, g) = (Sy, \psi(y)g) \). The resulting system \((X, T)\) is then a group extension of \((Y, S)\), or a skew product of \((Y, S)\) with \( G \).

Group extensions preserve unique ergodicity in the sense that if \((Y, S)\) is uniquely ergodic and \((X, T)\) is ergodic, then \((X, T)\) is also uniquely ergodic [17, p. 66]. Especially, an ergodic dynamical system obtained from a uniquely ergodic system by successive group extensions, is uniquely ergodic.

A transformation \( T \) on a group \( G \) is called affine if \( T x = aAx \), where \( A \) is an automorphism of the group \( G \) and \( a \in G \). We will now see how H. Furstenberg in [17] constructed uniquely ergodic affine transformations on \( d \)-dimensional tori \( K^d \) to show that polynomials \( p(n) \) of degree \( d \) which have irrational leading coefficients are uniformly distributed \( \text{(mod 1)} \). Let \( p(x) = p_d(x) \) be such a polynomial, and let \( p_{i-1}(x) = p_i(x + 1) - p_i(x), i = d, \ldots, 2 \). Denote the leading coefficient of \( p(x) \) by \( \alpha/d! \). Note that \( \deg(p_i) = i, i = 1, \ldots, d \), and that \( p_1(x) = \alpha x + \beta \) for some \( \beta \in \mathbb{R} \). Let \( T : K^d \to K^d \) be the affine transformation

\[
(x_1, \ldots, x_d) \mapsto (x_1 + \alpha, x_2 + x_1, \ldots, x_d + x_{d-1}).
\] (4.4)
$T$ can be seen as achieved by successive torus extensions, starting with the rotation $x_1 \mapsto x_1 + \alpha$ on $K$. Therefore, to prove that $T$ is uniquely ergodic, it is enough to show that $T$ is ergodic since any irrational rotation on $K$ is uniquely ergodic and unique ergodicity is preserved under torus extensions. This has been done in [17, p. 67]. Since $p_i(n + 1) = p_i(n) + p_{i-1}(n)$, $i = 1, \ldots, d$, we have

$$T^n(p_1(0), p_2(0), \ldots, p_d(0)) = (p_1(n), p_2(n), \ldots, p_d(n)).$$

If $f \in C(K)$, let $g \in C(K^d)$ be the function $(x_1, \ldots, x_d) \mapsto f(x_d)$. Then

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(p(n)) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} g(T^n(p_1(0), \ldots, p_d(0))) = \int g dx = \int f dx,$$

which shows that $p(n)$ is uniformly distributed (mod 1).

In [12] R. Fellgett has shown that uniform distribution of polynomials of many variables can be proved by using a class of affine actions of finitely generated abelian groups on finite dimensional tori.

We will use a similar approach, using affine transformations on nilmanifolds (see Section 4.3 for the definitions), to show uniform distribution of some generalized polynomials. It was observed implicitly in [2, Chapter IV] that there are some connections between generalized polynomials and nilmanifolds. In [8] J. Brezin used nilmanifold theory in an attempt to show uniform distribution of a class of generalized polynomials. We will show in Section 4.7 that his result is true, but that his proof is wrong because the transformations used are not uniquely ergodic. However, we will use his idea to show a very similar result, see Proposition 4.7.3.
4.2 Unique ergodicity and distality of generalized polynomials.

**Definition 4.2.1** A function $f \in l^\infty(\mathbb{Z})$ is called uniquely ergodic if there exists a uniquely ergodic system $(X, \mathcal{B}, \mu, T)$, a point $x \in X$ and a function $g \in C(X)$ such that $f(n) = g(T^n x)$.

Let $S$ denote the shift transformation $S f(n) = f(n + 1)$ on $l^\infty(\mathbb{Z})$ and for each $f \in l^\infty(\mathbb{Z})$, let $X_f = \overline{\{S^n f \mid n \in \mathbb{Z}\}}$, where the closure is taken in the product topology of $M^\mathbb{Z} \subset l^\infty(\mathbb{Z})$, $M = \sup_{n \in \mathbb{Z}} f(n)$. That makes $X_f$ a compact space so that $(X_f, S)$ is a dynamical system.

**Proposition 4.2.1** $f \in l^\infty(\mathbb{Z})$ is uniquely ergodic if and only if the dynamical system $(X_f, S)$ is uniquely ergodic.

**Proof:** Suppose $f \in l^\infty(\mathbb{Z})$ is uniquely ergodic and that $(Y, T)$ is a uniquely ergodic system, $y_0 \in Y$ and $g \in C(Y)$ such that $f(n) = g(T^n y)$. Let $\phi : Y \to l^\infty(\mathbb{Z})$ be the map $\phi(y) = (g(T^m y))_n$. Since $\phi(T^m y_0) = (g(T^{m+n} y_0))_n = S^m f$ and $\phi$ is continuous, $\phi$ maps $Y$ onto $X_f$ and $\phi \circ T = S \circ \phi$. So $(X_f, S)$ is a factor of $(Y, T)$. Let $x \in X_f$ and $y \in Y$ so that $x = \phi(y)$. Then for any $F \in C(X_f)$, we have $\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} F(S^n x) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} F \circ \phi(T^m y)$, which converges because $(Y, T)$ is uniquely ergodic. Hence $(X_f, S)$ is also uniquely ergodic by Theorem 4.1.1.
Now, suppose \((X_f, S)\) is uniquely ergodic. Let \(F \in C(X_f)\) be the function \(F(x) = x(0)\). Then \(f(n) = F(S^n f)\). So \(f(n)\) is uniquely ergodic.

\[\square\]

We shall call a generalized polynomial \(q(n)\) \textit{uniquely ergodic} if \(f(n) = e^{2\pi i q(n)}\) is uniquely ergodic. In the previous section we saw that any polynomial \(p(n)\) with irrational leading coefficient, is uniquely ergodic. Actually any polynomial is uniquely ergodic \([20]\). For example, if \(p(n) = \beta n^2 + \alpha n, \alpha, \beta \in \mathbb{R}\), then \((K^2, T)\) is a dynamical system for \(p(n)\), where \(T\) is defined on \(K^2\) by \(T(x, y) = (x + \beta, y + 2x + \beta + \alpha) \pmod{1}\). For then \(T^n(0, 0) = (\beta n, \beta n^2 + \alpha n) \pmod{1}\). However, \(T\) is uniquely ergodic only if \(\beta\) is irrational. If \(\alpha\) is irrational and \(\beta \in \mathbb{Q}\), say \(\beta = \frac{1}{3}\), then the orbit of \((0, 0)\) under \(T\) is \(D_3 \times K\), where \(D_3\) is the discrete subgroup \(\{0, \frac{1}{3}, \frac{2}{3}\}\) (written additively) of \(K\). We can therefore define \(T\) on the space \(D_3 \times K\) by \(T(x, y) = (x + \frac{1}{3}, y + 2x + \frac{1}{3} + \alpha) \pmod{1}\) and in this way make \(T\) uniquely ergodic. Let \(\nu\) be the atomic measure \(\nu(0) = \nu(\frac{1}{3}) = \nu(\frac{2}{3}) = \frac{1}{3}\) on \(D_3\) and let \(\mu\) be Haar measure on \(K\). Then \(\nu \times \mu\) is the unique \(T\)-invariant measure on \(D_3 \times K\). Note that uniform distribution of \(p(n)\) follows in this case since \(\mu\) is the Lebesgue measure on \(K\). However, if a polynomial \(p(n)\) has only rational coefficients then the corresponding uniquely ergodic system is a transformation on a discrete subgroup of a torus and the invariant measure is atomic.

Let \(q(n) = [\alpha n] \beta\), where \(1, \alpha, \alpha \beta\) are rationally independent. We have \(q(n) = \alpha \beta n - \{\alpha n\} \beta\). So if \(T\) is the rotation by \((\alpha \beta, \alpha)\) on the torus \(K^2\), then \(f(n) = e^{2\pi i q(n)} = g(T^n 0)\), where \(g(x, y) = e^{2\pi i (x - \{y\} \beta)}\). Since \((K^2, T)\) is uniquely ergodic
and $g$ is a Riemann-integrable periodic mod 1 function with integral equal 0, $q(n)$ is uniformly distributed (mod 1). It follows from the next theorem, which Furstenberg made me aware of, that $[\alpha n]\beta$ is also uniquely ergodic. I thank Prof. V. Lin for his crucial help in working out some parts of the proof.

**Theorem 4.2.2** Let $(X, T)$ be a uniquely ergodic system and $g : X \to \mathbb{C}$ a bounded function which is continuous on the complement of a set of measure 0. Then $f(n) = g(T^n x)$ is uniquely ergodic for each $x \in X$.

**Proof:** Let $\mathcal{A}$ be the smallest uniformly closed and conjugation closed algebra generated by $C(X) \cup \{T^n g \mid n \in \mathbb{Z}\}$, where $T^n g$ is the function $T^n g(x) = g(T^n x)$. Note that $\mathcal{A}$ is $T$-invariant. By the Gelfand theory (see for example [41]), $\mathcal{A} = C(\tilde{X})$, where

$$\tilde{X} = \mathcal{P}(\mathcal{A}) = \{\phi : \mathcal{A} \to \mathbb{C} \mid \phi \text{ is a multiplicative functional}\} \quad (4.5)$$

is a compact Hausdorff space when given the Gelfand topology. The isomorphism is given by the Gelfand transformation $f \mapsto \hat{f}$, where $\hat{f}(\phi) = \phi(f)$ for all $\phi \in \tilde{X}$. Let $\phi_x(f) = f(x)$. Since $\phi_x \in \tilde{X}$ for all $x \in X$, we have the inclusion $X \hookrightarrow \tilde{X}$. Moreover, $i(X)$ is dense in $\tilde{X}$. For otherwise, if $\overline{i(X)}$ is the closure of $i(X)$ in $\tilde{X}$ and $\overline{i(X)} \neq \tilde{X}$, there exists a non-zero continuous function $f$ on $\tilde{X}$ with supp$(f) \subset \tilde{X} \setminus \overline{i(X)}$. Let $F \in \mathcal{A}$ be such that $f = \hat{F}$. Then $F \equiv 0$ since $0 = \hat{F}(\phi_x) = \phi_x(F) = F(x)$ for all $x \in X$. Hence $f \equiv 0$, a contradiction. Therefore $\overline{i(X)} = \tilde{X}$. The inclusion $\mathcal{A} \supset C(X)$ gives rise to the map $\tilde{X} = \mathcal{P}(\mathcal{A}) \to \mathcal{P}(C(X))$ given by $\phi \mapsto \phi|_{C(X)}$, $\phi \in \tilde{X}$, and since $\mathcal{P}(C(X)) \cong X$, i.e., any multiplicative functional on $C(X)$ is of
the form \( \phi_x, x \in X, \phi_x(f) = f(x), ([23, Corollary 3.4.2]) \), there is a continuous map
\( \pi : \tilde{X} \to X \) such that \( \pi \circ i \) is the identity map on \( X \).

Define \( \tilde{T} : \tilde{X} \to \tilde{X} \) by \( \tilde{T}\phi = \phi(Tf) \). \( \tilde{T} \) is continuous. For suppose \( \phi_\eta \to \phi \) in \( \tilde{X} \) for a net \( \eta \), i.e., \( \hat{f}(\phi_\eta) \to \hat{f}(\phi) \) for all \( f \in \mathcal{A} \). Then \( \hat{f}(\tilde{T}\phi_\eta) = \phi_\eta(Tf) \to \phi(Tf) = \hat{f}(\tilde{T}\phi) \) for all \( f \in \mathcal{A} \), hence \( \tilde{T} \) is continuous. Since we also have \( \tilde{T}\phi_x = \phi_{Tx} \) for all \( x \in X \), it follows that \( T \circ \pi = \pi \circ \tilde{T} \). Hence \( (\tilde{X}, \tilde{T}) \) is a dynamical system having \( (X, T) \) as a factor. If we show that \( \pi \) is one-to-one on the set
\[
M = \{ x \in X \mid T^ng \text{ is continuous at } x \text{ for all } n \in \mathbb{Z} \} \tag{4.6}
\]
of measure 1, it will follow that \( (\tilde{X}, \tilde{T}) \) is uniquely ergodic. For suppose \( \nu_1 \) and \( \nu_2 \) are two \( \tilde{T} \)-invariant measures on \( \tilde{X} \). Then \( \nu_1 \) and \( \nu_2 \) are mutually singular, see [39, Theorem 6.10]. Since \( \nu_1 \circ \pi^{-1} \) and \( \nu_2 \circ \pi^{-1} \) are \( T \)-invariant measures on \( X \) and \( (X, T) \) is uniquely ergodic with measure \( \mu \), we have \( \nu_1 \circ \pi^{-1} = \nu_2 \circ \pi^{-1} = \mu \). Let \( A \subset M \subset X \) such that \( \mu(A) > 0 \) and let \( A = A_1 \cup A_2 \) with \( A_i \subset \text{support}(\nu_i), i = 1, 2 \).

Then \( \mu(A) = \mu(A_1) + \mu(A_2) = \nu_1(\pi^{-1}A_1) + \nu_2(\pi^{-1}A_2) = 2\mu(A) \) which implies that \( \mu(A) = 0 \), a contradiction.

So it is enough to prove that for any \( x \in M \), \( \pi^{-1}(x) = \{ \phi_x \} \). This follows from the next lemma since \( \pi(\phi) = x \) if and only if \( \hat{f}(\phi) = f(x) \) for all \( f \in C(X) \).

\[ \square \]

**Lemma** Let \( X, \tilde{X}, M \) be as in the above proof. Let \( x \in M \). If \( \phi \in \tilde{X} \) is such that for all \( f \in C(X) \), \( \hat{f}(\phi) = f(x) \), then \( \phi = \phi_x \).

**Proof:** Let \( \phi \in \tilde{X} \). There exists a net \( \phi_{x_\eta} \) such that \( \phi_{x_\eta} \to \phi \) in \( \tilde{X} \). Since \( X \) is
compact, we may suppose, by going to a subnet, that \(x_\eta \rightarrow x_1\) in \(X\) for some \(x_1 \in X\).

For any \(f \in C(X)\) we have

\[
f(x) = \hat{f}(\phi) = \lim_{\eta} \hat{f}(\phi_{x_\eta}) = \lim_{\eta} f(x_\eta) = f(x_1). \tag{4.7}
\]

Therefore \(x = x_1\). Let \(F \in \mathcal{A}\). Then since \(F\) is continuous at \(x\), we have

\[
\phi(F) = \hat{F}(\phi) = \lim_{\eta} \hat{F}(\phi_{x_\eta}) = \lim_{\eta} F(x_\eta) = F(x) = \phi_x(F).
\]

Hence, \(\phi = \phi_x\).

\square

**Definition 4.2.2** Let \((X, T)\) be a dynamical system with metric \(d\). A point \(x \in X\) is called distal if \(d(T^{n_i}x, T^{n_i}y) \rightarrow 0\) for some sequence \(n_i \rightarrow \infty\) and \(y \in X\), implies \(x = y\). If every point \(x\) in \(X\) is distal then \((X, T)\) is called distal.

**Definition 4.2.3** A function \(f \in l^\infty(\mathbb{Z})\) is point distal if there exists a dynamical system \((X, T)\), a distal point \(x \in X\) and \(g \in C(X)\) such that \(f(n) = g(T^n x)\) for all \(n \in \mathbb{Z}\). If in addition the system \((X, T)\) is distal, then \(f\) is called distal.

**Definition 4.2.4** We will call a function \(f \in l^\infty(\mathbb{Z})\) almost point distal if there exists a dynamical system \((X, T)\), a distal point \(x \in X\), a point \(y \in \{T^n x \mid n \in \mathbb{Z}\}\) and \(g \in C(X)\) such that \(f(n) = g(T^n y)\) for all \(n \in \mathbb{Z}\).

It will follow from the proposition below that the transformations that we will consider are distal.
Proposition 4.2.3 ([14]) A group extension of a distal dynamical system is distal.

The affine transformations on the tori which are used as a tool to show uniform distribution of polynomials, are obtained by successive torus extensions and are therefore distal. So the functions \( f(n) = e^{2\pi ip(n)} \), \( p(n) \) a polynomial, are distal functions. The transformations which we will use to show uniform distribution of generalized polynomials are also distal, but since the corresponding functions \( g \) are not everywhere continuous, the functions \( f(n) = e^{2\pi iq(n)} \) are in general not distal. It is unclear if any of them are distal. However, Furstenberg and Weiss have shown in [19] that "most" generalized polynomials are point distal, i.e., any generalized polynomial can be changed a little in a certain way in order to be made point distal. We shall describe this change by way of an example. Let \( q(n) = [(\alpha_1 n^2 + \alpha_2 n)\alpha_3 \alpha_4 \alpha_5] \) be a generalized polynomial. By adding a constant inside each bracket of \( q(n) \), we obtain a new generalized polynomial \( q_{\theta_1, \theta_2}(n) = [(\alpha_1 n^2 + \alpha_2 n + \theta_1)\alpha_3 n + \theta_2] \alpha_4 \). It follows from [19, Proposition 14] that for all but countably many \( \theta_i, i = 1, 2 \), \( q_{\theta_1, \theta_2}(n) \) is point distal. (In [19] it is shown that these generalized polynomials are \( IP^* \)-recurrent which is shown to be the same as point distal in [17].) We will show that such a change is necessary for some generalized polynomials in order to obtain a point distal generalized polynomial. However, when the generalized polynomials that we will consider are not point distal, they are almost point distal. We need the following theorem. The statements of the theorem are in [26] taken as definitions of point distal and distal functions, respectively.
Theorem 4.2.4 \( f \in l^\infty(\mathbb{Z}) \) is point distal if and only if, whenever
\[
g(n) = \lim_{i \to \infty} f(n + h_i) \quad \text{and} \quad g(n + k_i) = \lim_{i \to \infty} f(n + k_i)
\]
pointwise, it follows necessarily that \( g = f \).

\( f \in l^\infty(\mathbb{Z}) \) is distal if and only if, whenever
\[
g_1(n) = \lim_{i \to \infty} f(n + h_i), \quad g_2(n) = \lim_{i \to \infty} f(n + l_i) \quad \text{and} \quad g_1(n + k_i) = \lim_{i \to \infty} g_2(n + k_i)
\]
pointwise, it follows necessarily that \( g_1 = g_2 \).

**Proof:** Assume \( f \in l^\infty(\mathbb{Z}) \) has either of the two properties listed above. Let \( X_f = \{S^n f \mid n \in \mathbb{Z}\} \), where \( S \) is the shift transformation \( Sf(n) = f(n+1) \) and the closure is taken in the product topology. That makes \( X_f \) a compact space so that \((X_f, S)\) is a dynamical system. Let \( F \in C(X_f) \) be the function \( F(g) = g(0) \). Then \( f(n) = F(S^n f) \). The first property is equivalent to \( f \) being distal in \((X_f, S)\) and the second property is equivalent to \((X_f, S)\) being a distal system.

Suppose now that \( f \) is (point) distal by some other dynamical system \((X, T)\), i.e., there exists a distal point \( x \in X \) and \( g \in C(X) \) such that \( f(n) = g(T^n x) \). Then let \( \phi : X \to l^\infty(\mathbb{Z}) \) be the map \( \phi(y) = (g(T^n y))_n \). Since \( \phi(T^n x) = (g(T^{m+n} x))_n = S^m f \) and \( \phi \) is continuous, we have \( \phi|_Y : Y \to X_f \) is onto, where \( Y = \{T^n x \mid n \in \mathbb{Z}\} \), and \( \phi \circ T = S \circ \phi \), i.e., \( \phi \) is a homomorphism of dynamical systems. Therefore \( \phi \) maps any distal point of \( Y \) to a distal point of \( X_f \), [37, 17]. Hence, \( f = \phi(x) \) is distal in \( X_f \) and \( X_f \) is distal if \( X \) was.

It follows from the proof that if \( f \) in Theorem 4.2.4 is distal, the limit function \( g(n) = \lim_{i \to \infty} f(n + h_i) \) will also be distal. However, if \( f \) is just point distal, the limit
function \( g \) may fail to be point distal. We will see an example of this in the next proposition.

**Proposition 4.2.5** The generalized polynomials \([\alpha n]^k\beta, \alpha, \beta \text{ irrational, } k \geq 1\), are almost point distal, but not point distal. The generalized polynomials \([\alpha n + \theta]^k\beta\) are point distal for all but countably many \( \theta \) but are not distal.

**Remark:** As a matter of fact, \( [\alpha n + \theta]^k\beta \) fails to be point distal for any \( \theta = l\alpha, l \in \mathbb{Z} \), i.e., for countably many \( \theta \).

**Proof:** Let \( f(n) = e^{2\pi i [\alpha n]^k\beta} \). We shall use Theorem 4.2.4 to show that \( f \) is not point distal. Since \((\alpha n, [\alpha n]^2\beta, \ldots, [\alpha n]^k\beta)\) is uniformly distributed \( \text{mod } 1 \) (see Lemma 3.2.2 and Theorem 3.3.1) and \([\alpha n]\beta = \alpha\beta n - \{\alpha n\}\beta\), we can choose a sequence \( h_i \to \infty \) such that

\[
([\alpha h_i], [\alpha h_i]^2\beta, \ldots, [\alpha h_i]^k\beta) \to (0, \ldots, 0). \tag{4.8}
\]

Fix \( n \in \mathbb{Z} \), and let as before \( 1_A(x, y) \) be the function which is 1 if \( \{x\} + \{y\} \geq 1 \) and 0 otherwise. Since \( \{\alpha h_i\} \to 0 \) we have for sufficiently large \( h_i \), \( 1_A(\alpha n, \alpha h_i) = 0 \) and

\[
1_A(\alpha n, -\alpha h_i) = 1_A(\alpha n, 1 - \{\alpha h_i\}) = \begin{cases} 
1 & n \neq 0 \\
0 & n = 0
\end{cases} \tag{4.9}
\]

so that

\[
[\alpha(n + h_i)]^k\beta = ([\alpha n] + [\alpha h_i] + 1_A(\alpha n, \alpha h_i))^k\beta \\
= ([\alpha n] + [\alpha h_i])^k \\
= \sum_{j=0}^{k} \binom{k}{j} [\alpha n]^{k-j}[\alpha h_i]^{j}\beta \\
\to [\alpha n]^k\beta \pmod{1} \tag{4.10}
\]
and

\[
[\alpha(n-h_i)]^k \beta = \begin{cases} 
([\alpha n] - [\alpha h_i] - 1 + 1_A(\alpha n, -\alpha h_i))^k \beta \\
(0 - [\alpha h_i])^k \beta & n = 0
\end{cases}
\]

\[
= \begin{cases} 
\sum_{j=0}^{k} \binom{k}{i} [\alpha n]^{k-i} \beta (-1)^i [\alpha h_i]^i & n \neq 0 \\
(-1)^k (1 + [\alpha h_i])^k \beta & n = 0
\end{cases}
\]

\[
\rightarrow \begin{cases} 
[\alpha n]^k \beta \pmod{1} & n \neq 0 \\
(-1)^k \beta & n = 0
\end{cases}
\]  

(4.11)

Let

\[
g(n) = \lim_{i \to \infty} f(n - h_i) = \begin{cases} 
f(n) & n \neq 0 \\
2\pi i (-1)^k \beta & n = 0
\end{cases}
\]

(4.12)

Then

\[
\lim_{i \to \infty} g(n + h_i) = \lim_{i \to \infty} f(n + h_i) = f(n)
\]

(4.13)

which shows that \( f(n) \) is not point distal. It follows from [19] that \([\alpha n + \theta]^k \beta\) is point distal for all but countably many \( \theta \). If we let \( k_i \to \infty \) be a sequence such that

\[
\left\{ \alpha k_i + \theta \right\}, \left\{ \left[ \alpha k_i + \theta \right] \beta \right\}, \left\{ \left[ \alpha k_i + \theta \right]^2 \beta \right\}, \ldots, \left\{ \left[ \alpha k_i + \theta \right]^k \beta \right\} \to (0, \ldots, 0),
\]

(4.14)

then \([\alpha n + \alpha k_i + \theta]^k \beta \to [\alpha n]^k \beta \pmod{1}\). Hence, \( f(n) \) lies in the orbit closure of the point distal function \( \psi(n) = e^{2\pi i [\alpha n + \theta]^k \beta} \) and is therefore almost point distal. Since \( f(n) \) is not point distal it follows from the observations preceding the proposition that \( \psi(n) \) is not distal.

\[
\square
\]

**Definition 4.2.5** A subset \( B \) of \( \mathbb{Z} \) is called relatively dense if there exist \( l_1, \ldots, l_k \in \mathbb{Z} \) so that \( \bigcup_{i=1}^{k} (l_i + B) = \mathbb{Z} \).
Definition 4.2.6 (Bohr) A function $f \in \ell^\infty(\mathbb{Z})$ is almost periodic if for any $\varepsilon > 0$ there exists a relatively dense subset $B_\varepsilon \subset \mathbb{Z}$ such that

$$\text{for all } \tau \in B_\varepsilon, \sup_{n \in \mathbb{Z}} |f(n + \tau) - f(n)| < \varepsilon. \quad (4.15)$$

The following theorem gives a few characterizations of almost periodicity. See for example [4, p. 10–12, 29–31] for proof of $(2) \iff (1) \iff (3)$ and [32, p. 139–140] for the proof of $(1) \iff (4)$.

**Theorem 4.2.6** Let $f \in \ell^\infty(\mathbb{Z})$. The following are equivalent.

1. $f$ is (Bohr) almost periodic.
2. $f$ is Bochner almost periodic [6]: for any sequence $h_i \in \mathbb{Z}$ there exists a subsequence $l_i$ such that the sequence $f_i(n) = f(n + l_i)$ converges uniformly in $n$.
3. $f$ can be uniformly approximated by linear combinations of functions of the form $g(n) = e^{2\pi i \alpha n}$, $\alpha \in \mathbb{R}$.
4. There exists a rotation $T$ on a compact abelian group $G$, a point $x \in G$ and a continuous function $F$ on $G$ such that $f(n) = F(T^n x)$ for all $n \in \mathbb{Z}$.

S. Bochner introduced the more general notion of almost automorphy [5].

Definition 4.2.7 (Bochner) A function $f \in \ell^\infty(\mathbb{Z})$ is almost automorphic if for any sequence $h_i \in \mathbb{Z}$, there exists a subsequence $l_i$ such that both $\lim_{i \to \infty} f(n + l_i) = g(n)$
and $\lim_{i \to \infty} g(n - l_i) = f(n)$ hold for all $n \in \mathbb{Z}$ and some function $g$, but not necessarily uniformly.

Bochner observed that any almost periodic function is almost automorphic but the converse is not true. We will show below that $\psi(n) = e^{2\pi i (\alpha n + \frac{1}{2}) \beta}$ is almost automorphic but not almost periodic. Note that for example $f(n) = e^{2\pi i \alpha n^2}$, $\alpha$ irrational, is not almost automorphic. For take a sequence $h_i \to \infty$ such that $\{\alpha h_i\} \to 0$ and $\{\alpha h_i^2\} \to \theta \neq 0$ as $h_i \to \infty$. Then $g(n) = \lim_i f(n + h_i) = \lim_i e^{2\pi i (\alpha n^2 + 2\alpha h_i + \alpha h_i^2)} \to e^{2\pi i (\alpha n + \theta)}$ and $\lim_i g(n - h_i) = \lim_i e^{2\pi i (\alpha (n - h_i)^2 + \theta)} = e^{2\pi i (\alpha n^2 + 2\theta)} \neq f(n)$. Similarly, by for example taking a sequence $h_i \to \infty$ such that $\{\alpha h_i\} \to 0$, $\{[\alpha h_i] \beta\} \to 0$ and $\{[\alpha h_i]^2 \beta\} \to \theta \neq 0$, one can show that $e^{2\pi i (\alpha n + b)^2 \beta}$, $\alpha, \beta$ irrational, $b \in \mathbb{R}$, is not almost automorphic. In [36] Veech has developed much of the theory of almost automorphic functions.

We have the following theorem [36, Theorem 2.2.1].

**Theorem 4.2.7 ([36])** A function $f \in \ell^\infty(\mathbb{Z})$ is (Bochner) almost automorphic if and only if it is Bohr almost automorphic, i.e., if for each finite subset $F \subset \mathbb{Z}$ and $\varepsilon > 0$ there exists a relatively dense set $B_\varepsilon = B_\varepsilon(F) \subset \mathbb{Z}$ such that $B_\varepsilon = -B_\varepsilon$ and

1. for any $\tau \in B_\varepsilon$, $\max_{n \in F} |f(n + \tau) - f(n)| < \varepsilon$ and
2. for any $\tau_1, \tau_2 \in B_\varepsilon$, $\max_{n \in F} |f(n + \tau_1 - \tau_2) - f(n)| < 2\varepsilon$.

**Proposition 4.2.8** Let $\alpha, \beta$ be rational numbers. The function $\psi(n) = e^{2\pi i (\alpha n + \frac{1}{2}) \beta}$ is almost automorphic but not almost periodic. The function $e^{2\pi i (\alpha n) \beta}$ is not almost
automorphic.

**Proof:** We will check that \( \psi(n) \) is Bohr almost automorphic but not almost periodic. Let \( F = \{n_1, \ldots, n_l\} \) and \( \varepsilon \) be given. Note that

\[
[\alpha(n + k) + \frac{1}{2}]\beta = [\alpha n + \frac{1}{2}]\beta + [\alpha k]\beta + 1_A(\alpha n + \frac{1}{2}, \alpha k)\beta \quad (4.16)
\]

where \( 1_A(x, y) \) is the indicator function which is 1 if \( \{x\} + \{y\} \geq 1 \) and 0 otherwise.

Let

\[
c = \min\left\{\{n_i\alpha + \frac{1}{2}\}, 1 - \{n_i\alpha + \frac{1}{2}\} \mid i = 1, \ldots, l\right\}. \quad (4.17)
\]

Since \( \alpha \) is irrational, \( c > 0 \). Let \( a = c/2 \). Define

\[
B^+_\varepsilon = \left\{k \in \mathbb{Z} \mid |e^{2\pi i[k\alpha]\beta} - 1| < \varepsilon \text{ and } \{k\alpha\} < a \right\} \quad (4.18)
\]

and

\[
B^-_\varepsilon = \left\{k \in \mathbb{Z} \mid |e^{2\pi i[k\alpha]\beta + \beta} - 1| < \varepsilon \text{ and } \{k\alpha\} > 1 - a \right\}. \quad (4.19)
\]

Then \( B^-_\varepsilon = -B^+_\varepsilon \). If \( k \in B^+_\varepsilon \) then \( 1_A(\alpha n + \frac{1}{2}, \alpha k) = 0 \) for all \( n \in F \) since

\[
\{n\alpha + \frac{1}{2}\} + \{k\alpha\} \leq \max_i \{n_i\alpha + \frac{1}{2}\} + \{k\alpha\} < (1 - 2a) + a = 1 - a. \quad (4.20)
\]

If \( k \in B^-_\varepsilon \) then \( 1_A(\alpha n + \frac{1}{2}, \alpha k) = 1 \) for all \( n \in F \) since

\[
\{n\alpha + \frac{1}{2}\} + \{k\alpha\} \geq \min_i \{n_i\alpha + \frac{1}{2}\} + \{k\alpha\} > 2a + 1 - a = 1 + a. \quad (4.21)
\]

So if \( B_\varepsilon = B^+_\varepsilon \cup B^-_\varepsilon \), then

\[
|\psi(n + k) - \psi(n)| = |e^{2\pi i[k\alpha]\beta + 1_A(\alpha n + \frac{1}{2}, \alpha k)\beta} - 1| < \varepsilon \quad (4.22)
\]
for all $k \in B_{\varepsilon}$ and $n \in F$. Note that there is no $k \in \mathbb{Z} \setminus \{0\}$ for which (4.22) holds for any $n \in \mathbb{Z}$. Therefore $\psi(n)$ is not almost periodic.

Since $[\alpha k] \beta = \alpha \beta k - \{\alpha k\} \beta$, there exists for each $\varepsilon > 0$ some $\varepsilon' > 0$ such that $k \in B_{\varepsilon}$ if both $|e^{2 \pi i \alpha \beta k} - 1| < \varepsilon'$ and $\{\alpha k\} < \varepsilon'$. Therefore it follows from for example [17, Theorem 1.21] that $B_{\varepsilon}$ is relatively dense. Note that

$$[k_{1} \alpha - k_{2} \alpha] = \begin{cases} [k_{1} \alpha] - [k_{2} \alpha] & \text{if } \{k_{1} \alpha\} > \{k_{2} \alpha\} \\ [k_{1} \alpha] - [k_{2} \alpha] - 1 & \text{if } \{k_{1} \alpha\} < \{k_{2} \alpha\} \end{cases}$$

(4.23)

So we have

$$|\psi(n + k_{1} - k_{2}) - f(n)| = |e^{2 \pi i ([k_{1} \alpha - k_{2} \alpha] \beta + 1 \delta (\alpha n + \frac{1}{2} k_{1} \alpha - k_{2} \alpha) \beta)} - 1|$$

$$= \begin{cases} |e^{2 \pi i ([k_{1} \alpha] \beta - [k_{2} \alpha] \beta) - 1}| < 2\varepsilon & \text{if } k_{1}, k_{2} \in B_{\varepsilon}^{+}, \{k_{1} \alpha\} > \{k_{2} \alpha\} \\ |e^{2 \pi i ([k_{1} \alpha] \beta - [k_{2} \alpha] \beta + \beta) - 1}| < 2\varepsilon & \text{if } k_{1}, k_{2} \in B_{\varepsilon}^{+}, \{k_{1} \alpha\} < \{k_{2} \alpha\} \\ |e^{2 \pi i ([k_{1} \alpha] \beta + \beta - (k_{2} \alpha) \beta) - 1}| < 2\varepsilon & \text{if } k_{1}, k_{2} \in B_{\varepsilon}^{-}, \{k_{1} \alpha\} > \{k_{2} \alpha\} \\ |e^{2 \pi i ([k_{1} \alpha] \beta + \beta - (k_{2} \alpha) \beta) - 1}| < 2\varepsilon & \text{if } k_{1}, k_{2} \in B_{\varepsilon}^{+}, \{k_{1} \alpha\} < \{k_{2} \alpha\} \end{cases}$$

(4.24)

which shows that $\psi(n)$ is almost automorphic.

Consider now $g(n) = e^{2 \pi i \alpha n \beta}$. Let $F = \{0\}$ and $\varepsilon > 0$. Since there is no $k \in \mathbb{Z}$ for which both $|g(k) - g(0)| = |e^{2 \pi i \alpha k \beta} - 1| < \varepsilon$ and $|g(-k) - g(0)| = |e^{2 \pi i [-\alpha k] \beta} - 1| = |e^{-2 \pi i [-\alpha k] \beta} - 1| < \varepsilon$, $g(n)$ is not almost automorphic.

\[ \Box \]

**Remark:** The same proof works for any function $e^{2 \pi i (\alpha n + \theta) \beta}$ for which $\theta \neq l \alpha$, $l \in \mathbb{Z}$.

It follows from the following theorem, due to Veech, that $\psi(n)$ is not distal.

**Theorem 4.2.9** If a function $f \in \ell^{\infty}(\mathbb{Z})$ is almost automorphic but not almost periodic then $f(n)$ is not distal.
Proof: Recall that \( X_f = \{ S^n f \mid n \in \mathbb{Z} \} \) is a compact space in the product topology, where \( S \) is the shift transformation on \( \ell^\infty(\mathbb{Z}) \). Let \( X = X_f \). Denote by \( X^X \) the space of all transformations \( X \to X \). \( X^X \) is a compact space in the product topology and \( S^n \in X^X \) for all \( n \in \mathbb{Z} \). Let \( E \) be the closure of \( \{ S^n \mid n \in \mathbb{Z} \} \) in \( X^X \). \( E \) is then a compact semi-group of transformations on \( X = X_f \). Denote the elements of \( E \) by \( S_\alpha \).

Then \( S_\alpha g(n) = \lim_\alpha g(n + \alpha) \) pointwise, where \( \alpha \) is a sequence in \( \mathbb{Z} \) and \( g \in X_f \). By [36, Theorem 2.3.1], a function \( f \) on \( \mathbb{Z} \) is almost periodic if and only if it is almost automorphic and \( S_\alpha f \) is almost automorphic for all \( S_\alpha \in E \). So if \( f \) is not almost periodic, there exists \( g = S_\alpha f \) which is not almost automorphic. Let \( \beta \) be a sequence in \( \mathbb{Z} \). Since \( f \) but not \( g \), is almost automorphic, \( \beta \) has a subsequence which we also will call \( \beta \), such that both \( S_\beta g \) and \( S_{-\beta} S_\beta g \) converge and \( S_{-\beta} S_\beta g \neq g \). Therefore, since \( S_\beta \in E \), \( E \) is not a group. By a result of Ellis [11, Proposition 5.3], \( f \) is distal if and only if \( E \) is a group of transformations on \( X_f \). Hence, \( f \) is not distal.

\[ \square \]

Remark: By [17, Chapter 9], any almost automorphic function is point distal. It therefore follows from Proposition 4.2.8 that \( [\alpha n + \frac{1}{2}] \beta \) is point distal (see Proposition 4.2.5). Note, however, that the function \( e^{2\pi i [\alpha n + \frac{1}{2}] k} \beta \) is not almost automorphic if \( k > 1 \), see discussion after Definition 4.2.7.

Another generalization of almost periodicity is the following.

**Definition 4.2.8** A function \( f : \mathbb{Z} \to \mathbb{C} \) is a bounded Besicovitch function (or Besicovitch almost periodic function) if \( f \in \ell^\infty(\mathbb{Z}) \) and for each \( \varepsilon > 0 \) there exists a trigonometric polynomial \( P(n) = P_\varepsilon(n) \) (i.e., \( P(n) \) is a linear combination of func-
tions of form $g(n) = e^{2\pi i \alpha n}, \alpha \in \mathbb{R}$) such that

$$
\|f - P\|_1 = \limsup_{N} \frac{1}{N} \sum_{n=0}^{N-1} |f(n) - P(n)| \leq \varepsilon.
$$

(4.25)

**Theorem 4.2.10 ([3])** Let $T$ be a rotation by $a$ on an abelian group $G$ such that

$\{a^n \mid n \in \mathbb{Z}\}$ is dense in $G$, and let $g : G \to \mathbb{R}$ be a Riemann integrable function (i.e., the set $\{x \in G \mid g \text{ is discontinuous at } 0\}$ has Haar measure 0). Then for any $x \in G$, the sequence $f(n) = g(T^nx)$, $n = 1, 2, \ldots$, is bounded Besicovitch.

Since any linear generalized polynomial $q(n)$ is a sum of generalized polynomials of form $[\cdots [\alpha n + \theta_1] \lambda_1 + \theta_2] \cdots \lambda_{k-1} + \theta_{k-1}] \lambda_k$ and the identity

$$
[q_1(n) + \theta]\lambda = q_1(n) \lambda + \theta \lambda - \{q_1(n) + \theta\} \lambda
$$

(4.26)

can be used repeatedly, we can find $\beta_1, \ldots, \beta_l \in \mathbb{R}$ such that 1, $\beta_1, \ldots, \beta_l$ are rationally independent and so that $q(n) = g(\beta_1 n, \ldots, \beta_l n)$ for some Riemann-integrable periodic (mod 1) function $g$ on $\mathbb{R}^l$. Hence, we have

**Corollary 4.2.11** Any function $f(n) = e^{2\pi i q(n)}$, where $q(n)$ is a generalized polynomial of degree one, is bounded Besicovitch.

Remark that in general a linear generalized polynomial is not almost periodic (see Proposition 4.2.8). The bounded Besicovitch functions have the nice property of being *good universal weights* in the following sense [33, Theorem 5] or [3].
Definition 4.2.9 A sequence $a_n$, $n = 1, 2, \ldots$, is a good universal weight if for any measure preserving system $(X, B, \mu, T)$ and every $f \in L^1(X)$,
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} a_n f(T^n x) \quad \text{exists} \quad \mu\text{-a.e.} \quad (4.27)
\]

Remark: Let us summarize the properties of the function $f(n) = e^{2\pi i q(n)}$ when $q(n)$ is linear. $f(n)$ is bounded Besicovitch (Corollary 4.2.11) and almost point distal for any linear $q(n)$. (See Proposition 4.2.5 for the case $[\alpha n] \beta$. Similarly, any linear generalized polynomial can be shown to be almost point distal). If $q(n) = [\alpha n + \theta] \beta$, where $\theta \neq l \alpha$ for all $l \in \mathbb{Z}$ then $q(n)$ is almost automorphic (Proposition 4.2.8) and point distal (Proposition 4.2.5). If $q(n) = \alpha n$ then $q(n)$ is almost periodic and distal.

In Section 4.5 we will construct uniquely ergodic systems corresponding to generalized polynomials of degree 2 to show that these generalized polynomials are uniformly distributed (mod 1). It will follow that any generalized polynomial of degree two is uniquely ergodic. In Section 4.6 and Section 4.7 we will construct uniquely ergodic systems for some generalized polynomials of higher degrees. However, it is still an open question if every generalized polynomial can be obtained this way.

Conjecture 1 Any generalized polynomial is uniquely ergodic.

This conjecture is supported by the fact that the big class of generalized polynomials which was described right before Theorem 4.2.4, is point distal. Distality together with minimality do not in general imply unique ergodicity. This was shown by Furstenberg [16] who constructed an example of a minimal distal transformation
on the 2-torus which is not uniquely ergodic. However, most often minimal point
distal systems are also uniquely ergodic.

4.3 Nilmanifold theory

We will give definitions and facts about nilmanifolds and affine transformations on
nilmanifolds, which are necessary for our next sections.

Definition 4.3.1 A group $G$ is a Lie group if it is a finite dimensional analytic
manifold with analytic group structure.

Let $G$ be a Lie group. If $H_1$ and $H_2$ are subgroups of $G$ then the commutator
subgroup $[H_1, H_2]$ denotes the group generated by all elements $h_1 h_2 h_1^{-1} h_2^{-1}$, where
$h_1 \in H_1$ and $h_2 \in H_2$. Let

$$G \supset G^1 \supset \cdots \supset G^k \supset \cdots \supset \{e\}$$

(4.28)

be the lower central series of $G$, where $G^1 = [G, G]$ and $G^k = [G, G^{k-1}], k = 2, 3, \ldots$.
Each $G^k$ is a normal subgroup, and if $G$ is simply connected, then each $G^k$ is closed
and simply connected [35, Theorem 3.18.8].

Definition 4.3.2 $G$ is called nilpotent if $G^k = \{e\}$ for some $k$, where $e$ is the unit
element of $G$. 
If $G^{k-1} \neq \{e\}$ and $G^k = \{e\}$, then we say that $G$ is a $k$-step nilpotent Lie group.

It is easy to check that any group of matrices of the form
\[
\begin{pmatrix}
1 & * \\
\vdots & \\
0 & 1
\end{pmatrix}
\]
(4.29)
is nilpotent. Actually any simply connected nilpotent Lie group is isomorphic to such a group or a closed subgroup of such a group ([35, Theorem 3.6.6]).

We will be using the groups
\[
G_{ud} = G_{ud}(k) = \left\{ \begin{pmatrix}
1 & x_{12} & \cdots & x_{1,k+1} \\
1 & \ddots & \ddots & \\
0 & 1 & x_{k,k+1} & \\
0 & \cdots & \cdots & 1
\end{pmatrix} \mid x_{ij} \in \mathbb{R}, 1 \leq i < j \leq k+1 \right\} (4.30)
\]
(ud stands for upper diagonal), and subgroups of them. Note that $\dim(G_{ud}) = \frac{1}{2} k(k+1)$, which we will denote by $m_k$. Since for each $i$, $1 \leq i \leq k$, $G^i_{ud}$ is the group of matrices
\[
\begin{pmatrix}
1 & 0 & \cdots & 0 & x_{i,i+2} & \cdots & x_{1,k+1} \\
1 & 0 & \ddots & \ddots & \ddots & \\
\vdots & \ddots & \ddots & x_{k-i,k+1} & \\
\vdots & & \ddots & \ddots & \ddots & \\
0 & \cdots & \cdots & 1 & 0 \\
0 & \cdots & \cdots & 0 & 1
\end{pmatrix}
\]
(4.31)
where $x_{ij} \in \mathbb{R}, 1 \leq l \leq k - i, i + l + 1 \leq j \leq k + 1$}, $G_{ud}$ is a $k$-step nilpotent group.

**Definition 4.3.3** A subgroup $H$ of a connected group $G$ is called uniform in $G$ if the coset space $G/H$ is compact, where $\overline{H}$ is the closure of $H$ in $G$. 
Theorem 4.3.1 ([25]) Suppose that $G$ is a connected simply-connected nilpotent Lie group and $H$ is a subgroup of $G$. In order that $H$ be uniform in $G$, it is necessary and sufficient that the quotient space $G/HG^1$ is compact, where $G^1$ is the commutant of $G$.

Definition 4.3.4 Let $G$ be a connected, simply connected nilpotent Lie group and let $\Gamma$ be a uniform discrete subgroup of $G$ such that $G/\Gamma$ is compact. Then $G/\Gamma$ is called a nilmanifold.

If $G$ is the group $G_{ud}$ and $\Gamma$ the subgroup of $G_{ud}$ consisting of all matrices having integer entries, then $G_{ud}/\Gamma$ is a nilmanifold. There exist nilpotent Lie groups which do not have a uniform discrete subgroup. An example of such a group is given in [25], where it is proved that a necessary and sufficient condition for a nilpotent Lie group $G$ to have a uniform discrete subgroup is that the Lie algebra of $G$ has rational structure constants relative to some basis (see for example [35] for definitions). However, all the groups we will consider do have a uniform discrete subgroup, and we will usually denote it by $\Gamma$.

Each $G^i\Gamma$ is a closed subgroup of $G$ since $G^i$ is a closed normal subgroup of $G$. Since $\Gamma$ is countable and $G^i\Gamma/G^i \cong \Gamma/(G^i \cap \Gamma)$ is the image of $\Gamma$ in $G/G^i$, the subgroup $G^i\Gamma/G^i$ of $G/G^i$ is discrete. We have $G/G^i\Gamma \cong (G/G^i)/(G^i\Gamma/G^i)$ and the quotient space $G/G^i\Gamma$ is the continuous image of the compact space $G/\Gamma$ under the projection $G/\Gamma \rightarrow G/G^i\Gamma$ so that $G/G^i\Gamma$ is compact. Therefore $G^i\Gamma/G^i$ is a uniform discrete
subgroup of $G/G^i$ and $G/G^i\Gamma$ is a nilmanifold. Especially important for us will be the torus group $G/G^{i_1}\Gamma$, see for example Theorem 4.3.3.

**Definition 4.3.5** Let $G/\Gamma$ be a nilmanifold. Then the torus $G/G^{i_1}\Gamma$ is called the maximal torus factor of $G/\Gamma$.

If $A$ is an automorphism of $G$ such that $A\Gamma = \Gamma$ and $a \in G$, then the affine transformation $Tx = aAx$ on $G$ induces an affine transformation $T(x\Gamma) = aA(x)\Gamma$ on $G/\Gamma$. Since $A(G^i\Gamma) \subset G^i\Gamma$, $T$ induces an affine transformation on each $G/G^i\Gamma$, which also will be denoted by $T$, $T(xG^i\Gamma) = TxG^i\Gamma$. We will now show why affine transformations on nilmanifolds are distal. Each $G^i\Gamma/G^{i+1}\Gamma$ is a torus group which acts on $G/G^{i+1}\Gamma$ by

$$g_iG^{i+1}\Gamma : yG^{i+1}\Gamma \to g_i yG^{i+1}\Gamma, \quad g_i \in G^i, \ y \in G.$$  

(4.32)

Since $G^i\Gamma/G^{i+1}\Gamma$ is in the center of $G/G^{i+1}\Gamma$, we have $g_i yG^{i+1}\Gamma = yG^{i+1}\Gamma$ only if $g_iG^{i+1}\Gamma = G^{i+1}\Gamma$, i.e. $G^i\Gamma/G^{i+1}\Gamma$ acts freely on $G/G^{i+1}\Gamma$. Let $T = aA$ be the induced affine transformation on $G/G^{i+1}\Gamma$. Then for $g_iG^{i+1}\Gamma \in G^i\Gamma/G^{i+1}\Gamma$ we have

$$T(g_i yG^{i+1}\Gamma) = aA(g_i yG^{i+1}\Gamma) = a(Ag_i)(Ay)G^{i+1}\Gamma = (Ag_i)TyG^{i+1}\Gamma.$$  

(4.33)

So $T(g_i yG^{i+1}\Gamma) = g_i T(yG^{i+1}\Gamma)$ if

$$A(g_iG^{i+1}\Gamma) = g_iG^{i+1}\Gamma \quad \text{for all} \quad g_iG^{i+1}\Gamma \in G^i\Gamma/G^{i+1}\Gamma.$$  

(4.34)

In the case (4.34) holds, $(G/G^{i+1}\Gamma, T)$ is a group extension of $(G/G^i\Gamma, T)$ (see Definition 4.1.3), since $(G/G^{i+1}\Gamma)/(G^i\Gamma/G^{i+1}\Gamma) \cong G/G^i\Gamma$. Furthermore, if $T = aA$ is
an affine transformation on $G/\Gamma$ for which (4.34) holds for all $i = 1, \ldots, k - 1$, where $G$ a $k$-step nilpotent group, then $(G/\Gamma, T)$ is the result of a finite sequence of group extensions, starting from the trivial one point space. It follows from Proposition 4.2.3 that such systems are distal. Especially, if $T$ is a rotation, then this is the case. By refining the lower central series of $G$ to a series $G \supset G_1 \supset \cdots \supset G_m \supset \{e\}$, where $m > k$ so that $G_i/G_{i+1} \cong \mathbb{R}$ (see Theorem 4.3.9), it can be proved similarly that any affine transformation on $G/\Gamma$ is distal. So we have

**Proposition 4.3.2** Any affine transformation $T = aA$ on a nilmanifold $G/\Gamma$ is distal.

Since $G$ is a nilpotent Lie group, $G$ is unimodular, i.e., left and right Haar measures on $G$ coincide. The normalized Haar measure on $G$ induces a unique $G$-invariant probability measure $\mu$ on $G/\Gamma$ which is also $T$-invariant [27]. We will need some properties of this measure. Suppose $(X, T)$ is a dynamical system and that $H$ is a compact abelian group which acts freely on $X$, and such that

$$T(hx) = \sigma(h)Tx \quad \text{for all } x \in X, \ h \in H. \quad (4.35)$$

Let $S$ be the induced transformation on the orbit space $X/H$ such that $S(xH) = (Tx)H$. For any $f \in C(X)$ we can define a function $\overline{f} \in C(X/H)$ by

$$\overline{f}(\overline{x}) = \int_H f(hx)dh, \quad (4.36)$$

where $dh$ denotes Haar measure on $H$. Let $\nu$ be an $S$-invariant measure on $X/H$, and define a measure $\mu$ on $X$ by

$$\int_X f(x)d\mu = \int_{X/H} \overline{f}(\overline{x})d\nu = \int_{X/H} \left( \int_H f(hx)dh \right) d\nu. \quad (4.37)$$
The measure $\mu$ is $T$-invariant. For, using the invariance of Haar measure $dh$ and of the measure $\nu$, we have

$$\int_X f(Tx)d\mu = \int_{X/H} \left( \int_H f(Thx)dh \right) d\nu = \int_{X/H} \left( \int_H f(\sigma(h)Tx)dh \right) d\nu$$

$$= \int_{X/H} \left( \int_H f(hTx)dh \right) d\nu = \int_{X/H} \overline{f}(S\pi) d\nu = \int_{X/H} \overline{f}(\pi) d\nu = \int_X f(x) d\mu. \quad (4.38)$$

So if $(X,T)$ is uniquely ergodic, then $\mu$ is the unique invariant measure. Now, let $X = G/G^{i+1} \Gamma$, $T$ an affine transformation on $G/G^{i+1} \Gamma$ and $H = G^{i} \Gamma / G^{i+1} \Gamma$. Then we have $X/H \cong G/G^i \Gamma$. Let $\nu = \mu_i$ be the unique invariant measure on $G/G^i \Gamma$, induced by Haar measure on $G/G^i$. From (4.33) we see that the condition (4.35) is satisfied. Therefore the measure $\mu_{i+1}$ on $G/G^{i+1} \Gamma$ defined by

$$\int_{G/G^{i+1} \Gamma} f(x) d\mu_{i+1} = \int_{G/G^i \Gamma} \left( \int_{G^i \Gamma / G^{i+1} \Gamma} f(hx)dh \right) d\mu_i \quad (4.39)$$

is the unique invariant measure on $G/G^{i+1} \Gamma$. Note that this is true for any $i = 1, \ldots, k - 1$ if $G$ is a $k$-step nilpotent group.

The following two theorems give a useful criterion for an affine transformation to be uniquely ergodic.

**Theorem 4.3.3** ([29]) The affine transformation $T(x \Gamma) = aA(x) \Gamma$ on $G/\Gamma$ is uniquely ergodic if and only if the induced transformation on the maximal torus factor $G/G^1 \Gamma$ of $G/\Gamma$ is uniquely ergodic. Moreover, minimality is equivalent to unique ergodicity for such transformations.
Theorem 4.3.4 ([22, 21])  Let $G$ be a compact metric abelian and connected group. Then the affine transformation $T = aA$ on $G$ is uniquely ergodic if and only if
\[
\bigcap_{n=0}^{\infty} B^n G = \{ e \} \quad \text{and} \quad [a, BG] = G,
\] (4.40)
where $B$ is the endomorphism of $G$ defined by $B(x) = x^{-1} A(x)$ and $[a, BG]$ denotes the smallest closed subgroup of $G$ containing $a$ and $BG$.

So if $G = K^d$, then $T = \tilde{\alpha} A$ is uniquely ergodic if and only if there exists $n > 0$ such that $(A - I)^n K^d = \{ 0 \}$ and $[\tilde{\alpha}, (A - I) K^d] = K^d$, where $I$ is the identity matrix.

Note that any torus $K^d$ can be seen as a nilmanifold $H^d/\Gamma$, where $H^d$ is the group of matrices
\[
H^d = \left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 & x_1 \\ & \ddots & \ddots & \vdots \\ & & 0 & \vdots \\ 1 & x_d & & 1 \end{pmatrix} \mid x_i \in \mathbb{R}, i = 1, \ldots, d \right\}
\] (4.41)
and $\Gamma$ is the subgroup of $H^d$ with matrices having integer coordinates. This way we may say that the usual polynomials are coming from affine transformations on nilmanifolds. If $T_1$ is an affine transformation on a nilmanifold $G/\Gamma$ and $T_2$ is an affine transformation on the torus $K^d$, then the direct product $(G/\Gamma \times K^d, T_1 \times T_2)$ is isomorphic to $(G_1/\Gamma_1, S)$, where
\[
G_1 = \begin{pmatrix} G & 0 \\ 0 & H^d \end{pmatrix}
\] (4.42)
is a nilpotent Lie group, $\Gamma_1$ is the corresponding uniform discrete subgroup and $S = T_1 \times T_2$ is an affine transformation on the nilmanifold $G_1/\Gamma_1$. So by Theorem 4.3.3, $(G/\Gamma \times K^d, T_1 \times T_2)$ is uniquely ergodic if and only the induced affine transformation $T_1 \times T_2$ on the torus $G/G^1 \Gamma \times K^d$ is uniquely ergodic.
We will now show how uniquely ergodic transformations on \( G_{ud}/\Gamma \) give rise to uniformly distributed \( \text{mod} 1 \) sequences in \( \mathbb{R}^m \). Define functions \( \theta_{ij} : G_{ud} \to \mathbb{R} \), \( 1 \leq i < j \leq k + 1 \), successively by

\[
\theta_{ij}(x) = x_{ij} - \sum_{l=i+1}^{j-1} x_{il}[\theta_{lj}(x)], \quad 1 \leq i < j \leq k + 1.
\]  

(4.43)

Lemma 4.3.5 Let \( x = (x_{ij}) \in G_{ud} \) and \( \gamma = (\gamma_{ij}) \in \Gamma \) and let \( \theta_{ij} \) be given by (4.43). Then

\[
\theta_{ij}(x\gamma) = \theta_{ij}(x) + \theta_{ij}(\gamma) - \sum_{l=i+1}^{j-1} \theta_{il}(\gamma)[\theta_{lj}(x)], \quad 1 \leq i < j \leq k + 1.
\]  

(4.44)

So each \( \theta_{ij} \) gives rise to a function \( \overline{\theta}_{ij} : G_{ud}/\Gamma \to K \), \( \overline{\theta}_{ij}(x\Gamma) = \theta_{ij}(x) \pmod{1} \), which is continuous outside a set of measure \( \theta \).

Proof: The proof is by induction on \( j - i \). We have

\[
\theta_{i,i+1}(x\gamma) = x_{i,i+1} + \gamma_{i,i+1} = \theta_{i,i+1}(x) + \theta_{i,i+1}(\gamma)
\]  

(4.45)

so (4.44) is true when \( j - i = 1 \). Suppose (4.44) is true when \( j - i < m \) and let \( j - i = m \). Then

\[
\begin{align*}
\theta_{ij}(x\gamma) &= x\gamma - \sum_{l=i+1}^{j-1} (x\gamma)_{il}[\theta_{lj}(x\gamma)] \\
&= x_{ij} + \gamma_{ij} + \sum_{l=i+1}^{j-1} x_{il}\gamma_{lj} \\
&\quad - \sum_{l=i+1}^{j-1} \left( x_{il} + \gamma_{il} + \sum_{s=i+1}^{l-1} x_{is}\gamma_{sl} \right) \left[ \theta_{lj}(x) + \theta_{lj}(\gamma) - \sum_{r=l+1}^{j-1} \theta_{lr}(\gamma)[\theta_{lj}(x)] \right] \\
&= x_{ij} - \sum_{l=i+1}^{j-1} x_{il}[\theta_{lj}(x)] + \gamma_{ij} - \sum_{l=i+1}^{j-1} \gamma_{il}[\theta_{lj}(\gamma)]
\end{align*}
\]
\[
\begin{align*}
&+ \sum_{l=i+1}^{j-1} x_{il} \gamma_{il} - \sum_{l=i+1}^{j-1} x_{il} \theta_{lj}(\gamma) - \sum_{l=i+1}^{j-1} \sum_{s=i+1}^{l-1} x_{is} \gamma_{sl} \theta_{lj}(\gamma) \\
&+ \sum_{l=i+1}^{j-1} x_{il} \sum_{r=l+1}^{j-1} \theta_{lr}(\gamma) [\theta_{rj}(x)] + \sum_{l=i+1}^{j-1} \sum_{s=i+1}^{l-1} x_{is} \gamma_{sl} \sum_{r=l+1}^{j-1} \theta_{lr}(\gamma) [\theta_{rj}(x)] \\
&- \sum_{l=i+1}^{j-1} \sum_{s=i+1}^{l-1} x_{is} \gamma_{sl} [\theta_{lj}(x)] \\
&+ \sum_{l=i+1}^{j-1} \gamma_{il} \sum_{r=l+1}^{j-1} \theta_{lr}(\gamma) [\theta_{rj}(x)] - \sum_{l=i+1}^{j-1} \gamma_{il} [\theta_{lj}(x)] \\
&= \theta_{ij}(x) + \theta_{ij}(\gamma) \\
&+ \sum_{l=i+1}^{j-1} x_{il} \sum_{s=l+1}^{j-1} \gamma_{is} \theta_{sj}(\gamma) - \sum_{l=i+1}^{j-1} \sum_{s=i+1}^{l-1} x_{is} \gamma_{sl} \theta_{lj}(\gamma) \\
&+ \sum_{l=i+1}^{j-1} \sum_{r=l+1}^{j-1} x_{il} \left( \gamma_{lr} - \sum_{s=l+1}^{r-1} \gamma_{ls} \theta_{sr}(\gamma) \right) [\theta_{rj}(x)] \\
&- \sum_{l=i+1}^{j-1} \sum_{s=i+1}^{l-1} x_{is} \gamma_{sl} [\theta_{lj}(x)] + \sum_{l=i+1}^{j-1} \sum_{s=i+1}^{l-1} x_{is} \gamma_{sl} \sum_{r=l+1}^{j-1} \theta_{lr}(\gamma) [\theta_{rj}(x)] \\
&- \sum_{l=i+1}^{j-1} \left( \gamma_{il} - \sum_{s=i+1}^{l-1} \gamma_{is} \theta_{sl}(\gamma) \right) [\theta_{lj}(x)].
\end{align*}
\]

In the last step we used that \( \theta_{lj}(\gamma) = \gamma_{lj} - \sum_{s=i+1}^{j-1} \gamma_{ls} \theta_{sj}(\gamma) \). By checking the limits of summations and rewriting them, we see that line (4.47) and also line (4.48) together with line (4.49) sum up to 0. Hence

\[
\theta_{ij}(x\gamma) = \theta_{ij}(x) + \theta_{ij}(\gamma) - \sum_{l=i+1}^{j-1} \theta_{dl}(\gamma) [\theta_{lj}(x)],
\]

which was to be proved. It is only left to prove that \( \bar{\theta}_{ij} \) is continuous outside a set of measure 0. Clearly, \( \bar{\theta}_{ij} \) is continuous outside the set of \( x \) for which \( \theta_{ij}(x) \equiv 0 \) (mod 1), \( l = i + 1, \ldots, j - 1 \). So it is enough to check that the set

\[
E_{ij} = \{ x\Gamma \in G_{ud}/\Gamma \mid \theta_{ij}(x) \equiv 0 \pmod{1} \}
\]
has measure 0 for each \( i, j \). This follows since for each fixed \( \mathbf{n} = (n_{i+1,j}, \ldots, n_{j-1,j}) \in \mathbb{Z}^{j-i-1} \), the set

\[
D_{ij}(\mathbf{n}) = \{ x\Gamma \mid x_{ij} - \sum_{l=i+1}^{j-1} n_{lj}x_{il} \equiv 0 \pmod{1} \}
\] (4.53)

has measure 0 and \( E_{ij}(\mathbf{n}) \subset \bigcup_{\mathbf{n} \in \mathbb{Z}^{j-i-1}} D_{ij}(\mathbf{n}) \).

\[ \square \]

\textbf{Lemma 4.3.6} Let \( \tilde{\theta}(\mathbf{x}) = (\theta_{12}(\mathbf{x}), \ldots, \theta_{k,k+1}(\mathbf{x})) \) and let \( \mathbf{l} = (l_{12}, \ldots, l_{k,k+1}) \in \mathbb{Z}^{m_k} \), \( m_k = \frac{1}{2}k(k + 1) \). Let

\[
g(\mathbf{x}\Gamma) = \exp\left(2\pi i \sum_{i=1}^{k} \sum_{j=i+1}^{k+1} l_{ij}\theta_{ij}(\mathbf{x})\right).
\] (4.54)

Then \( \int_{G_{ud}/\Gamma} g d\mu = 0 \) if \( \mathbf{l} \neq 0 \).

\textbf{Proof:} Let \( s \) be the maximal number for which there exists \( l_{ij} \neq 0 \) with \( j - i = s \). Let

\[
\theta_{ij}(\mathbf{x}) = x_{ij} - \sum_{l=i+1}^{j-1} x_{il}[\theta_{ij}(\mathbf{x})] = x_{ij} - \rho_{ij}(\mathbf{x}).
\] (4.55)

Note that \( \theta_{ij}(\mathbf{x}) \) is constant on \( G^{j+1}_{ud} \) and that \( \rho_{ij}(\mathbf{x}) \) is constant on \( G^{j-i}_{ud} \). Therefore,

\[
\int_{G^{j+1}_{ud}} g(hx)dh = g(x) \quad \text{for all} \quad l > s, \quad \text{so that by (4.39)} \quad \text{we have}
\]

\[
\int_{G_{ud}/\Gamma} g(x) d\mu = \int_{G_{ud}/G^{j+1}_{ud}\Gamma} g(x) d\mu = \int_{G_{ud}/G^{s}_{ud}\Gamma} \left( \int_{G^{s+1}_{ud}} g(hx)dh \right) d\mu_s.
\] (4.56)

If \( h \in G^{s}_{ud}\Gamma/G^{s+1}_{ud}\Gamma \), then \( \theta_{ij}(hx) = \begin{cases} h_{ij} + x_{ij} + \rho(x) & \text{if} \quad j - i = s \\
_{ij}(x) & \text{if} \quad j - i \neq s \end{cases} \) so that

\[
g(hx) = \exp 2\pi i \left( \sum_{i} l_{i,i+s}h_{i,i+s} + \psi(x) \right).
\] (4.57)
Therefore
\[
\int_{G_{ud}/G_{ud}^{+1}} g(hx) dh = \int_{G_{ud}/G_{ud}^{+1}} \exp 2\pi i \left( \sum_i t_{i,i+s} h_{i,i+s} + \psi(x) \right) dh = 0. \quad (4.58)
\]
Hence, \( \int_{G_{ud}/G} g(x) d\mu = 0 \).

As was remarked earlier we have the following lemma.

**Lemma 4.3.7** Let \((X, T)\) be a uniquely ergodic system with measure \(\mu\), and let \(g\) be a function on \(X\) which is continuous outside a set of measure 0. Then
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} g(T^n x) = \int_X g d\mu. \quad (4.59)
\]

**Proof:** Let \(A\) be the smallest uniformly closed and conjugation closed algebra generated by \(C(X) \cup \{T^n g \mid n \in \mathbb{Z}\}\). We showed in the proof of Theorem 4.2.2 that there is a uniquely ergodic dynamical system \((\tilde{X}, \tilde{T})\) such that \(C(\tilde{X}) \cong A\) and for each \(x \in X\), \(g(x) = \hat{g}(\phi_x)\), \(\hat{g} \in C(\tilde{X})\). The invariant measure \(\tilde{\mu}\) on \(\tilde{X}\) is so that \(\int_{\tilde{X}} \hat{f} d\tilde{\mu} = \int_X f d\mu\) for all \(f \in A\). Therefore, we have for any \(x \in X\),
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} g(T^n x) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \hat{g}(T^n \phi_x) = \int_{\tilde{X}} \hat{g} d\tilde{\mu} = \int_X g d\mu, \quad (4.60)
\]
which was to be shown.

**Theorem 4.3.8** If \(T\) is a uniquely ergodic transformation on \(G_{ud}/\Gamma\), then
\[
\hat{\theta}(T^n x) = (\theta_{12}(T^n x), \ldots, \theta_{k,k+1}(T^n x)), \ n = 1, 2, \ldots \quad (4.61)
\]
is uniformly distributed \((\mod 1)\) for all \(x \in G_{ud}/\Gamma\).
Proof: We will use Weyl’s criterion (Theorem 1.2.1). So let
\[ f(x_{12}, \ldots, x_{k,k+1}) = \exp\left(2\pi i \sum_{i=1}^{k} \sum_{j=i+1}^{k+1} l_{ij} x_{ij}\right), \] (4.62)
and define \( g : G/\Gamma \to \mathbb{C} \) to be the function \( g(x) = f(\theta_{12}(x), \ldots, \theta_{k-1,k}(x)) \). Since \( f \) is continuous, \( g \) is continuous outside a set of measure 0 by Lemma 4.3.5, and by Lemma 4.3.7,
\[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\theta_{12}(T^n x), \ldots, \theta_{k-1,k}(T^n x)) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} g(T^n x) = \int_{G_{ad}/\Gamma} g d\mu. \] (4.63)
Since \( \int_{G_{ad}/\Gamma} g d\mu = 0 \) by Lemma 4.3.6, \( \tilde{\theta}(T^n x) \) is uniformly distributed (mod 1) by Weyl’s criterion.

\[ \square \]

Remark: The function \( f(n) = e^{2\pi i \theta_{ij}(T^n x)} \) is uniquely ergodic by Theorem 4.2.2 since \( g(x) = e^{2\pi i \theta_{ij}(x)} : G/\Gamma \to K \) is continuous outside a set of measure 0.

Definition 4.3.6 ([25]) Let \( G \) be a connected, simply-connected nilpotent Lie group and suppose \( G \) has a system of one-parameter subgroups \( x_1(t), x_2(t), \ldots, x_k(t) \) such that

1. the map \((t_1, t_2, \ldots, t_k) \mapsto g = x_1(t_1) \cdot x_2(t_2) \cdots x_k(t_k)\) is a diffeomorphism of \( \mathbb{R}^k \) onto \( G \).

2. each subset \( \{x_i(t_i) \cdot x_{i+1}(t_{i+1}) \cdots x_k(t_k) \mid t_j \in \mathbb{R}, j = i, \ldots, k\} \) for \( i = 1, \ldots, k \) is a closed normal subgroup \( G_i \) of \( G \).

3. the factor groups \( G_i/G_{i+1}, i = 1, \ldots, k \) (and \( G_{k+1} = \{e\} \)) are each \( \mathbb{R} \).
Then the subgroups $x_1(t), x_2(t), \ldots, x_k(t)$ are called a system of global coordinates of $G$. (In \cite{25} and \cite{2} $x_1(t), x_2(t), \ldots, x_k(t)$ are called a system of (canonical) coordinates of the second kind of $G$.)

**Theorem 4.3.9** (\cite{25}) Let $G$ be a connected, simply connected nilpotent Lie group and $\Gamma$ a uniform discrete subgroup of $G$. Then there exists a system of global coordinates $x_1(t), x_2(t), \ldots, x_k(t)$ of $G$ such that if $\gamma_i = x_i(1)$, $i = 1, \ldots, k$, then any $\gamma \in \Gamma$ can be represented in the form $\gamma = \gamma_1^{n_1} \gamma_2^{n_2} \cdots \gamma_k^{n_k}$, where $n_i \in \mathbb{Z}, i = 1, \ldots, k$.

**Example:** Consider the Heisenberg group

$$N_1 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\} \quad (4.64)$$

which is a 3-dimensional 2-step nilpotent Lie group. Then the one-parameter subgroups

$$x_1(t) = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_2(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, \quad x_3(t) = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (4.65)$$

is a system of global coordinates of $H_1$. The discrete group

$$\Gamma_1 = \left\{ \begin{pmatrix} 1 & n & l \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix} \mid n, m, l \in \mathbb{Z} \right\} \quad (4.66)$$

is a uniform subgroup of $N_1$ and we have

$$\begin{pmatrix} 1 & n & l \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix} = x_1(1)^n \cdot x_2(1)^m \cdot x_3(1)^{l-nm}. \quad (4.67)$$
Definition 4.3.7 ([2]) A subset $F \subset G$ is a fundamental domain for $G/\Gamma$ if

(a) each point of $G/\Gamma$ is represented by exactly one point of $F$, and

(b) the natural projection $\pi : G \to G/\Gamma$ is continuous on $F$ and is a homeomorphism when restricted to the interior of $F$.

For the nilmanifold $G_{ud}/\Gamma$, $G_{ud}$ given by (4.30), we have

Lemma 4.3.10 The subset $F$ of $G_{ud}$ of matrices having $0 \leq x_{ij} < 1$, $1 \leq i < j \leq k + 1$, is a fundamental domain for $G_{ud}/\Gamma$, and the map $\phi = (\phi_{ij}) : G_{ud}/\Gamma \to F$, where $\phi_{ij}(x\Gamma) = \{\theta_{ij}(x)\}$ (fractional part of $\theta_{ij}(x)$), $\theta_{ij}$ given by (4.43), is the inverse map of $\pi|_F : F \to G_{ud}/\Gamma$, $\pi$ the natural projection $G_{ud} \to G_{ud}/\Gamma$.

Remark: Note that we use this fundamental domain $F$ of $G_{ud}/\Gamma$ in order to obtain sequences $\phi_{ij}(x_n\Gamma)$, $n = 1, 2, \ldots$, in $[0, 1)$ which are uniformly distributed (mod 1). This is the only fundamental domain of $G_{ud}/\Gamma$ that we will use.

Proof: Since the natural projection $G_{ud} \to G_{ud}/\Gamma$ is a continuous open map, we need only prove that each point of $G_{ud}/\Gamma$ is represented by exactly one point of $F$ in order to show that $F$ is a fundamental domain. We will show that for each $x = (x_{ij}) \in G_{ud}$ there exists a unique $\tilde{\gamma}(x) = (\gamma_{ij}(x)) \in \Gamma$ so that $x\tilde{\gamma}(x) \in F$ and that $\gamma_{ij}(x) = -[\theta_{ij}(x)]$. This will be done by induction on $j - i$. Let $y_{ij} = x\tilde{\gamma}(x)$. When $j = i + 1$, then $y_{i,i+1} = x_{i,i+1} + \gamma_{i,i+1}(x)$. So $y_{i,i+1} \in [0, 1)$ if and only if $\gamma_{i,i+1} = -[x_{i,i+1}] = -[\theta_{i,i+1}(x)]$. We have

$$y_{ij} = \sum_{s=i}^{j} x_{is}\gamma_{sj}(x) = x_{ij} + \sum_{s=i+1}^{j-1} x_{is}\gamma_{sj}(x) + \gamma_{ij}(x).$$ (4.68)
Suppose that $\gamma_{sj}(x) = -[\theta_{sj}(x)]$ for $s > i$. Then

$$y_{ij} = x_{ij} - \sum_{s=i+1}^{j-1} x_{is}[\theta_{sj}(x)] + \gamma_{ij}(x)$$

(4.69)

so that $y_{ij} \in [0, 1)$ if and only if $\gamma_{ij}(x) = -[x_{ij} - \sum_{s=i+1}^{j-1} x_{is}[\theta_{sj}(x)]] = -[\theta_{ij}(x)]$, which was to be proved. Since

$$\left(x_{\tilde{\gamma}}(x)\right)_{ij} = x_{ij} - [\theta_{ij}(x)] - \sum_{s=i+1}^{j-1} x_{is}[\theta_{sj}(x)] = \theta_{ij}(x) - [\theta_{ij}(x)] = \{\theta_{ij}(x)\},$$

(4.70)

the map $\phi(x) = \{\{\theta_{ij}(x)\}\}$ is the inverse map of $\pi|_F$.

$\square$

**Remark:** The first part of Lemma 4.3.10 is a special case of the following more general result which is proved in [2, p. 54].

*Let $G$ be a connected simply-connected nilpotent Lie group and $\Gamma$ a uniform discrete subgroup of $G$. Let $(t_1, \ldots, t_k) \mapsto x_1(t_1) \cdots x_k(t_k)$ be global coordinates of $G$ which carries the integral lattice $\mathbb{Z}^k$ onto $\Gamma$ and let $x_1(a_1) \cdots x_k(a_k)$ be a point in $G$. Then

$$\{x_1(b_1) \cdots x_k(b_k) \mid a_i \leq b_i < a_i + 1, i = 1, \ldots, k\}$$

(4.71)

is a fundamental domain for $G/\Gamma$.*

Note that if $G = G_{ud}$, the fundamental domain $F$ described in Lemma 4.3.10 is the one with $a_i = 0$ for all $i$ and this is one of the easiest fundamental domain to describe in terms of the coordinates of the matrices.

**Example:** If $G$ is the Heisenberg group $N_1$, let $n, m, l \in \mathbb{Z}$ be such that

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & n & l \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x + n & z + xm + l \\ 0 & 1 & y + m \\ 0 & 0 & 1 \end{pmatrix} \in F.$$  

(4.72)
Then \( n = -[x] \), \( m = -[y] \), \( l = -[z - [y]x] \) and \( \phi_{12}(x\Gamma) = \{x\} \), \( \phi_{23}(x\Gamma) = \{y\} \) and \( \phi_{13}(x\Gamma) = \{z - [y]x\} \). Note that \((x, y, z) \mapsto (x, y, z - [y]x)\) is discontinuous when \( y \equiv 0 \pmod{1} \).

By combining Theorem 4.3.8 and Lemma 4.3.10 and by identifying the fundamental domain \( F \) by \([0, 1) m_k \), \( m_k = \frac{1}{2} k(k + 1) \), we have the following corollary, where \( \phi = (\phi_{ij}) = (\{\theta_{ij}\}) \) and \( \theta_{ij} \) is given by (4.43).

**Corollary 4.3.11** If \( T \) is a uniquely ergodic transformation on \( \text{Gud}/\Gamma \), then the sequence \( \phi(T^n x) \), \( n = 1, 2, \ldots \), is uniformly distributed (mod 1) in \( \mathbb{R}^{m_k} \), \( m_k = \frac{1}{2} k(k + 1) \), for any \( x \in \text{Gud}/\Gamma \), where \( \phi(T^n x) \) is the sequence in the fundamental domain \( F \) of \( \text{Gud}/\Gamma \) uniquely determined by the sequence \( T^n x \) in \( \text{Gud}/\Gamma \).

**Remark:** In the next section fundamental domains and the corresponding function \( \phi \) are given for some nilmanifolds \( G/\Gamma \), where \( G \) is a subgroup of \( \text{Gud} \).

The notion of uniform distribution mod 1 extends in a natural way to more general settings. In [1] uniform distribution mod a uniform discrete subgroup of a solvable Lie group is defined and studied. The following definitions, which we have adjusted to our nilpotent case, are taken from [1]. Let \( G \) be a connected simply-connected nilpotent Lie group, \( \Gamma \) a uniform discrete subgroup of \( G \) and let \( x_1(t), x_2(t), \ldots, x_k(t) \) be global coordinates \( G \). By a rectangle in \( G \) we shall mean a subset of \( G \) of the form

\[
\left\{ \left( x_1(t_1), x_2(t_2), \ldots, x_k(t_k) \right) \mid |t_i - a_i| \leq c_i, i = 1, \ldots, k \right\}
\]

(4.73)

where \( x_1(a_1) \cdots x_k(a_k) \) is a point in \( G \) and \( c_1, \ldots, c_k \) are arbitrary positive real numbers. A subset \( A \) of \( G/\Gamma \) is called a small rectangle if (1) \( A \) is evenly covered by the
natural map $G \to G/\Gamma$ and (2) there exists a rectangle $B$ in $G$ such that $G \to G/\Gamma$ carries $B$ homeomorphically onto $A$.

**Definition 4.3.8 ([1])** A sequence $x_n, n = 1, 2, \ldots$ in $G$ is uniformly distributed mod $\Gamma$ if

$$\lim_{N \to \infty} \frac{1}{N} \text{card}(\{n \leq N \mid x_n \Gamma \in V\}) = \mu(V)$$

(4.74)

for any small rectangle $V$ in $G/\Gamma$, where $\mu$ is the probability measure on $G/\Gamma$ induced by Haar measure on $G$.

Note that if $G = \mathbb{R}^d$, uniform distribution mod $\mathbb{Z}^d$ is just uniform distribution mod 1. The following theorem is an analog of Theorem 1.2.1.

**Theorem 4.3.12 ([1])** The following three conditions are equivalent.

(a) $x_n, n = 1, 2, \ldots$ is uniformly distributed mod $\Gamma$.

(b) Whenever $f : G/\Gamma \to \mathbb{C}$ is continuous, we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(x_n \Gamma) = \int_{G/\Gamma} f d\mu.$$  

(4.75)

(c) Whenever $I$ is a closed subspace of $L^2(G/\Gamma, \mu)$ that is invariant and irreducible under the action of $G$ and $f \in I$ is a non-constant $C^\infty$-function on $G/\Gamma$, then

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(x_n \Gamma) = 0.$$  

(4.76)
Note that Theorem 4.3.12 shows that $T$ is a uniquely ergodic transformation on a nilmanifold $G/\Gamma$ if and only if the sequence $x_n = T^nx$ is uniformly distributed in $G \mod \Gamma$ for any $x \in G/\Gamma$. We have already seen in Corollary 4.3.11 that if $G = G_{ud}$ then the sequence $\phi(T^nx), n = 1, 2, \ldots,$ is uniformly distributed (mod 1) if $T$ is uniquely ergodic, where $\phi(T^nx)$ is the representative of $T^nx \in G_{ud}/\Gamma$ in the fundamental domain $F$ of $G_{ud}/\Gamma$ defined in Lemma 4.3.10.

4.4 Examples of some nilmanifolds and their fundamental domains

In the previous section we showed how uniquely ergodic transformations on $G_{ud}/\Gamma$ give rise to uniformly distributed sequences in $\mathbb{R}^{m_k}$, $m_k = \frac{1}{2}k(k + 1)$ (see Theorem 4.3.8 and Corollary 4.3.11). This was done by finding a function $\phi$ from $G_{ud}/\Gamma$ to the fundamental domain $F = F_{ud}$ of $G_{ud}/\Gamma$ which is the inverse of $\pi|_{F_{ud}} : F_{ud} \to G_{ud}/\Gamma$ (the natural projection) and by identifying $F_{ud}$ with $[0, 1)^{m_k}$.

In later sections we will often be using nilmanifolds $G/\Gamma$ where $G$ is a subgroup of $G_{ud}$ (and $\Gamma$ is a uniform discrete subgroup of $G$). The purpose of this section is therefore to find an appropriate fundamental domain $F$ for each such nilmanifold $G/\Gamma$ and also to find the corresponding map $\phi : G/\Gamma \to F$ which is the inverse of the projection $\pi|_F : F \to G/\Gamma$. For, as in the case when $G = G_{ud}$, it then follows that for any $x \in G/\Gamma$, $\phi(T^nx)$ is uniformly distributed (mod 1) in $\mathbb{R}^d$ for some $d \in \mathbb{N}$ if $T$ is a uniquely ergodic transformation on $G/\Gamma$.

Example 1
Let
\[ N_k = \left\{ \begin{pmatrix} 1 & x_1 & \cdots & x_k & z \\ 1 & 0 & \ddots & & y_1 \\ & & \ddots & & \\ 0 & & & 1 & y_k \\ & & & & 1 \end{pmatrix} \mid x_i, y_i \in \mathbb{R}, i = 1, \ldots, k \right\} \] (4.77)
be the \((2k + 1)\)-dimensional \textit{Heisenberg group}, and let \(\Gamma_k\) be the uniform discrete subgroup of matrices having integer coordinates. We will denote the elements of \(N_k\) by
\[ (x, y, z), \] (4.78)
where \(y = (y_1, \ldots, y_k)\) and \(x = (x_1, \ldots, x_k)\). \(N_k\) is an example of a subgroup of \(G_{ud}\) for which there are no relations between the non-zero coordinates of the matrices. Therefore, the set
\[ F = F_{ud} \cap N_k = \{ (x, y, z) \mid 0 \leq x_i, y_i < 1, i = 1, \ldots, k \} \] (4.79)
is a fundamental domain for \(N_k/\Gamma_k\) and \(\phi : N_k/\Gamma_k \to F\) is the map \(\phi(x, y, z) = (\{\theta_{ij}(x, y, z)\})\), where \(\theta_{1i}(x, y, z) = x_{i-1}, \theta_{ik+1}(x, y, z) = y_{i-1}, i = 2, \ldots, k + 1\) and \(\theta_{1k+1}(x, y, z) = z - \sum_{i=1}^{k} x_i[y_i]\).

**Example 2**

Let \(a_{ij}, b_{ij}, c_{ij}, d_{ij} \in \mathbb{Z}\) and let
\[
M = \left\{ (x, y, z) \in N_k \mid x_i = \sum_{j=1}^{k} \frac{a_{ij}}{c_{ij}} y_j + \sum_{j=1}^{s} \frac{b_{ij}}{d_{ij}} x_j, i = s + 1, \ldots, k, \right. \\
y_j, x_i, z \in \mathbb{R}, j = 1, \ldots, k, i = 1, \ldots, s \right\} \\
= \left\{ (x, y, z) \in N_k \mid x_i = \sum_{j=1}^{k} \frac{c_j}{c_{ij}} a_{ij} w_j + \sum_{j=1}^{s} \frac{d_j}{d_{ij}} b_{ij} v_j, i = s + 1, \ldots, k, \right. \\
y_j = c_j w_j, x_i = d_i v_i, w_j, v_i, z \in \mathbb{R}, j = 1, \ldots, k, i = 1, \ldots, s \right\}, \] (4.80)
where $c_i = \text{lcm}_j(c_{ij})$ (least common multiple of $c_{ij}$, $j = 1, \ldots, k$) and $d_i = \text{lcm}_j(d_{ij})$. $M$ is a subgroup of $N_k$. If we denote the elements of $M$ by $(v, w, z)$ then

$$\Gamma = M \cap \Gamma_k = \{(v, w, z) \mid v_i, w_i, z \in \mathbb{Z}\} \quad (4.81)$$

is a uniform discrete subgroup of $M$. So a fundamental domain for $M/\Gamma$ is

$$F = \{(v, w, z) \mid 0 \leq v_i, w_i, z < 1\} = \{(x, y, z) \in N_k \mid 0 \leq x_i < \sum_{j=1}^{k} c_{ij}a_{ij} + \sum_{j=1}^{s} d_{ij}b_{ij}, i = s + 1, \ldots, k; \quad 0 \leq x_i < d_i, i = 1, \ldots, s; \quad 0 \leq z < 1\} \quad (4.82)$$

Define $\phi = (\phi_{ij} : M/\Gamma \to F$ by

$$\phi_{ij}(v, w, z) = \begin{cases} d_i\{v_i\} & \text{if } l = 1, j = i, i \leq s \\ \sum_{j=1}^{k} c_{ij}a_{ij}\{w_j\} + \sum_{j=1}^{s} d_{ij}b_{ij}\{v_j\} & \text{if } l = 1, j = i, i \leq s \\ c_i\{w_i\} & \text{if } l = i, j = k + 1, 1 \leq i \leq k \\ \{z - k \sum_{i=1}^{k} c_i[w_i]\} & \text{if } l = 1, j = k + 1 \end{cases} \quad (4.83)$$

Let $(x, y, z) \in M$ and $(n, m, l) \in \Gamma$ where $m_i = c_i m'_i$. Then

$$\phi_{1k+1}((x, y, z)(n, m, l)) = \{z + \sum_{i=1}^{k} x_i c_i m'_i - \sum_{i=1}^{k} x_i c_i [w_i + m'_i] + l\} = \{z - \sum_{i=1}^{k} x_i c_i[w_i]\} = \phi_{1k+1}(x, y, z), \quad (4.84)$$

which shows that $\phi$ is constant on the cosets $(x, y, z)\Gamma$ and hence is well defined on $M/\Gamma$.

**Example 3**

So far we have only dealt with left coset spaces $G/\Gamma$, where the cosets are of the form...
However, right coset spaces $\Gamma \backslash \Gamma$ can equally well be used. For the group $G_{AB}$ defined below, we will consider both left and right coset spaces. Let

$$G_{AB} = \left\{ \begin{pmatrix} 1 & x & \frac{x^2}{2!} & \cdots & \frac{x^{k-1}}{(k-1)!} & y_k \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 1 & x & y_2 & y_1 & 1 \end{pmatrix} \mid x, y_i \in \mathbf{R}, i = 1, \ldots, k \right\},$$

(4.85)

which is a 2-step nilpotent group with $\dim(G_{AB}^1) = k - 1$. Denote the elements of $G_{AB}$ by

$$(x, y) = (x, y_1, \ldots, y_k).$$

(4.86)

The group operation in $G_{AB}$ is then

$$(x, y_1, \ldots, y_k)(x', y'_1, \ldots, y'_k) = (x+x', y_1+y'_1, \ldots, y_i + \sum_{j=0}^{i-1} \frac{x^j}{j!} y_{i-j}, \ldots, y_k + \sum_{j=0}^{k-1} \frac{x^j}{j!} y'_{k-j}).$$

(4.87)

Let $\Gamma$ be the discrete subgroup of matrices $(m_0 k!, m_1, \ldots, m_k)$, $m_i \in \mathbf{Z}$. Note that $m_0$ must be multiplied by an integer which is divisible by $(k-1)!$ in order to make $\Gamma$ a subgroup of $G_{AB}$. Then

$$F = \{(x, y_1, \ldots, y_k) \mid x \in [0, k!), y_i \in [0, 1)\}$$

(4.88)

is a fundamental domain of $G_{AB}/\Gamma$ and $\Gamma \backslash G_{AB}$.

Let us first consider the left coset space $G_{AB}/\Gamma$. We will find $\phi = (\phi_{ij}) : G_{AB}/\Gamma \to F$. Since $x + mk! \in [0, k!)$ if and only if $m = -\lfloor \frac{x}{k!} \rfloor$, we have $\phi_{12}(x, y) = \{\frac{x}{k!}k!\}$. Let $\phi_{ik+1}(x, y) = \theta_{k-i+1}(x, y)$, $i = 1, \ldots, k$, where

$$\theta_i(x, y) = y_i - \sum_{j=1}^{i-1} [\theta_j(x, y)] \frac{x^{i-j}}{j!}, \quad i = 1, \ldots, k.$$

(4.89)
Note that \( \theta_i \) is the restriction to \( G_{AB} \) of the function \( \theta_{k-i+1} \) defined on \( G_{ud} \) by (4.43). Therefore it is easily seen that \( \phi \) has the right properties.

Now, consider the space \( \Gamma \setminus G_{AB} \). Let \((x, y) \in G_{AB}\) and \((mk!, n) \in \Gamma\) such that 
\[
(mk!, n)(x, y) \in F.
\]
By (4.87), we have
\[
(mk!, n)(x, y) = (x + mk!, n_1 + y_1, \ldots, n_i + \sum_{j=0}^{i-1} \frac{(mk!)^j}{j!} y_{i-j}, \ldots, n_k + \sum_{j=0}^{k-1} \frac{(mk!)^j}{j!} y_{k-j}).
\]
So \( m = -\left\lfloor \frac{x}{k} \right\rfloor \) and 
\[
n_i = -\left\lfloor y_i + \sum_{j=1}^{i-1} \frac{(-\left\lfloor \frac{x}{k} \right\rfloor)!}{j!} y_{i-j} \right\rfloor, \quad i = 1, \ldots, k
\]
so that \( \phi_{12}(x, y) = \left\{ \frac{x}{k} \right\}k! \) and 
\[
\phi_{k-i+1,k+1}(x, y) = \left\{ y_i + \sum_{j=1}^{i-1} \frac{(-\left\lfloor \frac{x}{k} \right\rfloor)!}{j!} y_{i-j} \right\}, \quad i = 1, \ldots, k.
\]
Since this shows that for each \((x, y) \in G_{AB}\) there exists exactly one \( \gamma \in \Gamma \) such that \( \gamma (x, y) = \phi(x, y) \in F \), the function \( \phi \) is well defined on \( \Gamma \setminus G_{AB} \) and is the function \( \Gamma \setminus G_{AB} \to F \) that we wanted.

**Remark:**

It is now easy to see that the results concerning uniform distribution of sequences in \( G_{ud} \) obtained in the previous section, are also true for the groups above. For let \( g : N \to C, \) \( N \) any of the nilmanifolds above, be defined by 
\[
g(x) = \exp 2\pi i \sum_{i,j} l_{ij} \phi_{ij}(x).
\]
Then \( g \) is continuous outside a set of measure 0. It follows as in Lemma 4.3.6 that 
\[
\int_N g \, d\mu = 0 \quad \text{and therefore} \quad \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} g(T^n x) = 0 \quad \text{for any} \ x \in N, \] which proves that
the sequence \((\phi_{ij}(T^n x)), n = 1, 2, \ldots\), is uniformly distributed (mod 1) in \(\mathbb{R}^d\) for some \(d \in \mathbb{N}\).

### 4.5 Dynamical approach to generalized polynomials of degree two

We will show in this section how to construct uniquely ergodic dynamical systems corresponding to uniformly distributed generalized polynomials of degree 2. This gives an alternative proof of the fact that these generalized polynomials are uniformly distributed (mod 1).

Let \(\tilde{\alpha} = \begin{pmatrix} 1 & \beta & \gamma \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix}\) be an element of the Heisenberg group \(N_1\) (see 4.64), and let \(T\) be the rotation by \(\tilde{\alpha}\) on \(N_1/\Gamma_1\). Then

\[
T^n(\Gamma_1) = \begin{pmatrix}
1 & \beta n & \gamma n + \frac{n(n-1)}{2} \alpha \beta \\
0 & 1 & \alpha n \\
0 & 0 & 1
\end{pmatrix} \Gamma_1
\]

and the corresponding sequence \(\phi(T^n \Gamma_1), n = 1, 2, \ldots\), in the fundamental domain \(F\) of \(N_1/\Gamma_1\), where \(\phi\) is given in Lemma 4.3.10, is

\[
(\alpha n, \beta n, \gamma n + \frac{n(n-1)}{2} \alpha \beta - [\alpha n] \beta n) \quad (\text{mod } 1).
\]

Note that \(\phi(T^n \Gamma_1), n = 1, 2, \ldots\), is uniformly distributed (mod 1) by Corollary 4.3.11 if \(T\) is uniquely ergodic. By Theorem 4.3.3, \(T\) is uniquely ergodic if \(1, \alpha, \beta\) are rationally independent. If in addition, \(\gamma = 0\) and \(\alpha \beta \in \mathbb{Z}\), then we get that the sequence \([\alpha n] \beta n, n = 1, 2, \ldots\), is uniformly distributed (mod 1). In order to obtain \([\alpha n] \beta n\) by itself for other \(\alpha, \beta\)'s we will need to use affine transformations on nilmanifolds \(G/\Gamma\) where \(G\) is either a Heisenberg group \(N_k\) defined by (4.77) or a subgroup of some \(N_k\).
Note that $\mathbb{R}^d \cong H^d$, where $H^d$ was given by (4.41), can be regarded as a subgroup of the Heisenberg group $N_d$ and that $N_k \times \mathbb{R}^d$ is isomorphic to the group of matrices

$$
\begin{pmatrix}
1 & x_1 & \cdots & x_k & 0 & \cdots & 0 & z \\
1 & \ddots & & \ddots & \ddots & 0 & \vdots \\
& & 1 & & \ddots & \ddots & \ddots & y_k \\
& & \ddots & 1 & & \ddots & \ddots & w_1 \\
& & \vdots & & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & & & 0 & \cdots & 1 & w_d \\
& & & & & 1 & & 1
\end{pmatrix}
$$

(4.96)

where $x_i, y_i, w_j \in \mathbb{R}, i = 1, \ldots, k, j = 1, \ldots, d$. **Notation:** Denote the elements of $N_k/\Gamma_k$ by $(x, y, z)$ and those of $K^d$ by $w$, where $x = (x_1, \ldots, x_k)$, $y = (y_1, \ldots, y_k)$ and $w = (w_1, \ldots, w_d)$.

**Lemma 4.5.1** Let $A_1$ be an automorphism of $K^d$. A transformation $A$ on $N_k/\Gamma_k \times K^d$ defined by

$$
A(x, y, z, w) = (x, y, z + \sum_{i=1}^{k} b_ix_i + \sum_{i=1}^{k} c_iy_i + \sum_{i=1}^{d} d_iw_i, A_1w),
$$

(4.97)

where $b_i, c_i, d_i \in \mathbb{Z}$, is an automorphism of $N_k/\Gamma_k \times K^d$ and the affine transformation

$$
T(x, y, z, w) = (\tilde{\beta}, \tilde{\alpha}, \gamma, \tilde{\lambda})A(x, y, z, w)
$$

(4.98)

is uniquely ergodic if and only if the transformation $(x, y, w) \mapsto (\tilde{\beta}, \tilde{\alpha}, \tilde{\lambda})(x, y, A_1w)$ is uniquely ergodic on the maximal torus factor $K^{d+2k}$ of $N_k/\Gamma_k \times K^d$.

**Proof:** Since

$$
A((x, y, z, w)(x', y', z', w'))
$$
\[ = A(x + x', y + y', z + z' + \sum_{i=1}^{k} x_i y'_i, w + w') \]
\[ = (x + x', y + y', z + z' + \sum_{i=1}^{k} x_i y'_i + \sum_{i=1}^{k} b_i (x_i + x'_i) + \sum_{i=1}^{k} c_i (y_i + y'_i) + \sum_{i=1}^{d} d_i (w_i + w'_i), A_1 (w_i + w'_i)) \]
\[ = A(x, y, z, w) A(x', y', z', w') \]

(4.99)

it follows that \( A \) is an automorphism. We have \( N^1_k = \{(0, 0, z) \in N_k \mid z \in \mathbb{R}\} \).

So the maximal torus factor of \( N_k/\Gamma_k \times K^d \) is \( N_k/(N^1_k \Gamma_k) \times K^d \cong K^{d+2k} \). By Theorem 4.3.3, \( T \) is uniquely ergodic if and only if the induced transformation \( (x, y, w) \mapsto (\tilde{\beta}, \tilde{\alpha}, \tilde{\lambda})(x, y, A_1 w) \) on \( K^{d+2k} \) is uniquely ergodic.

\[ \square \]

**Remark:** In order for \( A \) to be well defined on \( N_k/\Gamma_k \times K^d \) it is necessary that \( b_i, c_i, d_i \)
in Lemma 4.5.1 be integers and not just rational numbers since any automorphism of \( N_k/\Gamma_k \times K^d \) leaves \( \Gamma_k \times \mathbb{Z}^d \) invariant.

In Proposition 3.1.3 we proved that a generalized polynomial \([\alpha n] \beta n\) is uniformly distributed (mod 1) if and only if either \( \alpha^2 \not\in \mathbb{Q} \) and \( \beta \) is irrational or \( \alpha^2 \in \mathbb{Q} \) and \( \beta \) is rationally independent of \( 1, \alpha \). We will now show how to generate dynamically the uniformly distributed generalized polynomials \([\alpha n] \beta n\).

Let \( H_b, b \in \mathbb{Z}, \) be the abelian subgroup

\[ \left\{ \begin{pmatrix} 1 & by & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid y, z \in \mathbb{R} \right\} \]

(4.100)
of \( N_1 \). Then \( \Gamma_b = \Gamma_1 \cap H_b \) is a uniform discrete subgroup of \( H_b \).
Theorem 4.5.2 Any generalized polynomial \([\alpha n] \beta n\), where \(\alpha, \beta\) are irrational numbers such that if \(\alpha = \sqrt{c}\) then \(\beta \neq k_1 \sqrt{c} + k_0\) for all \(k_1, k_0 \in \mathbb{Q}\), is uniformly distributed (mod 1) and can be generated by a uniquely ergodic dynamical, which is the system

\[
\begin{align*}
(G/\Gamma \times K, T_1) & \quad \text{if } \alpha \beta \text{ is rationally independent of } 1, \alpha, \beta, \text{ and} \\
(G/\Gamma, T_2) & \quad \text{if } \alpha \beta = e + c \alpha + d \beta \text{ for some } e, c, d \in \mathbb{Z}
\end{align*}
\]

(4.101)

where \(G/\Gamma = \begin{cases} N_1/\Gamma_1 & \text{if } 1, \alpha, \beta \text{ are rationally independent} \\
H_b/\Gamma_b & \text{if } \beta = a + b \alpha \text{ for some } a, b \in \mathbb{Z}
\end{cases}\) and

\[
T_1 \left( \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \Gamma, w \right) = \left( \begin{pmatrix} 1 & \beta & 0 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & z - w \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \Gamma, w + \alpha \beta \right)
\]

(4.102)

and

\[
T_2 \left( \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \Gamma = \left( \begin{pmatrix} 1 & \beta & 0 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & z - c y - d x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \Gamma \right)
\]

(4.103)

(where \(x = by\) when \(G = H_b\)).

Proof: Suppose first that \(1, \alpha, \beta\) are rationally independent. Then both \(T_1\) and \(T_2\) are uniquely ergodic affine transformations by Lemma 4.5.1 since they induce rotations by \((\beta, \alpha, \alpha \beta)\) and \((\beta, \alpha)\) on the respective maximal torus factors. These rotations are both uniquely ergodic. Note that we subtract \(w\) in (4.102) and \(cy + dx\) in (4.103) from \(z\) in order to get rid of the quadratic polynomial \(\alpha \beta^{n(n-1)}\) in the \(z\)-coordinate of \(T^n(0)\) which comes from the rotation by \(\begin{pmatrix} 1 & \beta & 0 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix}\). So we have

\[
T^n_1(\Gamma_1, 0) = \left( \begin{pmatrix} 1 & \beta n & \alpha \beta^{n(n-1)} - \alpha \beta^{n(n-1)} \\ 0 & 1 & \frac{an}{2} \\ 0 & 0 & 1 \end{pmatrix} \Gamma_1, \alpha \beta n \right)
\]
\[
= \left( \begin{array}{ccc}
1 & \beta n & 0 \\
0 & 1 & \alpha n \\
0 & 0 & 1
\end{array} \right) \Gamma_1, \alpha \beta n \right)
\] (4.104)

and
\[
T_2^n(\Gamma_1) = \left( \begin{array}{ccc}
1 & \beta n & \frac{n(n-1)}{2}(\alpha \beta - ba - c\beta) \\
0 & 1 & \alpha n \\
0 & 0 & 1
\end{array} \right) \Gamma_1 = \left( \begin{array}{ccc}
1 & \beta n & 0 \\
0 & 1 & \alpha n \\
0 & 0 & 1
\end{array} \right) \Gamma_1. \quad (4.105)
\]

So we get the sequence \([\alpha n \beta n]\) in the fundamental domain of \(N_1/\Gamma_1\).

Now, let \(\beta = a + ba, a, b \in \mathbb{Z}\). We may assume \(a = 0\) since \([\alpha n](a + ba)n \equiv [\alpha n]ban \pmod{1}\). Note that \(H_b/\Gamma_b\) is an abelian group with group operation \((y, z)(y', z') = (y + y', z + z' + yy')\). Therefore we can apply Theorem 4.3.4 directly to \(T_1\) and \(T_2\) to show that these transformations are uniquely ergodic. When \(\alpha^2\) is rationally independent of \(1, \alpha\), then \(T_1\) is the affine transformation
\[
T_1(y, z, w) = (\alpha, 0, \alpha^2)A_1(y, z, w), \quad (4.106)
\]
where \(A_1(y, z, w) = (y, z - w, w)\). So
\[
B(y, z, w) = (y, z, w)^{-1}A_1(y, z, w) = (-y, -z + y^2, -w)(y, z - w, w) = (0, -w, 0).
\] (4.107)

Therefore, \((\alpha, 0, \alpha^2)\) together with \(B(H_b/\Gamma_b \times K)\) span \(H_b/\Gamma_b \times K\) so that \(T_1\) is uniquely ergodic by Theorem 4.3.4. If \(ba^2 = e + ca\) then \(T_2(y, z) = (\alpha, 0)A_2(y, z)\), where \(A_2(y, z) = (y, z - cy)\). So
\[
B(y, z) = (y, z)^{-1}A_2(y, z) = (-y, -z + y^2)(y, z - cy) = (0, -cy) \quad (4.108)
\]
which shows that \((\alpha, 0)\) together with \(B(H_b/\Gamma_b)\) span \(H_b/\Gamma_b\) if and only if \(c \neq 0\). Hence, \(T_2\) is uniquely ergodic if and only if \(c \neq 0\). It follows as in the case when
1, $\alpha, \beta$ were rationally independent, that $T_1$ and $T_2$ generate $[\alpha n] \beta n$. One can also find uniquely ergodic systems for $[\alpha n] \beta n$ when $\alpha \beta = e + c\alpha + d\beta$ or $\beta = a + b\alpha$ and some of the rational numbers $a, b, c, d, e$ are not integers, see remarks below.

\[ \square \]

**Remark:** Since $2[\alpha n]\alpha n = \alpha^2 n^2 + [\alpha n]^2 - \{\alpha n\}^2 = g(\alpha^2 n^2, \alpha n)$, where $g(x, y) = x - \{y\}^2$, we may say that the sequence $2[\alpha n]\alpha n, n = 1, 2, \ldots$, is coming from the 2-torus. This is consistent with the fact that $H_1/\Gamma_1$ is isomorphic to $K^2$ (since any compact abelian nilmanifold is isomorphic to a torus).

**Remarks regarding rational relations between coefficients:**

Recall that a uniquely ergodic system $(X, T)$ generating a polynomial with rational but not integer leading coefficient, is disconnected. However, the transformation $T$ induces a transformation on each connected component of $X$, each generating a polynomial with irrational leading coefficient. One therefore shows that it is enough to consider these connected spaces in order to prove uniform distribution. In our discussion about rational numbers appearing in relations between coefficients of a generalized polynomial in Remark of Section 3.1 we found that we may without loss of generality assume that the rational numbers are integers. We will assume the same here. However, we do this only to simplify things. We will now describe briefly by way of examples how to handle the cases when the rational numbers are not integers.

If $a$ or $e$ in Theorem 4.5.2 is not an integer then we may have to use a disconnected
space. Say \( e = \frac{1}{3} \) and \( \alpha \beta = \frac{1}{3} + \alpha + \beta \). Then we can use the group

\[
G = \left\{ \begin{pmatrix} 1 & x & 0 & z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & w \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R}, w \in D_3 \right\}
\]

where \( D_3 = \{0, \frac{1}{3}, \frac{2}{3}\} \), and \( \Gamma \) is the subgroup of \( G \) for which \( x, y, z \) are integers. Then

\[
T \begin{pmatrix} 1 & x & 0 & z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & w \\ 0 & 0 & 0 & 1 \end{pmatrix} \Gamma = \begin{pmatrix} 1 & \beta & 0 & 0 \\ 0 & 1 & 0 & \alpha \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & 0 & z - x - y - w \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & w \\ 0 & 0 & 0 & 1 \end{pmatrix} \Gamma \quad (4.110)
\]

generates \([\alpha n] \beta n\).

If \( b, c \) or \( d \) in Theorem 4.5.2 are not integers, then we may use a subgroup \( \bar{\Gamma} \) of \( \Gamma \) for which the \( z \)-coordinate take any integer value. Say \( b = \frac{r}{s}, r, s \in \mathbb{Z} \), then by Example 2, Section 4.4,

\[
\bar{\Gamma} = \left\{ \begin{pmatrix} 1 & rm & l \\ 0 & 1 & sm \\ 0 & 0 & 1 \end{pmatrix} \mid m, l \in \mathbb{Z} \right\}
\]

and by using \( \alpha' = \frac{\alpha}{s} \) we obtain the sequence \( s[\alpha' n] \alpha' n = s[\frac{\alpha}{s} n] \beta n \). Since

\[
s[\alpha' n] \beta n = [s \alpha' n] \beta n - \sum_{i=0}^{s-1} i 1_{[\frac{1}{s}, \frac{s}{s}]}(\alpha' n) \beta n \quad (4.112)
\]

we also get \([\alpha n] \beta n\).

If \( c = \frac{r}{s}, e = 0, d \in \mathbb{Z} \), then we can use

\[
\bar{\Gamma} = \left\{ \begin{pmatrix} 1 & m_1 & l \\ 0 & 1 & sm_2 \\ 0 & 0 & 1 \end{pmatrix} \mid m_1, m_2, l \in \mathbb{Z} \right\}
\]

and

\[
T_2 \begin{pmatrix} 1 & x & z \\ 0 & 1 & sy \\ 0 & 0 & 1 \end{pmatrix} \Gamma = \begin{pmatrix} 1 & \beta & 0 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & z - ry - dx \\ 0 & 1 & sy \\ 0 & 0 & 1 \end{pmatrix} \Gamma \quad (4.114)
\]

which gives \( s[\frac{\alpha}{s} n] \beta n \).
Lemma 4.5.3 Let $G$ be a Heisenberg group $N_k$ or a non-abelian subgroup of $N_k$, and let $T$ be a uniquely ergodic affine transformation on $G/\Gamma$. Let $T^n(\Gamma) = (x(n), y(n), z(n))$. Then for any polynomial $p(n)$,

$$q(n) = \sum_{i=1}^{k} [y_i(n)]x_i(n) + p(n) \quad (4.115)$$

can be generated by a uniquely ergodic transformation on $G/\Gamma \times K^d \times D$ for some $d$ and where $D$ is a discrete space.

Proof: The sequence $T^n(\Gamma), n = 1, 2, \ldots$ give rise to the sequence

$$(x(n), y(n), z(n) - \sum_{i=1}^{k} [y_i(n)]x_i(n)) \mod 1. \quad (4.116)$$

So $T$ generates $\theta(n) = \sum_{i=1}^{k} [y_i(n)]x_i(n) - z(n)$. Note that $x_i(n), y_i(n)$ and $z(n)$ are polynomials and that $q(n) = \theta(n) + p_0(n)$, where $p_0(n)$ is a polynomial of degree say $d + 1$. If $p_0(n)$ is a linear combination over $\mathbb{Z}$ of $1, x_i(n), y_i(n)$ then $(G/\Gamma, T)$ is already a system for $q(n)$. Let $1, \alpha_1, \ldots, \alpha_l$ be rationally independent so that the leading coefficient of each $x_i(n), y_i(n)$ is a linear combination over $\mathbb{Q}$ of $1, \alpha_1, \ldots, \alpha_l$. Let $\lambda$ be the leading coefficient of $p_0(n)$. If $\lambda$ is rationally independent of $1, \alpha_1, \ldots, \alpha_l$ then let $(K^{d+1}, T_1)$ be a uniquely ergodic system generating $p_0(n)$ such that $T_1(w_1, \ldots, w_d, w_{d+1}) = (T_2(w_1, \ldots, w_d), w_{d+1} + w_d)$. Then the transformation $S$ on $G/\Gamma \times K^d$ defined by $S(x, y, z, w) = (T(x, y, z - w_d), T_2w)$ is uniquely ergodic and generates $q(n)$. Now, if $\lambda$ is rationally dependent of $1, \alpha_1, \ldots, \alpha_l$ but $p_0(n)$ is not a linear combination of $1, x_i(n), y_i(n), i = 1, \ldots, k$, then the system may be extended in such a way that $p_0(n)$ will be a linear combination of the new terms. By the remarks preceding the lemma,
we are done.

\[ T \] by

\[
(\beta_1, \ldots, \beta_k, \alpha_1, \ldots, \alpha_k, \gamma) = \begin{pmatrix}
1 & \beta_1 & \cdots & \beta_k & \gamma \\
1 & 0 & \alpha_1 & & \\
\vdots & & \ddots & \vdots & \\
0 & 1 & \alpha_k & & \\
& & & 1 & \\
\end{pmatrix} \tag{4.117}
\]

on \( N_k/\Gamma_k \) generates the generalized polynomial

\[
\sum_{i=1}^{k} \alpha_i \beta_i n^{(n-1)/2} + \gamma n - \sum_{i=1}^{k} [\alpha_i n] \beta_i n
\]

because this is the sequence in the fundamental domain \( F \) of \( N_k/\Gamma_k \) corresponding to

\[
T^n(\Gamma_k) = (\beta_1 n, \ldots, \beta_k n, \alpha_1 n, \ldots, \alpha_k, \gamma n + \sum_{i=1}^{k} \alpha_i \beta_i n^{(n-1)/2}). \tag{4.118}
\]

Since the commutator subgroup \( N_k^1 \) is the subgroup of \( N_k \) with all \( x_i = y_i = 0 \), the maximal torus factor of \( N_k/\Gamma_k \) is the torus \( K^{2k} \cong \{ (x_1, \ldots, x_k, y_1, \ldots, y_k) \mid x_i, y_i \in K \} \). Therefore \( T \) is uniquely ergodic only if \( 1, \alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k \) are rationally independent. However, if we restrict \( T \) to the nilmanifold corresponding to the smallest \( T \)-invariant subgroup \( M \) of \( N_k \), then the restriction of \( T \) to this new space will be uniquely ergodic if \( M \) is not abelian. If \( M \) is abelian, it will depend on \( \gamma \) if \( T \) is uniquely ergodic or not.

Suppose that \( 1, \alpha_1, \ldots, \alpha_k \) are rationally independent and that there are relations among \( 1, \alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k \). Let \( \sigma_1, \ldots, \sigma_s \) be such that \( 1, \alpha_1, \ldots, \alpha_k, \sigma_1, \ldots, \sigma_s \) are rationally independent and such that

\[
\beta_i = a_{i0} + \sum_{j=1}^{k} a_{ij} \alpha_j + \sum_{j=1}^{s} b_{ij} \sigma_j \tag{4.119}
\]
for all \( i = 1, \ldots, k \), where \( a_{ij}, b_{ij} \in \mathbb{Q} \). We will assume that \( a_{ij}, b_{ij} \in \mathbb{Z} \). If \( T \) is the rotation by (4.117) and each \( \beta_i \) satisfies the equation (4.119), then

\[
M = \left\{ \begin{pmatrix} 1 & x_1 & \cdots & x_k & z \\ 1 & 0 & \ddots & \vdots \\ 0 & 1 & \cdots & 1 \\ y_1 & & & 1 \end{pmatrix} \right| x_i = \sum_{j=1}^{k} a_{ij} y_j + \sum_{j=1}^{s} b_{ij} w_j, i = 1, \ldots, k, \ w_j, y_j \in \mathbb{R} \right\}
\]

(4.120)

is the smallest subgroup of \( N_k \) invariant under \( T \). Let \( \Gamma \) be the uniform discrete subgroup \( M \cap \Gamma_k \).

**Notation:** We will say that \( q_1(n) \sim q_2(n) \) if there exist linear polynomials \( \lambda_1 n, \ldots, \lambda_l n \) and a Riemann-integrable periodic mod 1 function \( g \) on \( \mathbb{R}^l \) such that \( q_1(n) - q_2(n) = g(\lambda_1 n, \ldots, \lambda_l n) \). By the symbol \( \not\sim \) we will mean the negation of \( \sim \).

**Lemma 4.5.4** Let \( \alpha_i, \beta_i \) and \((M/\Gamma, T)\) be as above. Then \( q(n) = \sum_{i=1}^{k} [\alpha_i n][\beta_i n] \sim \lambda n^2 \) for some \( \lambda \in \mathbb{R} \) if and only if \( M \) is abelian.

**Proof:** Recall that \( [a]b \equiv -[b]a + ab - \{a\}\{b\} \pmod{1} \) for all \( a, b \in \mathbb{R} \) and if \( a = b \) then \( 2[a]a \equiv a^2 - \{a\}^2 \pmod{1} \). So we have

\[
q(n) = \sum_{i=1}^{k} [\alpha_i n] \left( \sum_{j=1}^{k} a_{ij} \alpha_j + \sum_{j=1}^{s} b_{ij} \sigma_j \right) n
= \sum_{i<j} a_{ij}[\alpha_i n] \alpha_j n + \sum_{i>j} a_{ij}[\alpha_i n] \alpha_j n + \sum_{i=j} a_{ii}[\alpha_i n] \alpha_i n + \sum_{i,j} b_{ij}[\alpha_i n] \sigma_j n
\sim \sum_{i<j} (a_{ij} - a_{ji})[\alpha_i n] \alpha_j n + \sum_{i,j} b_{ij}[\alpha_i n] \sigma_j n + \lambda n^2
\]

(4.121)

for some \( \lambda \in \mathbb{R} \), so that \( q(n) \sim \lambda n^2 \) if and only if \( a_{ij} = a_{ji} \) and \( b_{ij} = 0 \) for all \( i, j \).
The commutator subgroup $M^1$ of $M$ is generated by the set

$$\{(0,0,\sum_{i=1}^{k}(x_iy'_i-x'_iy_i)) \mid (x,y,z),(x',y',z') \in M\}, \quad (4.122)$$

where $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$. So $M$ is abelian if and only if

$$0 \equiv \sum_{i=1}^{k}(x_iy'_i-x'_iy_i)$$

$$= \sum_{i=1}^{k}\left(\sum_j(a_{ij}y_j+b_{ij}w_j)y'_i-\sum_j(a_{ij}y'_j+b_{ij}w'_j)y_i\right)$$

$$= \sum_{i,j}(a_{ij}-a_{ji})y_jy'_i+\sum_{i,j}b_{ij}(w_jy'_i-w'_jy_i), \quad (4.123)$$

hence, if and only if $a_{ij} = a_{ji}$ and $b_{ij} = 0$ for all $i,j$.

\[\square\]

**Corollary 4.5.5** If $q(n) = \sum_{i=1}^{l}[\tau_i n]\delta_i n + \rho_2 n^2 + \rho_1 n \not\sim p(n)$ for any polynomial $p(n)$, then $q(n)$ is uniformly distributed (mod 1) and is generated by a uniquely ergodic affine transformation on a nilmanifold $M/\Gamma \times K^d$ which is not a torus, where $M$ is of form (4.120) and $d \geq 0$.

**Proof:** Let $1, \alpha_1, \ldots, \alpha_k, \sigma_1, \ldots, \sigma_s$ rationally independent and such that $\tau_i = c_{i0} + \sum_{j=1}^{k}c_{ij}\alpha_j$ and $\beta_i = a_{i0} + \sum_{j=1}^{k}a_{ij}\alpha_j + \sum_{i=1}^{s}b_{ij}\sigma_j$ for some $c_{ij}, a_{ij}, b_{ij} \in \mathbb{Q}$, and

$$q(n) = \sum_{i=1}^{k}[\alpha_i n]\beta_i n + \gamma_2 n^2 + \gamma_1 n + g(\lambda_1 n, \ldots, \lambda_r n) \quad (4.124)$$

for some irrational numbers $\gamma_i, \lambda_i$. We will assume that $c_{ij}, a_{ij}, b_{ij} \in \mathbb{Z}$. By adjusting the function $g$, we can let $1, \lambda_1, \ldots, \lambda_r$ be rationally independent. Let $M$ be the
group (4.120) and $\Gamma$ the corresponding discrete subgroup. By Lemma 4.5.4, $M$ is not abelian. So if $T_1$ is the rotation by

$$
\begin{pmatrix}
1 & \beta_1 & \cdots & \beta_k & -\gamma_2 - \gamma_1 \\
1 & 0 & \cdots & 0 & \alpha_1 \\
0 & \cdots & \cdots & \cdots & \cdots \\
0 & 1 & \cdots & \cdots & \cdots \\
1 & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\Gamma
$$

(4.125)
on $M/\Gamma$, then the induced transformation on the maximal torus factor is a uniquely ergodic rotation. So by Theorem 4.3.3, $T_1$ is uniquely ergodic. It follows that

$$\sum_{i=1}^{k} [\alpha_i n] \beta_i n - \frac{n(n-1)}{2} \sum_{i=1}^{k} \alpha_i \beta_i + (\gamma_2 + \gamma_1) n$$

is uniformly distributed (mod 1). If $2\gamma_2 - \sum_{i=1}^{k} \alpha_i \beta_i$ is rationally dependent of $1, \alpha_i, \beta_i, i = 1, \ldots, k$, say $2\gamma_2 - \sum_{i=1}^{k} \alpha_i \beta_i = \psi(\tilde{\beta}, \tilde{\alpha})$, where $\psi$ is a linear transformation with integer coefficients, $\tilde{\beta} = (\beta_1, \ldots, \beta_k)$ and $\tilde{\alpha} = (\alpha_1, \ldots, \alpha_k)$, then we let $T$ be the affine transformation $T_1 \circ A$ on $M/\Gamma$, where $A$ is the automorphism $(x, y, z) \mapsto (\tilde{\beta}, \tilde{\alpha}, -2(\gamma_2 + \gamma_1), \sigma(x, y, z - w, w))$. For then we get the sequence

$$\sum_{i=1}^{k} [\alpha_i n] \beta_i n - \frac{n(n-1)}{2} \sum_{i=1}^{k} \alpha_i \beta_i + (\gamma_2 + \gamma_1) n + \frac{n(n-1)}{2} (2\gamma_2 + \sum_{i=1}^{k} \alpha_i \beta_i)$$

$$= \sum_{i=1}^{k} [\alpha_i n] \beta_i n + \gamma_2 n^2 + \gamma_1 n$$

(4.126)
in the fundamental domain of $M/\Gamma$. If $\sigma = 2\gamma_2 - \sum_{i=1}^{k} \alpha_i \beta_i$ is rationally independent of $1, \alpha_i, \beta_i, i = 1, \ldots, k$, then we let $T$ be the uniquely ergodic affine transformation

$$(x, y, z, w) \mapsto (\tilde{\beta}, \tilde{\alpha}, -(\gamma_2 + \gamma_1), \sigma)(x, y, z - w, w)$$

(4.127)
on $M/\Gamma \times K$. This generates $\sum_{i=1}^{k} [\alpha_i n] \beta_i n + \gamma_2 n^2 + \gamma_1 n$.

Note that in both cases $T$ is an affine transformation $T = \tilde{\rho} A$ and that $T$ generates $\alpha n$ for any coordinate $\alpha$ of $\tilde{\rho}$ except $\alpha = -(\gamma_2 + \gamma_1)$. Let $C$ be the set of coordinates
of $\hat{\rho}$, excluding $-(\gamma_2 + \gamma_1)$. $(G/\Gamma, T)$ is already a system for any $\lambda_i$ which is rationally dependent of $C \cup \{1\}$. Let $t$ be the smallest $i$ such that $\lambda_i$ is not rationally dependent of $C \cup \{1\}$, and let $G_1 = G \times \mathbb{R}$, $\Gamma_1 = \Gamma \times \mathbb{Z}$ and $T_1 = T \times T_t$ where $T_t$ is the rotation by $\lambda_t$ on $K = \mathbb{R}/\mathbb{Z}$. By construction, this system is uniquely ergodic. Now, compare $\lambda_{t+1}$ with $C \cup \{1, \lambda_t\}$ and proceed as above. By repeating this for $\lambda_i, i > t$, we end up with a system $(M/\Gamma \times K^d, T)$ for $(q(n), \lambda_1n, \ldots, \lambda_rn)$, which is uniquely ergodic.

\[ \square \]

**Theorem 4.5.6** Any uniformly distributed generalized polynomial $q(n)$ of degree two can be generated by a uniquely ergodic affine transformation $T$ on a nilmanifold $G/\Gamma$. If $q(n)$ has no nested brackets with only constants on the outside, then $G$ is a subgroup of a Heisenberg group.

**Proof:** Let $q(n)$ be any uniformly distributed generalized polynomial of degree two. Before we can find a corresponding dynamical system we need to write $q(n)$ in an appropriate form. The idea is to find generalized polynomials $q_0(n), q_1(n), \ldots, q_l(n)$ of the form $\sum_{i=1}^{r} [\alpha_in] \beta n + \gamma n^2 + \lambda n$ such that $q(n) = q_0(n) + g(q_1(n), \ldots, q_l(n))$ for some Riemann-integrable periodic mod 1 function $g$ and such that $(q_0(n), q_1(n), \ldots, q_l(n))$ is generated by a uniquely ergodic system $(G/\Gamma, T)$. We will use the following steps to find the $q_i(n)$’s.

(1) Use the identity $[v(n)] \beta = v(n) \beta - \{v(n)\} \beta$ to get rid of any bracket with only a constant on the outside.
(2) Since we have

\[ [v_1(n)] \beta + v_2(n) \] \[ v_3(n) \] (4.128)

\[ = \left( v_1(n) \beta - \{ v_1(n) \right) \right] \beta + v_2(n) \] \[ v_3(n) \] (4.129)

\[ = [v_1(n) \beta + v_2(n)] v_3(n) - \sum_{i=0}^{[\beta]+1} I_C \left( v_1(n) \beta + v_2(n), v_1(n) \right) v_3(n) \] (4.130)

for some subset \( C_I \subset [0,1)^2 \), write any term of form \( \left[ [v_1(n)] \beta + v_2(n) \] \[ v_3(n) \right] \), where \( \deg(v_1) = \deg(v_3) = 1 \), in the form (4.130).

(3) Use the identity

\[ [a \gamma] [b \gamma] = [a \gamma] b \gamma + [b \gamma] a \gamma - a \beta \gamma n^2 + \{ a \gamma \} \{ b \gamma \} \] (4.131)

to get rid of any product of brackets.

After going through steps (1)-(3), \( q(n) \) is of the form \( u_0(n) + g(u_1(n), \ldots, u_l(n)) \) where each \( u_j(n) \) is of the form \( \sum_{i=1}^{\beta} [a_i, n] \beta_i n + \lambda n^2 + \gamma n \). But most likely, \( (u_1(n), \ldots, u_l(n)) \) is not uniformly distributed (mod 1). Let \( U_j \) be the set of \( u_i(n) \)'s, \( i > 0 \), of degree \( j \), \( j = 1, 2 \). Note that any \( u_i(n) \in U_2 \) was obtained in step (1) and if \( [a \gamma] [b \gamma] \) is a term of some \( u_i(n) \in U_2 \), then \( u_0(n) \) has a term \( [a \gamma] \gamma n \). Proceed as follows.

(4) Let \( 1, \alpha_1, \ldots, \alpha_k \) be rationally independent numbers so that \( u_j(n) = \sum_{i=1}^{k} [a_i, n] \beta_j n + \gamma_j n^2 + \lambda_j n + g(\delta_1 n, \ldots, \delta_l) \) for some \( \beta_j, \gamma_j, \lambda_j, \delta_i \in \mathbb{R}, j = 1, \ldots, l \). We will assume that \( \beta_j, \gamma_j, \lambda_j \) are irrational numbers. Because of the identity (3.3), we may also assume that if \( \beta_j = a_{j0} + \sum_{r=1}^{k} a_{jr} \alpha_r \) then \( a_{jr} = 0 \) for \( i \geq r \). Now, exchange \( u_j(n) \) for \( u_j(n) - g(\delta_1 n, \ldots, \delta_l) \) and call this new generalized polynomial \( u_j(n) \). Adjust \( U_1 \) accordingly.
Choose a subset \( V = \{v_1(n), \ldots, v_{l_1}(n)\} \) of \( U_2 \) such that

(i) for any \( u(n) \in U_2 \) there exist \( a_i \in \mathbb{Q} \) with \( u(n) \sim \sum_{i=1}^{l_1} a_i v_i(n) \), and

(ii) \( \sum_{i=1}^{l_1} c_i v_i(n) \not\sim 0 \) for any \( c_i \in \mathbb{Q} \) not all 0.

In order to minimize the number of non-polynomials, change the set \( V \) in the following way. If \( w_i(n) \in V, i = 1, \ldots, s \), are non-polynomials and \( c_i \in \mathbb{Q} \setminus \{0\} \) such that \( \sum_{i=1}^{s} c_i w_i(n) \sim \lambda n^2 \) for some irrational \( \lambda \in \mathbb{R} \), then exchange one \( w_i(n) \) for \( \lambda n^2 \). Do this until no linear combination of non-polynomials in \( V \) reduces to a polynomial. Also, if there exist \( c_i \in \mathbb{Q} \) and \( \gamma, \lambda \in \mathbb{R} \) such that \( u_0(n) \sim \sum_{i=1}^{l_1} c_i v_i(n) + \gamma n^2 + \lambda n \), then let \( q_0(n) = \gamma n^2 + \lambda n \). Otherwise, let \( q_0(n) = u_0(n) \). Also, let \( q_i(n) = u_i(n), i = 1, \ldots, l_1 \) and let \( q_{l_1+1}(n), \ldots, q_l(n) \) be linear polynomials spanning the set \( U_1 \).

We will now find a uniquely ergodic system for \((q_0(n), q_1(n), \ldots, q_l(n))\). Let \( q_{s_1}(n), \ldots, q_{s_t}(n) \) be the non-polynomials in \( V \cup \{q_0(n)\} \). Recall that

\[
q_j(n) = \sum_{i=1}^{k} [\alpha_i n] \beta_{ij} n + \gamma_i n^2 + \lambda_j n, \quad j = s_1, \ldots, s_t. \tag{4.132}
\]

Fix a basis \( 1, \alpha_1, \ldots, \alpha_k, \sigma_1, \ldots, \sigma_t \) for the vector space spanned by \( \{1, \alpha_i, \beta_{ij} \mid i = 1, \ldots, k, \ j = s_1, \ldots, s_t\} \) so that \( \beta_{ij} = \sum_{r=1}^{k} a_{ijr} \alpha_r + \sum_{r=1}^{t} b_{ijr} \sigma_r \). By Corollary 4.5.5 there exists for each such \( q_i(n) \) a corresponding uniquely ergodic dynamical system \((M_i/\Gamma_i \times K^{d_i}, T_i)\), where \( M_i \) is a non-abelian nilpotent group of form (4.120). Let

\[
M = \begin{pmatrix}
M_{s_1} & & & 0 \\
& \ddots & & \\
0 & & \ddots & \\
0 & & & M_{s_t}
\end{pmatrix} \tag{4.133}
\]
and let $\Gamma$ be the corresponding discrete subgroup. By construction and Theorem 4.3.3, the transformation $T$ induced on $M/\Gamma$ by the transformations $T_i$ on $M_i/\Gamma_i \times K^{d_i}$, $i = s_1, \ldots, s_l$, is uniquely ergodic if the commutator subgroup $M^1$ is the direct product $M^1_{s_1} \times \cdots \times M^1_{s_l}$. Since no linear combination of these $q_i(n)$'s reduces to a polynomial, it follows, as in the proof of Lemma 4.5.4, that $M^1 = M^1_{s_1} \times \cdots \times M^1_{s_l}$. The system can now be extended to a uniquely ergodic system for $(q_0(n), q_1(n), \ldots, q_l(n))$ in a similar way as in Corollary 4.5.5.

We will end this section with an example.

**Example:** Let $q(n) = [\sqrt{2}n][\sqrt{3}n]\sqrt{5}$. We will show that $(N_2/\Gamma_2, T)$, where $T$ is the rotation by

$$
\begin{pmatrix}
1 & \sqrt{15} & \sqrt{10} & \sqrt{30} \\
0 & 1 & 0 & \sqrt{2} \\
0 & 0 & 1 & \sqrt{3} \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

(4.134)
on $N_2/\Gamma_2$, is a uniquely ergodic system for $q(n)$. Note first that we can write

$$q(n) = [\sqrt{2}n]\sqrt{15n} + [\sqrt{3}n]\sqrt{10n} - \sqrt{30n^2} + \{\sqrt{2n}\}\{\sqrt{3n}\}\sqrt{5}.
$$

(4.135)

Since $1, \sqrt{2}, \sqrt{3}, \sqrt{15}, \sqrt{10}$ are rationally independent, $T$ is uniquely ergodic. We have

$$T^n(\Gamma_2) = \begin{pmatrix}
1 & \sqrt{15n} & \sqrt{10n} & \sqrt{30n + 2\sqrt{30n(n-1)}} \\
0 & 1 & 0 & \sqrt{2n} \\
0 & 0 & 1 & \sqrt{3n} \\
0 & 0 & 0 & 1
\end{pmatrix}_{\Gamma_2}
$$

$$= \begin{pmatrix}
1 & \sqrt{15n} & \sqrt{10n} & \sqrt{30n^2} \\
0 & 1 & 0 & \sqrt{2n} \\
0 & 0 & 1 & \sqrt{3n} \\
0 & 0 & 0 & 1
\end{pmatrix}_{\Gamma_2}.
$$

(4.136)
Therefore,
\[
\phi(T^n \Gamma_2) = \left( \sqrt{15}n, \sqrt{10}n, \sqrt{2}n, \sqrt{3}n, \sqrt{30}n^2 - [\sqrt{2}n]\sqrt{15}n - [\sqrt{3}n]\sqrt{10}n \right) \pmod{1} \quad (4.137)
\]
which is the sequence in the fundamental domain \( \{ x \in N_3 \mid 0 \leq x_i, y_i, z < 1, i = 1, 2 \} \) of \( N_2/\Gamma_2 \) corresponding to \( T^n(\Gamma_2) \), is uniformly distributed \( \pmod{1} \). It follows that

the function
\[
f(n) = e^{2\pi i[\sqrt{2}n]\sqrt{15}n} = g\left( [\sqrt{2}n]\sqrt{15}n + [\sqrt{3}n]\sqrt{10}n - \sqrt{30}n^2, \sqrt{2}n, \sqrt{3}n \right), \quad (4.138)
\]
where \( g(x, y, z) = e^{2\pi i(x + y\{z\})\sqrt{5}} \), is uniquely ergodic. Note that there are nothing special with the numbers 2, 3, 5 except that 1, \( \sqrt{2}, \sqrt{3}, \sqrt{15}, \sqrt{10} \) are rationally independent. So the same result and proof is true for any \( a, b, c \in \mathbb{Q} \) for which 1, \( \sqrt{a}, \sqrt{b}, \sqrt{bc}, \sqrt{ac} \) are rationally independent.

### 4.6 Use of Heisenberg groups to generate generalized polynomials of higher degrees

We showed in the previous section that almost any uniformly distributed generalized polynomial of degree 2 can be generated by an affine transformation on a nilmanifold \( G/\Gamma \), where \( G \) is a subgroup of a Heisenberg group (see Theorem 4.5.6). Since the commutator subgroup \( N_{k,1} \) of \( N_k \) is one-dimensional, the generalized polynomials generated by a transformation on \( N_k/\Gamma_k \) have at most one nested bracket. In this section we will present a class of generalized polynomials of degree \( k \), \( k \) any natural number, and show that this class is generated by affine transformations on nilmanifolds.
The following theorem due to W. Parry classifies all uniquely ergodic affine transformation on the nilmanifold $N_1/\Gamma_1$.

**Theorem 4.6.1 ([30])** An affine transformation $T(x\Gamma_1) = \tilde{\alpha}Ax\Gamma_1$ of the nilmanifold $N_1/\Gamma_1$ is uniquely ergodic if and only if it is of the following form,

$$
\tilde{\alpha} = \begin{pmatrix}
1 & \beta & \gamma \\
0 & 1 & \alpha \\
0 & 0 & 1
\end{pmatrix}
\quad \text{and} \quad
A = \begin{pmatrix}
1 & x & z \\
0 & 1 & y \\
0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & x + 2by & z + cx^2 + \lambda x + by^2 + \mu y \\
0 & 1 & y + 2cx \\
0 & 0 & 1
\end{pmatrix}
$$

(4.139)

where $b, c, \lambda, \mu \in \mathbb{Z}$, $bc = 0$ and such that $(\beta, \alpha)$ together with $(bx, cy)$ generates the 2-torus $K^2$.

Since we already have Theorem 4.3.3 and Theorem 4.3.4 at our disposal, theorems which Parry also used in his proof of Theorem 4.6.1, the most interesting part of this theorem is the classification of all automorphisms of $N_1/\Gamma_1$. We will use techniques similar to those Parry used in his proof of Theorem 4.6.1 to prove Theorem 4.6.2 below.

Let us fix some notation. An element of the Heisenberg group $N_k$ will be denoted $(x, y, z)$, where $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$. Let $\langle x, y \rangle = \sum_{i=1}^{k} x_iy_i$ be the inner product on $\mathbb{R}^k$. Then the group operation of $N_k$ can be written

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + \langle x, y' \rangle).$$

(4.140)

An automorphism of $N_k/\Gamma_k$ will be denoted $A(x, y, z) = (\phi, \psi, \theta)$, where $\phi = \phi(x, y, z)$, $\psi = \psi(x, y, z)$ and $\theta = \theta(x, y, z)$. For terminology, see Section 4.3.
Theorem 4.6.2 Any automorphism \( A(x, y, z) = (\phi, \psi, \theta) \) of \( N_k/\Gamma_k \) has the following properties,

(i) \( \phi \) and \( \psi \) are independent of \( z \) and \( A_1(x, y) = (\phi, \psi) \) is the induced automorphism on the maximal torus factor \( K^{2k} \) of \( N_k/\Gamma_k \).

There exists \( \varepsilon \in \{-1, 1\} \) so that

(ii) The identity
\[
\langle \phi(x, y), \psi(x', y') \rangle - \varepsilon \langle x, y' \rangle = \langle \phi(x', y'), \psi(x, y) \rangle - \varepsilon \langle x', y \rangle \quad (4.141)
\]
holds for all \((x, y, z), (x', y', z') \in N_k\).

(iii) \( \frac{1}{2} \left( \langle \phi(n, m), \psi(n, m) \rangle - \varepsilon \langle n, m \rangle \right) \in Z \) for all \( n, m \in Z^k \).

(iv) \( \theta(x, y, z) = \varepsilon z + \frac{1}{2} \left( \langle \phi(x, y), \psi(x, y) \rangle - \varepsilon \langle x, y \rangle \right) + L(x, y) \),

where \( L(x, y) \) is a linear transformation from \( K^{2k} \) to \( K \).

Furthermore, any automorphism \( A_1(x, y) = (\phi, \psi) \) on \( K^{2k} \) satisfying (ii) and (iii) can be extended to an automorphism \( A \) on \( N_k/\Gamma_k \) by (iii).

Proof: Let \((x, y, z) \) and \((x', y', z') \) be two elements of \( N_k \). Since \( A(x, y, z) = (\phi, \psi, \theta) \) is an automorphism of \( N_k \), we have

\[
A(x + x', y + y', z + z' + \langle x, y' \rangle) = \left( \phi(x, y, z) + \phi(x', y', z'), \psi(x, y, z) + \psi(x', y', z'), \theta(x, y, z) + \theta(x', y', z') + \langle \phi(x, y, z), \psi(x', y', z') \rangle \right). \quad (4.144)
\]
So
\[ \phi(x + x', y + y', z + z' + (x, y')) = \phi(x, y, z) + \phi(x', y', z'), \tag{4.145} \]
\[ \psi(x + x', y + y', z + z' + (x, y')) = \psi(x, y, z) + \psi(x', y', z') \tag{4.146} \]

and
\[ \theta(x + x', y + y', z + z' + (x, y')) = \theta(x, y, z) + \theta(x', y', z') + \langle \phi(x, y, z), \psi(x', y', z') \rangle. \tag{4.147} \]

Since \( AN_1^k = N_1^k \) and \( N_1^k = \{ (0, 0, z) \in N_k \mid z \in R \} \), we have
\[ \phi(0, 0, z) = \psi(0, 0, z) = 0. \tag{4.148} \]

By letting \( x' = y' = 0, z = 0 \) in (4.145) and (4.6), we get
\[ \phi(x, y, z) = \phi(x, y, 0) + \phi(0, 0, z) = \phi(x, y, 0) \tag{4.149} \]

and
\[ \psi(x, y, z) = \psi(x, y, 0) + \psi(0, 0, z) = \psi(x, y, 0). \tag{4.150} \]

Therefore, \( \phi \) and \( \psi \) are independent of \( z \) and we can write \( \phi(x, y, z) = \phi(x, y) \) and \( \psi(x, y, z) = \psi(x, y) \). Since the maximal torus factor of \( N_k/\Gamma_k \) is \( N_k/N_1^k \Gamma_k \cong K^{2k} \), the automorphism \( A_1(x, y) = (\phi, \psi) \) on \( K^{2k} \) is induced by \( A \). From (4.147) and (4.148) it follows that
\[ \theta(0, 0, z + z') = \theta(0, 0, z) + \theta(0, 0, z') \tag{4.151} \]
so that \( \theta \) restricted to \( N_1^k/(\Gamma_k \cap N_1^k) \) is a linear transformation and therefore \( \theta(0, 0, z) = \varepsilon z \). Since \( A \) is invertible, \( \varepsilon \in \{1, -1\} \). So by letting \( x' = y' = 0, z = 0 \) in (4.147), we
get by (4.148) that
\[ \theta(x, y, z) = \varepsilon z + \theta(x, y, 0) \]  
(4.152)

By using the equality (4.152) in equation (4.147) we get
\[ \theta(x + x', y + y', 0) - \theta(x, y, 0) - \theta(x', y', 0) = \langle \phi(x, y), \psi(x', y') \rangle - \varepsilon(x, y') \]  
(4.153)

and by switching the order of \((x, y, z)\) and \((x', y', z')\) we also get
\[ \theta(x + x', y + y', 0) - \theta(x, y, 0) - \theta(x', y', 0) = \langle \phi(x', y'), \psi(x, y) \rangle - \varepsilon(x', y'). \]  
(4.154)

Therefore,
\[ \langle \phi(x, y), \psi(x', y') \rangle - \varepsilon(x, y') = \langle \phi(x', y'), \psi(x, y) \rangle - \varepsilon(x', y) \]  
(4.155)

which shows (ii) of the theorem.

By setting \(z = z' = 0, y' = -\frac{y}{2}, x' = -\frac{x}{2}\) in (4.147) and using that \(\psi\) is linear so that \(\psi(-\frac{x}{2}, -\frac{y}{2}) = -\frac{1}{2}\psi(x, y)\), we get
\[ \theta(x, y, 0) = \frac{1}{2} \left( \langle \phi(x, y), \psi(x, y) \rangle - \langle x, y' \rangle \right) + \theta(x', y', 0) - \theta(x, y, 0). \]  
(4.156)

Let \(L(x, y) : K^{2k} \rightarrow K\) be a linear transformation. We will show that
\[ \theta(x, y, z) = \varepsilon z + \frac{1}{2} \left( \langle \phi, \psi \rangle - \varepsilon(x, y) \right) + L(x, y) \]  
(4.157)

satisfies equation (4.147). It then will follow that \(\theta(x, y, 0)\) given by (4.157) is a solution of (4.156). We have by using property (ii) of the theorem,
\[ \theta(x + x', y + y', z + z' + \langle x, y' \rangle) \]
\[ = \varepsilon(z + z' + \langle x, y' \rangle) + \frac{1}{2} \left( \langle \phi(x + x', y + y'), \psi(x + x', y + y') \rangle \right) \]
which shows (4.147). Note that it is necessary that \( L(x, y) \) be linear. Therefore \( \theta(x, y, z) \) must have the form (4.157). Also, in order that \( \theta \) be well defined on \( N_k / \Gamma_k \), (iii) must be satisfied.

Now, suppose \( A_1 \) is given and satisfies (i), (ii) and (iii). Define \( A \) by \((x, y, z) \mapsto (\phi, \psi, \theta)\), where \( \theta \) is defined by (iv) for some linear transformation \( L \) from \( K^{2k} \) to \( K \). We have already shown that \( A \) is an endomorphism of \( N_k / \Gamma_k \). So it is left to show that \( A \) is invertible. By assumption, \( A_1 \) is invertible. We need to check that \( A_1^{-1} \) has property (ii). Let \( A_1^{-1}(x, y) = (\tilde{\phi}, \tilde{\psi}) \) and let \( (x_0, y_0), (x'_0, y'_0) \) be two elements of \( K^{2k} \). Let \((x, y) = A_1^{-1}(x_0, y_0) = \tilde{\phi}(x_0, y_0), \tilde{\psi}(x_0, y_0) \) and \((x', y') = A_1^{-1}(x'_0, y'_0) = \tilde{\phi}(x'_0, y'_0), \tilde{\psi}(x'_0, y'_0) \). Then \( A_1(x, y) = (\phi(x, y), \psi(x, y)) = (x_0, y_0) \) and \( A_1(x', y') = (\phi(x', y'), \psi(x', y')) = (x'_0, y'_0) \). So we have, using (ii) for \( A_1 \),

\[
\langle \tilde{\phi}(x_0, y_0), \tilde{\psi}(x'_0, y'_0) \rangle - \varepsilon(x_0, y'_0) = \langle x, y' \rangle - \varepsilon(\phi(x, y), \psi(x', y'))
\]
\[ = \langle x, y' \rangle - \varepsilon \langle \phi(x', y'), \psi(x, y) \rangle + \varepsilon \langle x, y' \rangle - \varepsilon \langle x', y \rangle \]

\[ = \langle x', y \rangle - \varepsilon \langle \phi(x', y'), \psi(x, y) \rangle \]

\[ = \langle \tilde{\phi}(x_0', y_0'), \tilde{\psi}(x_0, y_0) \rangle - \varepsilon \langle x_0', y_0 \rangle \]

which shows that (ii) holds for \( A_1^{-1} \). Then

\[ A^{-1}(x, y, z) = (\tilde{\phi}, \tilde{\psi}, \tilde{\theta}), \]

where

\[ \tilde{\theta}(x, y, z) = \varepsilon z + \frac{1}{2} \left( (\tilde{\phi}(x, y), \tilde{\psi}(x, y)) - \varepsilon \langle x, y \rangle \right) - \varepsilon L(\tilde{\phi}(x, y), \tilde{\psi}(x, y)). \] (4.160)

For

\[ A^{-1}A(x, y, z) = A^{-1}(\phi, \psi, \theta) = (\tilde{\phi}(\phi, \psi), \tilde{\psi}(\phi, \psi), \tilde{\theta}(\phi, \psi, \theta)) = (x, y, z) \] (4.161)

since

\[ \tilde{\theta}(\phi, \psi, \theta) = \varepsilon \theta + \frac{1}{2} \left( (\tilde{\phi}(\phi, \psi), \tilde{\psi}(\phi, \psi)) - \varepsilon \langle \phi, \psi \rangle \right) - \varepsilon L(\tilde{\phi}(x, y), \tilde{\psi}(x, y)) \]

\[ = \varepsilon \left( \varepsilon z + \frac{1}{2} \left( (\phi, \psi) - \varepsilon \langle x, y \rangle \right) + L(x, y) \right) \]

\[ + \frac{1}{2} \left( (x, y) - \varepsilon \langle \phi, \psi \rangle \right) - \varepsilon L(x, y) \]

\[ = z. \] (4.162)

We will use Theorem 4.6.2 to prove the following theorem.

**Theorem 4.6.3** Let \( p_1(n) = r_1(n) = n \) and \( p_2(n) = r_2(n) = \frac{n(n-1)}{2} \).

(a) Let \( \alpha \) be an irrational number and \( k \) a natural number. If \( k \) is odd, define rational polynomials \( p_l(n) \) inductively by \( p_{2l+1}(n) = \sum_{i=1}^{n-1} p_{2l}(i) \), \( l \geq 1 \) and \( p_2(n) = \)
Let \( p_{2l-1}(i) + p_{2l-2}(i) \), \( l \geq 2 \), \( l \neq \frac{k+1}{2} \) and \( p_{k+1}(n) = \sum_{i=1}^{n-1} 2(p_{2l-1}(i) + p_{2l-2}(i)) \).

If \( k \) is even, define \( p_{2l+1}(n) = \sum_{i=1}^{n-1} p_{2l}(i) \), \( l \geq 1 \), \( l \neq k/2 \), and \( p_{2l}(n) = \sum_{i=1}^{n-1} (p_{2l-1}(i) + p_{2l-2}(i)) \), \( l \geq 2 \), \( l \neq \frac{k}{2} + 1 \) and \( p_{k+1}(n) = 2 \sum_{i=1}^{n-1} p_{k}(i) \), \( p_{k+2}(n) = \sum_{i=1}^{n-1} (p_{k+1}(i) + 2p_{k}(i)) \).

Then \( \deg(p_l) = l, l = 1, \ldots, 2k, \) and

\[
q(n) = \sum_{l=0}^{k-1} \left[ p_{2l+1}(n) \alpha \right] p_{2k-2l}(n) \alpha \quad (4.163)
\]

is uniformly distributed \((\text{mod } 1)\) and is generated by a uniquely ergodic affine transformation on a nilmanifold \( G/\Gamma \), where \( G = N_k \times \mathbb{R}^d \) for some \( d \in \mathbb{N} \).

(b) Let \( 1, \alpha, \beta \) be rationally independent and let \( k \) be an odd natural number. Define rational polynomials \( p_l(n) \), \( r_l(n) \) inductively by \( p_{2l+1}(n) = \sum_{i=1}^{n-1} p_{2l}(i) \), \( l \geq 1 \),

\[
p_{2l}(n) = \sum_{i=1}^{n-1} (p_{2l-1}(i) + p_{2l-2}(i)) \text{, } l \geq 2 \text{ and } p_{2l+1}(n) = \sum_{i=1}^{n-1} (r_{2l}(i) + p_{2l-1}(i)) \text{, } r_{2l}(n) = \sum_{i=1}^{n-1} r_{2l-1}(i) \text{, } l \geq 1.
\]

Then \( \deg(p_l) = \deg(r_l) = l, l = 1, \ldots, k, \) and

\[
q(n) = \sum_{l=0}^{k-1} \left[ p_{2l+1}(n) \alpha \right] r_{2l}(n) \beta + \sum_{l=1}^{k-1} \left[ r_{2l}(n) \beta \right] p_{k-2l+1}(n) \alpha \quad (4.164)
\]

is uniformly distributed \((\text{mod } 1)\) and is generated by a uniquely ergodic affine transformation on a nilmanifold \( G/\Gamma \), where \( G = N_k \times \mathbb{R}^d \).

(c) Let \( 1, \alpha, \beta \) be rational independent and let \( k \) be an even natural number. Define rational polynomials \( p_l(n) \) inductively by \( p_{2l+1}(n) = \sum_{i=1}^{n-1} (p_{2l}(i) + p_{2l-1}(i)) \) and \( p_{2l}(n) = \sum_{i=1}^{n-1} p_{2l-1}(i) \), \( l \geq 1 \). Then \( \deg(p_l) = l, l = 1, \ldots, k, \) and

\[
q(n) = \sum_{l=1}^{k/2} \left[ p_{2l}(n) \alpha \right] p_{k-2l+1}(n) \beta + \left[ p_{2l}(n) \beta \right] p_{k-2l+1}(n) \alpha \quad (4.165)
\]

is uniformly distributed \((\text{mod } 1)\) and is generated by a uniquely ergodic affine transformation on a nilmanifold \( G/\Gamma \), where \( G = N_k \times \mathbb{R}^d \).
**Proof: (a)** We will use the notation of Theorem 4.6.2 when defining an automorphism $A$ on $N_k/\Gamma_k$. Let the transformation $A_1$ on $K^{2k}$ be defined by

$$
\phi_l(x, y) = \begin{cases} 
x_l + y_{k-l+1} + x_{l+1} & \text{if } l \neq \frac{k+1}{2}, \frac{k}{2} \\
x_l + 2y_{k-l+1} + 2x_{l+1} & \text{if } l = \frac{k+1}{2} \\
x_l + y_{k-l+1} & \text{if } l = \frac{k}{2}
\end{cases}
$$

and

$$
\psi_l(x, y) = \begin{cases} 
y_l & \text{if } l = 1 \\
y_l + x_{k-l+2} & \text{if } l \neq 1
\end{cases}
$$

The matrix representation of $A_1$ with respect to the standard basis for

$$
R^{2k} = \{(y_1, x_k, y_2, x_{k-1}, \ldots, y_k, x_1) | y_i, x_i \in R\}
$$

is then

$$
\begin{pmatrix}
1 & 1 & 1 & & & & \\
1 & 1 & 1 & & & & \\
1 & 1 & 1 & & & & \\
& \ddots & & & & & \\
1 & 1 & 1 & & & & \\
2 & 2 & 1 & & & & \\
1 & 1 & & & & & \\
0 & 1 & 1 & & & & \\
1 & 1 & & & & & \\
1 & 1 & & & & & \\
\end{pmatrix}
$$

Therefore $\det(A_1) = 1$ and $A_1$ is an automorphism of $K^{2k}$.

We will show that $A_1 = (\phi, \psi)$ satisfies (ii) and (iii) of Theorem 4.6.2. Let $x_{k+1} = y_{k+1} = 0$. Then

$$
\langle \phi(x, y), \psi(x', y') \rangle - \langle x, y' \rangle
$$
and (ii) holds. From equation (4.174) we see that

$$\langle \phi(x, y), \psi(x', y') \rangle - \langle x, y' \rangle$$

$$= \sum_{l=1, l \neq \frac{k+1}{2}}^{k} (x_{l+1} + y_{l+1} + x_{l+1}) (y'_{l+1} + x'_{l+1})$$

$$+ (x_{k+1} + 2y_{k+1} + 2x_{k+1})(y'_{k+1} + x'_{k+1}) - \langle x, y' \rangle$$

$$= \sum_{l=2}^{k} x_{l}x'_{k-l+2} + \sum_{l=1, l \neq \frac{k+1}{2}}^{k} (y_{k-l+1}y'_{l+1} + y_{k-l+1}x'_{k-l+2} + x_{l+1}y'_{l+1} + x_{l+1}x'_{k-l+2})$$

$$+ 2(y_{k+1}y'_{k+1} + y_{k+1}x'_{k+1} + x_{k+1}y'_{k+1} + x_{k+1}x'_{k+1} + y'_{k+1}) (4.170)$$

We can write

$$\sum_{l=2}^{k} x_{l}x'_{k-l+2} = \sum_{l=2}^{k+1} x_{l}x'_{k-l+2} + \sum_{s=2}^{k+1} x_{l}x_{s}x_{s+2}, (4.171)$$

$$\sum_{l=1, l \neq \frac{k+1}{2}}^{k} y_{k-l+1}y'_{l+1} = \sum_{l=1}^{k+1-1} y_{k-l+1}y'_{l+1} + \sum_{s=1}^{k+1-1} y'_{s}y_{s+1} (4.172)$$

and

$$\sum_{l=1, l \neq \frac{k+1}{2}}^{k} y_{k-l+1}x'_{k-l+2} = \sum_{s=1, s \neq \frac{k+1}{2}}^{k} y_{s}x_{s+1} (4.173)$$

so that

$$\langle \phi(x, y), \psi(x', y') \rangle - \langle x, y' \rangle$$

$$= \sum_{l=1, l \neq \frac{k+1}{2}}^{k} (x_{l+1} + y_{l+1} + x_{l+1}) (y'_{l+1} + x'_{l+1})$$

$$+ \sum_{l=2}^{k+1} (x_{l}x'_{k-l+2} + x'_{l}x_{k-l+2}) + \sum_{l=1}^{k+1-1} (y_{k-l+1}y'_{l+1} + y_{k-l+1}y_{l+1} + y_{k-l+1}y_{l+1})$$

$$+ 2y_{k+1}y'_{k+1} + 2x_{k+1}x'_{k+1} + x_{k+1}x'_{k+1} + y'_{k+1}. (4.174)$$

Therefore,

$$\langle \phi(x, y), \psi(x', y') \rangle - \langle x, y' \rangle = \langle \phi(x', y'), \psi(x, y) \rangle - \langle x', y \rangle, (4.175)$$

and (ii) holds. From equation (4.174) we see that

$$\frac{1}{2} \left( \langle \phi(x, y), \psi(x, y) \rangle - \langle x, y \rangle \right) (4.176)$$
\[
= \sum_{l=1, l \neq \frac{k}{2} + 1}^{k} y_l x_{l+1} + 2y_{\frac{k}{2} + 1} x_{\frac{k}{2} + 1} + \sum_{l=2}^{\frac{k}{2} + 1} x_l x_{k-l+2} + \sum_{l=1}^{\frac{k}{2} + 1} y_{k-l+1} y_l + y_{\frac{k}{2} + 1}^2 + x_{\frac{k}{2} + 1}^2, \]

so that (iii) is also satisfied. Let
\[
\theta(x, y, z) = z + \frac{1}{2} \left( \langle \phi(x, y), \psi(x, y) \rangle - (x, y) \right). \tag{4.177}
\]

By Theorem 4.6.2, \(A(x, y, z) = (\phi, \psi, \theta)\) is an automorphism of \(N_k/\Gamma_k\).

Let \(\alpha\) be the given irrational number. Let \(\mathbf{\hat{\alpha}}\) be the element \((x, y)\) in \(R_{2k}\) for which \(x = 0\) and \(y = (\alpha, 0, \ldots, 0)\), and let \(\tilde{\alpha} = (\mathbf{\hat{\alpha}}, 0) \in N_k\). Define
\[
T(x, y, z) = \tilde{\alpha} A(x, y, z). \tag{4.178}
\]

\(T\) induces the affine transformation \(T_1(x, y) = \mathbf{\hat{\alpha}} A_1(x, y)\) on \(K_{2k}\). By Theorem 4.3.3, it is sufficient to check that \(T_1\) is uniquely ergodic in order to prove that \(T\) is uniquely ergodic. Since \(A_1\) has the matrix representation (4.169), we have \((A_1 - I)^3 = 0\) and
\[
(A_1 - I)K_{2k} = \{(0, z_2, \ldots, z_{2k}) \mid z_i \in K\}. \]

Hence, \(\left[(A_1 - I)K_{2k}, \tilde{\alpha}\right] = K_{2k}\). So by Theorem 4.3.4, \(T_1\) and hence \(T\) is uniquely ergodic.

If \(T^n(\Gamma_k) = (x(n), y(n), z(n))\) then we have
\[
y_1(n) = \alpha n
\]
\[
x_k(n) = \sum_{i=1}^{n-1} y_1(i) = \left(\sum_{i=1}^{n-1} i\right) \alpha = p_2(n) \alpha
\]
\[
y_2(n) = \sum_{i=1}^{n-1} x_k(i) = \left(\sum_{i=1}^{n-1} p_2(i)\right) \alpha = p_3(n) \alpha
\]
\[
x_{k-1}(n) = \sum_{i=1}^{n-1} (y_2(i) + x_k(i)) = \sum_{i=1}^{n-1} (p_3(i) + p_2(i)) \alpha \tag{4.179}
\]

and in general
\[
y_l(n) = p_{2l-1}(n) \alpha, \quad l = 1, \ldots, k \tag{4.180}
\]

and
\[
x_{l+1}(n) = p_{2k-2l}(n) \alpha, \quad l = 0, \ldots, k - 1. \tag{4.181}
\]
Also,

\[
z(n) = \sum_{i=1}^{n-1} \left( \sum_{l=1, l \neq \frac{k+1}{2}}^{k} y_l(i)x_{i+1}(i) + 2y_{\frac{k+1}{2}}(i)x_{\frac{k+1}{2}+1}(i) + \sum_{l=2}^{\frac{k-1}{2}} x_l(i)x_{k-l+2}(i) \right) \\
+ \sum_{l=1}^{\frac{k-1}{2}} y_{k-l+1}(i)y_l(i) + y_{\frac{k+1}{2}}^2(i) + x_{\frac{k+1}{2}+1}(i) \right) \\
= p(n)\alpha^2, \tag{4.182}
\]

where \( p(n) \) is a rational polynomial of degree \( k^2 \). The sequence corresponding to \( T^n(\Gamma_k) \) in the fundamental domain \( F \) of \( N_k/\Gamma_k \), is therefore

\[
p(n)\alpha^2 - \left( \sum_{l=0}^{k-1} [p_{2l+1}(n)\alpha]p_{2k-2l}(n)\alpha \right). \tag{4.183}
\]

By Lemma 4.5.3, we can get rid of the polynomial \( p(n)\alpha^2 \) by extending our system \( (N_k/\Gamma_k, T) \) to a system \( (G/\Gamma, S) \), where \( G = N_k \times \mathbb{R}^d \) and the value of \( d \) depends on whether \( 1, \alpha, \alpha^2 \) are rationally independent or not.

Let now \( k \) be even. Define \( A_1 \) by

\[
\phi_l(x, y) = \begin{cases} 
  x_l + y_{k-l+1} + x_{l+1} & \text{if } l \neq \frac{k}{2}, k \\
  x_l + y_{k-l+1} + 2x_{l+1} & \text{if } l = \frac{k}{2} \\
  x_l + y_{k-l+1} & \text{if } l = k
\end{cases} \tag{4.184}
\]

and

\[
\psi_l(x, y) = \begin{cases} 
  y_l & \text{if } l = 1 \\
  y_l + x_{k-l+2} & \text{if } l \neq 1, \frac{k}{2} + 1 \\
  y_l + 2x_{k-l+2} & \text{if } l = \frac{k}{2} + 1
\end{cases} . \tag{4.185}
\]

Note the similarity to the case when \( k \) is odd. So we see that \( A_1 \) is an automorphism and that

\[
\langle \phi(x, y), \psi(x', y') \rangle - \langle x, y' \rangle \tag{4.186}
\]

\[
= \sum_{l=1, l \neq \frac{k}{2}+1}^{k} (x_l + y_{k-l+1} + x_{l+1})(y'_l + x'_{k-l+2})
\]
\(+ (x_{\frac{k}{2}} + y_{\frac{k}{2}+1} + 2x_{\frac{k}{2}+1})(y'_{\frac{k}{2}} + x'_{\frac{k}{2}+2}) + (x_{\frac{k}{2}+1} + y'_{\frac{k}{2}} + x_{\frac{k}{2}+2})(y'_{\frac{k}{2}+1} + 2x'_{\frac{k}{2}+1}) - (x, y')\) \hfill (4.187)

\[ \begin{aligned}
&= \sum_{l=2, l \neq \frac{k}{2}+1}^{k} x_l x'_{k-l} + \sum_{l=1}^{k} y_{k-l+1}y'_l + \sum_{l=1, l \neq \frac{k}{2}+1}^{k} y_{k-l+1}x'_{k-l+2} \\
&+ \sum_{l=1, l \neq \frac{k}{2}+1}^{k} x_{l+1}y'_l + \sum_{l=1}^{k} x_{l+1}x'_{k-l+2} \\
&+ 2\left( x_{\frac{k}{2}+1} x'_{\frac{k}{2}+1} + y'_{\frac{k}{2}+1} x_{\frac{k}{2}+1} + x_{\frac{k}{2}+1} y'_{\frac{k}{2}+1} + x_{\frac{k}{2}+1} x'_{\frac{k}{2}+2} + x_{\frac{k}{2}+2} x'_{\frac{k}{2}+1} \right) \hfill (4.188) \\
&= \sum_{l=2}^{\frac{k}{2}} (x_l x'_{k-l+2} + x'_l x_{k-l+2}) + \sum_{l=1}^{\frac{k}{2}} (y_{k-l+1}y'_l + y'_{k-l+1}y_l) \\
&+ \sum_{l=1, l \neq \frac{k}{2}}^{\frac{k}{2}} (y_l x'_{l+1} + x_{l+1}y'_l) + \sum_{l=2}^{\frac{k}{2}} (x_{l+1}x'_{k-l+2} + x'_{l+1}x_{k-l+2}) \\
&+ 2\left( x_{\frac{k}{2}+1} x'_{\frac{k}{2}+1} + y'_{\frac{k}{2}+1} x_{\frac{k}{2}+1} + x_{\frac{k}{2}+1} y'_{\frac{k}{2}+1} + (x_{\frac{k}{2}+1} x'_{\frac{k}{2}+2} + x_{\frac{k}{2}+2} x'_{\frac{k}{2}+1}) \right). \hfill (4.189)
\end{aligned} \]

So \( A_1 \) has property (i), (ii) and (iii) of Theorem 4.6.2. The rest of the proof is the same as when \( k \) is odd.

(b) Define \( A_1 \) on \( K^{2k} \) by

\[ \phi_l(x, y) = \begin{cases} 
  x_l + y_{k-l} + x_{l+2} & \text{if } l \neq k-1, k \\
  x_l + y_{k-l} & \text{if } l = k-1 \\
  x_l & \text{if } l = k
\end{cases} \] \hfill (4.190)

and

\[ \psi_l(x, y) = \begin{cases} 
  y_l & \text{if } l = 1 \\
  y_l + x_{k-l+2} & \text{if } l \neq 1.
\end{cases} \] \hfill (4.191)

Let \( R^{2k} = S_1 \times S_2 \), where

\[ S_1 = \left\{ (y_1, x_{k-1}, y_3, x_{k-3}, \ldots, x_2, y_k) \mid y_l, x_i \in \mathbb{R} \right\} = \mathbb{R}^k \] \hfill (4.192)

and

\[ S_2 = \left\{ (x_k, y_2, x_{k-2}, y_4, \ldots, y_{k-1}, x_1) \mid y_l, x_i \in \mathbb{R} \right\} = \mathbb{R}^k. \] \hfill (4.193)
Note that $S_1$ and $S_2$, considered as subspaces of $\mathbb{R}^{2k}$ are invariant under $A_1$ and that a matrix representation of $A_1$ can be written as $B_1 \times B_2$ (where $B_i$ is a transformation on $S_i$, $i = 1, 2$), and

$$ B_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ & & \ddots \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ & & \end{pmatrix} \quad \text{(4.194)} $$

and

$$ B_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ & & \ddots \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ & & \end{pmatrix} \quad \text{(4.195)} $$

So $\det A_1 = 1$ and $A_1$ is therefore invertible. Let $y_0 = x_{k+1} = x_{k+2} = 0$. Then

$$ \langle \phi(x, y), \psi(x', y') \rangle - \langle x, y' \rangle $$

$$ = \sum_{l=1}^{k} (x_l + y_{k-l} + x_{l+2})(y'_l + x'_{k-l+2}) - \langle x, y' \rangle $$

$$ = \sum_{l=1}^{k} (x_l x'_{k-l+2} + y_{k-l} y'_l + y_{k-l} x'_{k-l+2} + x_{l+2} y'_l + x_{l+2} x'_{k-l+2}) $$

$$ = \sum_{l=2}^{k+1} (x_l x'_{k-l+2} + x'_l x_{k-l+2}) + \sum_{l=1}^{k-1} (y_{k-l} y'_l + y y'_{k-l}) $$
which shows that (ii) and (iii) of Theorem 4.6.2 are satisfied. Therefore $A_1$ can be extended to an automorphism $A$ of $K_{2k}$ by letting

$$
\theta(x, y, z) = z + \frac{1}{2} \left( \langle \phi(x, y), \psi(x, y) \rangle - \langle x, y \rangle \right). \tag{4.197}
$$

Let $\hat{\alpha} = (x, y) \in \mathbb{R}^{2k}$ where $x = (0, \ldots, 0, \beta)$ and $y = (\alpha, 0, \ldots, 0)$ and let $\tilde{\alpha} = (\hat{\alpha}, 0)$. Define the affine transformation $T$ on $N_k/\Gamma_k$ by

$$
T(x, y, z) = \tilde{\alpha}A(x, y, z). \tag{4.198}
$$

By using the matrix representation (4.194) of $A_1$ we observe that $(A_1 - I)^3 = 0$ and that $(A_1 - I)K_{2k} = \{(x, y) \mid y_1 = 0, x_k = 0, y_i \in K, i > 1, x_j \in K, j < k\}$. Therefore, $[(A_1 - I)K_{2k}, \hat{\alpha}] = K_{2k}$ and $T$ is uniquely ergodic by Theorem 4.3.3 and Theorem 4.3.4. Let $T^n(\Gamma_k) = (x(n), y(n), z(n))$. Since

\begin{align*}
y_1(n) &= \alpha n \\
x_{k-1}(n) &= \left( \sum_{i=1}^{n-1} i \right) \alpha = p_2(n) \alpha \\
y_2(n) &= \left( \sum_{i=1}^{n-1} i \right) \beta = r_2(n) \beta \\
x_k(n) &= \beta n \\
y_3(n) &= \left( \sum_{i=1}^{n-1} p_2(i) \right) \alpha = p_3(n) \alpha \\
x_{k-2}(n) &= \left( \sum_{i=1}^{n-1} \left( r_2(i) + i \right) \right) \beta = r_3(n) \beta
\end{align*}

and so on, where $p_l(n), r_l(n)$ are rational polynomials of degree $l$, we have

\begin{align*}
y_{2l+1}(n) &= p_{2l+1}(n) \alpha, \quad l = 0, \ldots, \frac{k-1}{2} \\
x_{k-2l+1}(n) &= p_{2l}(n) \alpha, \quad l = 1, \ldots, \frac{k-1}{2} \\
x_{k-2l}(n) &= r_{2l+1}(n) \beta, \quad l = 0, \ldots, \frac{k-1}{2} \\
y_{2l}(n) &= r_{2l}(n) \beta, \quad l = 1, \ldots, \frac{k-1}{2}.
\end{align*} \tag{4.200}

So we obtain the sequence

$$
z(n) - \sum_{l=1}^{k} [y_l(n)] x_l(n)
$$
\[
\begin{align*}
  &= z(n) - \left( \sum_{l=0}^{k-1} [y_{2l+1}(n)] x_{2l+1}(n) + \sum_{l=1}^{k-1} [y_{2l}(n)] x_{2l}(n) \right) \\
  &= p(n)\alpha\beta - \left( \sum_{l=0}^{k-1} [p_{2l+1}(n)\alpha] r_{k-2l}(n)\beta + \sum_{l=1}^{k-1} [r_{2l}(n)\beta] p_{k-2l+1}(n)\alpha \right) \quad (4.201)
\end{align*}
\]

in the fundamental domain \( F \) of \( N_k/\Gamma_k \), where \( p(n) \) is a rational polynomial. By Lemma 4.5.3 there is a system \( (N_k/\Gamma_k \times K^d, S) \) generating \( q(n) \).

(c) Define \( A_1 \) on \( K^{2k} \) by

\[
\phi_l(x, y) = \begin{cases} 
  x_l + y_{k-l-1} + x_{l+2} & \text{if } l \neq k-1, k \\
  x_l & \text{if } l = k-1, k
\end{cases} \quad (4.202)
\]

and

\[
\psi_l(x, y) = y_l + x_{k-l+1}. \quad (4.203)
\]

The subspaces

\[
S_1 = \left\{ (x_k, y_1, x_{k-2}, y_3, \ldots, x_2, y_{k-1}) \mid x_i, y_i \in K \right\} \quad (4.204)
\]

and

\[
S_2 = \left\{ (x_{k-1}, y_2, x_{k-3}, y_4, \ldots, x_1, y_k) \mid x_i, y_i \in K \right\} \quad (4.205)
\]

are invariant under \( A_1 \) and \( B \times B \) is a matrix representation of \( A_1 \), where

\[
B = \begin{pmatrix}
  1 & 1 & 0 \\
  1 & 1 & 1 \\
  1 & 1 & 1 \\
  1 & 1 & 1 \\
  \vdots \\
  1 & 1 & 1 \\
  0 & 1 & 1 \\
  1 & 1 & 1 \\
  1 & 1 & 1
\end{pmatrix}. \quad (4.206)
\]
We have

\[
\langle \phi(x, y), \psi(x', y') \rangle - \langle x, y' \rangle = \sum_{l=1}^{k} (x_l + y_{k-l-1} + x_{l+2})(y'_l + x'_{k-l+1}) - \langle x, y' \rangle
\]

\[
= \sum_{l=1}^{k} \left( x_l x'_{k-l+1} + y_{k-l-1} y'_l + y_{k-l-1} x'_{k-l+1} + x_{l+2} y'_l + x_{l+2} x'_{k-l+1} \right)
\]

\[
= \frac{b}{2} \left( x_l x'_{k-l+1} + x'_{k-l+1} \right) + \frac{b}{2} \left( \sum_{l=1}^{k-1} \left( y_{k-l-1} y'_l + y y'_{k-l-1} \right) + (y x'_{l+2} + x_{l+2} y'_l) \right)
\]

\[
+ \left( x_{l+2} x'_{k-l+1} + x'_{l+2} x_{k-l+1} \right).
\]

(4.207)

So (i), (ii) and (iii) of Theorem 4.6.2 are satisfied, and we can define an automorphism \(A\) on \(N_k/\Gamma_k\) by (4.197) as done in the previous cases. Let \(x_\alpha = (0, \ldots, 0, \beta, \alpha)\) and define \(T(x, y, z) = (x_\alpha, 0, 0)A(x, y, z)\). If \(T^n(\Gamma_k) = (x(n), y(n), z(n))\) then

\[
\begin{align*}
x_{k-2l}(n) &= p_{2l+1}(n)\alpha, \quad l = 0, \ldots, k/2 + 2 \\
y_{2l-1}(n) &= p_{2l}(n)\alpha, \quad l = 1, \ldots, k/2 \\
x_{k-2l+1}(n) &= p_{2l-1}(n)\beta, \quad l = 1, \ldots, k/2 \\
y_{2l}(n) &= p_{2l}(n)\beta, \quad l = 1, \ldots, k/2.
\end{align*}
\]

(4.208)

So we obtain the sequence

\[
\begin{align*}
z(n) - \sum_{l=1}^{k} \left[ y_l(n) \right] x_l(n) \\
= z(n) - \sum_{l=1}^{k/2} \left( \left[ y_{2l+1}(n) \right] x_{2l+1}(n) + \left[ y_{2l}(n) \right] x_{2l}(n) \right) \\
= z(n) - \sum_{l=1}^{k/2} \left( p_{2l}(n)\alpha p_{k-2l+1}(n) + p_{2l}(n)\beta p_{k-2l+1}(n)\alpha \right)
\end{align*}
\]

(4.209)

in the fundamental domain \(F\) of \(N_k/\Gamma_k\). By Lemma 4.5.3 there is a system \((N_k/\Gamma_k \times K^d, S)\) generating \(q(n)\).

\(\Box\)

Remarks:
(1) The coefficients of the polynomials \( p_i(n) \) and \( r_i(n) \) that we obtained in the proof of Theorem 4.6.3, are rational numbers but not integers except when \( i = 1 \). Is it possible to find a system for which all the coefficients of the polynomials \( p_i(n) \) and \( r_i(n) \) are integers?

(2) Observe that each generalized polynomial obtained in Theorem 4.6.3 is a sum of terms which all have the same degree. Only in (a) when \( k = 1 \) and in (b) when \( k = 1 \) do we get just one term, namely \( q(n) = [\alpha n]p_2(n)\alpha \) in (a) and \( q(n) = [\alpha n]\beta n \) in (b). Actually, the only simple generalized polynomials of the type \([\alpha n^k]\beta n^k\) which we know to generate dynamically are those of degree two and those of degree 3 for which \( \beta = b\alpha, b \in \mathbb{Z}, b[\alpha n]\alpha n^2 \) (see Proposition 4.6.4 below) and \( b[\alpha n^2]\alpha n \) (see remark after proof of Proposition 4.6.4).

(3) It is easy to use the techniques of Chapter III (van der Corput’s method) to prove that all the generalized polynomials of Theorem 4.6.3 are uniformly distributed \( \pmod{1} \).

In the proposition below we will give an example of how sums of generalized polynomials from Theorem 4.6.3 can be generated.

**Proposition 4.6.4** Let \( 1, \alpha_1, \ldots, \alpha_k \) be rationally independent and let \( \beta_i = \sum_{j=1}^{k} b_{ij}\alpha_j \neq 0 \), \( 2b_{ij} \in \mathbb{Z}, b_{ii} \in \mathbb{Z} \) and \( b_{ij} = b_{ji}, \ i = 1, \ldots, k \). Then

\[
\sum_{i=1}^{k} [\alpha_i n] \beta_i n^2 + p(n) \tag{4.210}
\]
is uniformly distributed (mod 1) for any polynomial \( p(n) \) and is generated by a uniquely ergodic affine transformation on a nilmanifold \( G/\Gamma \times K^d \), where \( G \) is a subgroup of \( N_k \).

**Proof:** Let \( B \) be the matrix \((b_{ij})\). Note that \( B \) is symmetric. Suppose first that 
\[
\det(B) \neq 0. 
\]
Define \( A_1 \) on \( K^{2k} \) by
\[
\phi_l(x, y) = x_l + 2 \sum_{j=1}^{k} b_{lj} y_j, \\
\psi_l(x, y) = y_l, \quad l = 1, \ldots, k. 
\]
So a matrix representation of \( A_1 \) with respect to the standard basis for \( R^{2k} = \{(x, y) : x, y \in R^k\} \) is 
\[
\begin{pmatrix}
I & 2B \\
0 & I
\end{pmatrix}. 
\]
We have
\[
\langle \phi(x, y), \psi(x', y') \rangle - \langle x, y' \rangle = 2 \sum_{l,j=1}^{k} b_{lj} y_j y'_j \\
= \sum_{l<j} 2b_{lj} (y_l y'_j + y_j y'_l) + \sum_{l=1}^{k} 2b_{ll} y_l y'_l. 
\]
So \( A_1 \) extends to an automorphism \( A \) on \( N_k/\Gamma_k \) by Theorem 4.6.2. Let \( \hat{\alpha} = (\alpha_1, \ldots, \alpha_k) \) and \( \hat{\beta} = (\beta_1, \ldots, \beta_k) \). Define
\[
T(x, y, z) = (\hat{\beta}, \hat{\alpha}, 0) A(x, y, z). 
\]
\( T \) induces the transformation \( T_1(x, y) = (\hat{\beta}, \hat{\alpha}) A_1(x, y) \) on the maximal torus factor \( K^{2k} \) of \( N_k/\Gamma_k \). We have \((A_1 - I)^n = \begin{pmatrix} 0 & 2B \\ 0 & 0 \end{pmatrix}^n = (0) \) for some \( n \) and 
\((A_1 - I)(x, y) = (2By, 0) \). So 
\((A_1 - I)K^{2k}, (\hat{\beta}, \hat{\alpha}) \) = \( K^{2k} \) since \( \text{rank}(B) = k \). Hence, \( T_1 \) and therefore \( T \) is uniquely ergodic. Since \( T^n(\Gamma_k) = (\hat{\beta} n^2, \hat{\alpha} n, z(n)) \) for some polynomial \( z(n) = p_1(n) \sum_{l=1}^{k} \alpha_l \beta_l \), \( p_1(n) \) a polynomial with rational coefficients, we get the sequence
\[
z(n) - \sum_{l=1}^{k} [\alpha_l n] \beta_l n^2. 
\]
in the fundamental domain $F$ of $N_k/\Gamma_k$. We can now apply Lemma 4.5.3 to finish this case.

If det$(B) = 0$, let $r = \text{rank}(B) < k$. Let $b_i = (b_{i1}, b_{i2}, \ldots, b_{ik})$, $i = 1, \ldots, k$, be the row vectors of the matrix $B$. In order to simplify writing, we will assume that $b_1, \ldots, b_r$ are rationally independent and that $b_i = \sum_{j=1}^r c_{ij}b_j$, $i = r+1, \ldots, k$, for some $c_{ij} \in \mathbb{Q}$. Let us assume that $c_{ij} \in \mathbb{Z}$. Then

$$H = \left\{ (x, y, z) \in N_k \mid x \in \mathbb{R}^k, x_i = \sum_{j=1}^r c_{ij}x_j, i = r+1, \ldots, k, x_j \in \mathbb{R}, j = 1, \ldots, r \right\}$$

(4.215)

is a $T$-invariant non-abelian subgroup of $N_k$ if $T$ is defined by (4.213). The maximal torus factor of $H/\Gamma$ is $K^{k+r}$ and $(A_1 - I)K^{k+r} \cong K^r$ so that $T$ restricted to $H/\Gamma$ is uniquely ergodic. The rest of the proof is the same as in the previous case.

\[\square\]

**Remark:** If we instead of equations (4.211) use the equations

$$\phi_l(x, y) = x_l$$

$$\psi_l(x, y) = y_l + 2 \sum_{j=1}^k b_{lj}x_j, \ l = 1, \ldots, k.$$  

(4.216)

and do other corresponding adjustments in the proof, we can show that

$$\sum_{i=1}^k [\beta_i n^2] \alpha_i n + p(n)$$

(4.217)

is uniformly distributed (mod 1) and generated by a uniquely ergodic transformation.

Since

$$\sum_{i=1}^k [\alpha_i n] \beta_i n^2 = \sum_{i=1}^k [\alpha_i n] \sum_{j=1}^k b_{ij} \alpha_j n^2$$

$$= \sum_{i<j} b_{ij} [\alpha_i n] \alpha_j n^2 + \sum_{i=1}^k b_{ii} [\alpha_i n] \alpha_i n^2 + \sum_{i<j} b_{ij} [\alpha_i n] \alpha_j n^2$$
\[ \sum_{i<j} b_{ij} \left( [\alpha_j n] \alpha_i n^2 + [\alpha_i n] \alpha_j n^2 \right) + \sum_{i=1}^k b_{ii} [\alpha_i n] \alpha_i n^2 \]  

(4.218)

we see that \[ \sum_{i=1}^k [\alpha_i n] \beta_i n^2 \] is a linear combination over \( \mathbb{Z} \) of generalized polynomials obtained in Theorem 4.6.3 (a) \((k = 1)\) and (c) \((k = 2)\). A natural question to ask is if any generalized polynomial

\[ q(n) = \sum_{i=1}^k \left( [\alpha_i n] \beta_i n^2 + [\beta_i n] \alpha_i n^2 \right) + \sum_{i=1}^l [\lambda_i n] \lambda_i n^2 \]  

(4.219)

is uniformly distributed \((\text{mod } 1)\). The answer is yes as long as \( q(n) \) does not reduce to zero. We will say that a generalized polynomial \((4.219)\) \textit{reduces to a polynomial} \( p(n) \) if there exist irrational numbers \( \sigma_i \) such that \( 1, \sigma_1, \ldots, \sigma_{k_1} \) are rationally independent,

\[ \alpha_i = a_{i0} + \sum_{j=1}^{k_1} a_{ij} \sigma_j \]
\[ \beta_i = b_{i0} + \sum_{j=1}^{k_1} b_{ij} \sigma_j \]
\[ \lambda_j = c_{j0} + \sum_{i=1}^{l} c_{ji} \sigma_i \]  

(4.220)

for some \( a_{ij}, b_{ij}, c_{ij} \in \mathbb{Q}, i = 1, \ldots, k, j = 1, \ldots, l \), so that

\[ p(n) = \left( \sum_{i=1}^k (a_{i0} \beta_i + b_{i0} \alpha_i) + \sum_{i=1}^l c_{i0} \lambda_i \right) n^3 \]  

(4.221)

and

\[ \begin{align*}
& \sum_{i=1}^k \left( \sum_{j=1}^{k_1} a_{ij} [\sigma_j n] \right) \left( \sum_{j=1}^{k_1} b_{ij} [\sigma_j n] \right) n^2 + \sum_{i=1}^k b_{ii} [\sigma_i n] \left( \sum_{j=1}^{k_1} a_{ij} [\sigma_j n] \right) n^2 \\
& \quad + \left( \sum_{j=1}^{k_1} c_{ij} [\sigma_j n] \right) \left( \sum_{j=1}^{k_1} c_{ij} [\sigma_j n] \right) n^2 \\
& = \sum_{i=1}^k \sum_{j,r} (a_{ij} b_{ir} + b_{ij} a_{ir} + c_{ij} c_{ir}) [\sigma_j n] [\sigma_r n^2] \\
& = \sum_{j,r} \sum_{i=1}^k (a_{ij} b_{ir} + b_{ij} a_{ir} + c_{ij} c_{ir}) [\sigma_j n] [\sigma_r n^2] \equiv 0,
\end{align*} \]  

(4.222)
i.e., \( \sum_{i=1}^{k} (a_{ij}b_{ir} + b_{ij}a_{ir} + c_{ij}c_{ir}) = 0 \) for all \( j, r \). Let \( d_{jr} = \sum_{i=1}^{k} (a_{ij}b_{ir} + b_{ij}a_{ir} + c_{ij}c_{ir}) \).

Note that

\[
q(n) \mapsto p(n) + \sum_{j,r} d_{jr} [\sigma_j n] \sigma_r n^2 \mapsto p(n) + \left[ \sum_{j,r} d_{jr} \sigma_j n \right] \sigma_r n^2
\]  

(4.223)

and that \( d_{jr} = d_{rj} \). So if \( a_{ij}, b_{ij}, c_{ij} \in \mathbb{Z} \) then it follows from Proposition 4.6.4 that \( q(n) \) is uniformly distributed (mod 1) and can be generated by a uniquely ergodic system. So we have the following corollary.

**Corollary 4.6.5** Let \( \alpha_i, \beta_i, \lambda_j, i = 1, \ldots, k; \ j = 1, \ldots, l \), be irrational numbers such that the generalized polynomial

\[
q(n) = \sum_{i=1}^{k} ([\alpha_i n] \beta_i n^2 + [\beta_i n] \alpha_i n^2) + \sum_{i=1}^{l} [\lambda_i n] \lambda_i n^2
\]  

(4.224)

does not reduce to a polynomial. Then for any polynomial \( p(n) \), \( q(n) + p(n) \) is uniformly distributed (mod 1) and can be generated by a dynamical system \((G/\Gamma, T)\) where \( G \) is a subgroup of a Heisenberg group and \( T \) is an affine transformation on \( G/\Gamma \).

### 4.7 More generalized polynomials of higher degrees

In order to generate dynamically generalized polynomials having nested brackets we will use nilpotent Lie groups \( G \) with \( \dim(G^1) > 1 \). We will first consider a one-
parameter subgroup of $G_{\text{ud}}$ which induces a rotation on $G_{\text{ud}}/\Gamma$. Recall that

$$G_{\text{ud}} = \left\{ \begin{pmatrix} 1 & x_{12} & \cdots & x_{1,k+1} \\ 1 & \ddots & \vdots \\ \vdots & \ddots & 1 \\ 0 & \cdots & x_{k,k+1} & 1 \end{pmatrix} \mid x_{ij} \in \mathbb{R}, 1 \leq i < j \leq k + 1 \right\}.$$  

(4.225)

Since

$$G_{\text{ud}}^1 = \{ (x_{ij}) \in G_{\text{ud}} \mid x_{i,i+1} = 0 \}$$

(4.226)

the maximal torus factor of $G_{\text{ud}}/\Gamma$ is $K^k = \{ (x_{12}, \ldots, x_{k,k+1}) \mid x_{i,i+1} \in K \}$. Let

$$\tilde{\alpha}(t) = \begin{pmatrix} 1 & \alpha_k t & \frac{\alpha_k \alpha(k-1) t^2}{2!} & \frac{\alpha_k \alpha(k-2) t^3}{3!} & \cdots & \frac{1}{k!} \prod_{i=1}^{k} \alpha_i t^k \\ 1 & \alpha_{k-1} t & \frac{\alpha_{k-1} \alpha(k-2) t^2}{2!} & \cdots & \frac{1}{(k-1)!} \prod_{i=1}^{k-1} \alpha_i t^{k-1} \\ \vdots & \ddots & \vdots \\ 1 & \alpha_2 t & \frac{\alpha_2 t^2}{2!} & \cdots & \frac{1}{1!} \alpha_1 t \\ 1 & \alpha_1 t \\ 1 \end{pmatrix},$$

(4.227)

i.e., $\tilde{\alpha}(t) = (x_{ij}(t))$, where

$$x_{ij}(t) = \begin{cases} \frac{1}{(j-i)!} \prod_{l=k-i+1}^{k-j+2} \alpha_l t^{j-i} & \text{if } i < j \\ 1 & \text{if } i = j \\ 0 & \text{if } i > j \end{cases}.$$  

(4.228)

We have

$$\sum_{s=1}^{k+1} x_{is}(t) x_{sj}(r) = \sum_{s=1}^{j} x_{is}(t) x_{sj}(r)$$

$$= \frac{1}{(j-i)!} \prod_{l=k-i+1}^{k-j+2} \alpha_l t^{j-i} \frac{1}{(j-s)!} \prod_{l=k-s+1}^{k-j+2} \alpha_l r^{j-s}$$

$$= \frac{1}{(j-i)!} \prod_{l=k-i+1}^{k-j+2} \alpha_l \sum_{l=0}^{j-i} \frac{(j-i)!}{l!(j-i-l)!} t^l r^{j-i-l}$$

$$= \frac{1}{(j-i)!} \prod_{l=k-i+1}^{k-j+2} \alpha_l (t+r)^{j-i} = x_{ij}(t+r),$$

(4.229)
so that \( \hat{\alpha}(t) \hat{\alpha}(r) = \hat{\alpha}(t + r) \) for all \( t, r \in \mathbb{R} \). So \( \hat{\alpha}(t), t \in \mathbb{R} \), is a one-parameter subgroup of \( G_{ud} \). Let \( T \) be the rotation on \( G_{ud}/\Gamma \) by \( \hat{\alpha}(1) \). If \( 1, \alpha_1, \ldots, \alpha_k \) are rationally independent, then \( T \) is uniquely ergodic by Theorem 4.3.3 since \( T \) induces the rotation by \( (\alpha_1, \ldots, \alpha_k) \) on the maximal torus factor \( K^k \). By Theorem 4.3.8, the sequence
\[
\left( \theta_{12}(T^n\Gamma), \ldots, \theta_{k,k+1}(T^n\Gamma) \right) \pmod{1} \quad (4.230)
\]
is uniformly distributed \( \pmod{1} \), where \( \theta(x) \) is given by (4.43). Let \( a_l(n) = \theta_{k-l+1,k+1}(n), l = 1, \ldots, k \). We have
\[
a_1(n) = \alpha_1 n \\
a_2(n) = \frac{\alpha_1 \alpha_2}{2!} n^2 - [\alpha_1 n] \alpha_2 n \\
a_3(n) = \frac{\alpha_1 \alpha_2 \alpha_3}{3!} n^3 - \left[ \frac{\alpha_1 \alpha_2}{2!} n^2 - [\alpha_1 n] \alpha_2 n \right] \alpha_3 n - \left[ \alpha_1 n \right] \frac{\alpha_2 \alpha_3}{2!} n^2 \quad (4.231)
\]
and in general
\[
a_l(n) = \frac{\alpha_1 \cdots \alpha_l}{l!} n^l - \sum_{j=1}^{l-1} \left[ a_j(n) \right] \frac{\alpha_{j+1} \cdots \alpha_l}{(l-j)!} n^{l-j}, \quad (4.232)
\]
l = 1, \ldots, k. With this we have proved the following theorem.

**Theorem 4.7.1** Let \( 1, \alpha_1, \ldots, \alpha_k \) be rationally independent. Then the rotation by \( \hat{\alpha}(1) \) on \( G/\Gamma \) defined by (4.227), is uniquely ergodic and generates the generalized polynomial
\[
q(n) = \frac{\alpha_1 \cdots \alpha_k}{k!} n^k - \sum_{j=1}^{k-1} \left[ a_j(n) \right] \frac{\alpha_{j+1} \cdots \alpha_k}{(k-j)!} n^{k-j}, \quad (4.233)
\]
where \( a_j(n), j = 1, \ldots, k - 1, \) are given by (4.232).
Note that $a_k(n)$ given by (4.232) has a term $-\left[q_1(n)-\cdots\left[\alpha_1n\right]\alpha_2n\cdots\left[\alpha_{k-1}n\right]\alpha_kn\right]$ for some generalized polynomial $q_1(n)$. It is not clear how to obtain

$$\left[\cdots\left[\alpha_1n\right]\alpha_2n\cdots\left[\alpha_{k-1}n\right]\alpha_kn\right]$$

(4.234)

by itself by this method. However, we have been able to construct a system for a special class of generalized polynomials of type (4.234) when $k = 3$.

**Proposition 4.7.2** Let $1, \alpha_1, \alpha_2, \alpha_3$ be rationally independent and such that

\[
\begin{align*}
\alpha_1\alpha_2 &= b_0 - (b_1\alpha_1 + b_2\alpha_2 + 2b_3\alpha_3) & \text{and} \\
\alpha_2\alpha_3 &= c_0 - (2c_1\alpha_1 + c_2\alpha_2 - b_1\alpha_3)
\end{align*}
\]

(4.235)

(4.236)

for some $b_i, c_i \in \mathbb{Z}$, $i = 0, 1, 2, 3$. Then

$$q(n) = \left[\alpha_3n\right]\alpha_2n\alpha_1n$$

(4.237)

is generated by a uniquely ergodic affine transformation on a nilmanifold $G/\Gamma$.

**Remark:** If we let $b_0 = b_1 = b_2 = c_0 = c_2 = 0$ and $c_1 = -1, b_3 = -c$ for some $c > 0$ such that $\sqrt{c} \notin \mathbb{Z}$ in the equations (4.235) and (4.236), then $\alpha_1\alpha_2 = 2c\alpha_3$ and $\alpha_2\alpha_3 = 2\alpha_1$ so that $\frac{\alpha_1}{\alpha_3} = \sqrt{c}$. Let $\alpha_3 = \alpha$. Then $\alpha_1 = \alpha\sqrt{c}$ and $\alpha_2 = 2\sqrt{c}$. It follows from Proposition 4.7.2 that

$$\left[\alpha n\right]\alpha\sqrt{c}n$$

(4.238)

is uniformly distributed (mod 1) when $1, \alpha, \sqrt{c}, \alpha\sqrt{c}$ are rationally independent.
Proof: Let
\[
G_{ud}(3) = \left\{ x = \begin{pmatrix} x_1 & y_1 & z \\ 1 & x_2 & y_2 \\ 1 & x_3 \\ 1 \end{pmatrix} \mid x_i, y_i, z \in \mathbb{R} \right\}
\] (4.239)
and let \( \Gamma \) be the subgroup of \( G_{ud}(3) \) of matrices having integer coordinates. It is straightforward to check that
\[
Ax = \begin{pmatrix} 1 & x_1 & y_1 + b_1 x_1 + b_2 x_2 + 2 b_3 x_3 & z + c_1 x_1^2 + b_3 x_3^2 + c_2 y_1 + b_2 y_2 \\ 1 & x_2 & y_2 + 2 c_1 x_1 + c_2 x_2 - b_1 x_3 \\ 1 & x_3 \end{pmatrix}
\] (4.240)
is an automorphism of \( G_{ud}(3) \). Let
\[
\tilde{\alpha} = \begin{pmatrix} 1 & \alpha_1 & 0 & 0 \\ 1 & \alpha_2 & 0 & 0 \\ 1 & \alpha_3 \\ 1 \end{pmatrix}
\] (4.241)
and \( T_1 x \Gamma = \tilde{\alpha} A x \Gamma \). Then
\[
T_1^n(\Gamma) = \begin{pmatrix} 1 & \alpha_1 n & p_1(n) & p_3(n) \\ 1 & \alpha_2 n & p_2(n) \\ 1 & \alpha_3 n \end{pmatrix} \Gamma,
\] (4.242)
where
\[
\begin{align*}
p_1(n) &= (b_1 \alpha_1 + b_2 \alpha_2 + 2 b_3 \alpha_3 + \alpha_1 \alpha_2)^{n(n-1)/2} = b_0^{n(n-1)/2} \\
p_2(n) &= (2 c_1 \alpha_1 + c_2 \alpha_2 - b_1 \alpha_3 + \alpha_2 \alpha_3)^{n(n-1)/2} = c_0^{n(n-1)/2} \\
p_3(n) &= (c_1 \alpha_1^2 + b_3 \alpha_3^2)^{n-1} \sum_{k=1}^n k^2 + \left( c_2 (b_1 \alpha_1 + b_2 \alpha_2 + 2 b_3 \alpha_3 + \alpha_1 \alpha_2) \\
&\quad + (2 c_1 \alpha_1 + c_2 \alpha_2 - b_1 \alpha_3 + \alpha_2 \alpha_3)(b_2 + \alpha_1) \right) \sum_{k=1}^{n-1} \frac{k(k-1)}{2} \\
&\quad + (c_1 \alpha_1^2 + b_3 \alpha_3^2) \sum_{k=1}^{n-1} k^2 + (c_2 b_0 + c_0 (b_2 + \alpha_1)) \sum_{k=1}^{n-1} \frac{k(k-1)}{2}
\end{align*}
\] (4.243)
Note that the maximal torus factor of \( G_{ud}(3)/\Gamma \) is \( K^3 = \{(x_1, x_2, x_3) \mid x_i \in \mathbb{R}/\mathbb{Z}, i = 1, 2, 3\} \). So, since \( 1, \alpha_1, \alpha_2, \alpha_3 \) are rationally independent, \( T_1 \) is uniquely ergodic.
Therefore the sequence

\[
\theta_{14}(T^n \Gamma) = p_3(n) - [p_2(n) - [\alpha_3 n] \alpha_2 n] \alpha_1 n - [\alpha_3 n] p_1(n) = p_3(n) - [c_0 \frac{n(n-1)}{2} - [\alpha_3 n] \alpha_2 n] \alpha_1 n = p(n) + [\alpha_3 n] \alpha_2 n \alpha_1 n,
\]

is uniformly distributed (mod 1), where

\[
p(n) = (c_1 \alpha_1^2 + b_3 \alpha_3^2) \sum_{k=1}^{n-1} k^2 + c_0 \alpha_1 \sum_{k=1}^{n-1} \frac{k(k-1)}{2} - c_0 \alpha_1 \frac{n^2(n-1)}{2} + \alpha_1 n
\]

is a polynomial of degree 3. We need to get rid of \(p(n)\) in (4.246), i.e., we want to find a system \((G/\Gamma, T)\) for which \(\theta_{14}(T^n \Gamma) = [\alpha_3 n] \alpha_2 n \alpha_1 n\). Let \(\beta_i \in \mathbb{R}, \, i = 1, 2, 3\), be such that \(p(n) = \beta_3 n^3 + \beta_2 n^2 + \beta_1 n\).

Let us first find transformations \(S_1\) and \(S_2\) on the tori \(K^6\) and \(K^5\), respectively, which both generate \((\alpha_1 n, \alpha_2 n, \alpha_3 n, p(n))\). If \(\beta_3\) is rationally independent of \(1, \alpha_1, \alpha_2, \alpha_3\), then let

\[
S_1(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1 + \alpha_1, \ x_2 + \alpha_2, \ x_3 + \alpha_3, \ x_4 + \beta_3, \ x_5 + 6x_4 + 6\beta_3 + 2\beta_2, \ x_6 + x_5 + \beta_3 + \beta_2 + \beta_1).
\]

For then

\[
S_1^n(0) = (\alpha_1 n, \ \alpha_2 n, \ \alpha_3 n, \ 3\beta_3 (n^2 + n) + 2\beta_2 n, \ \beta_3 n^3 + \beta_2 n^2 + \beta_1 n). \quad (4.249)
\]

If \(\beta_3 = a_0 + \sum_{i=1}^{3} a_i \alpha_i, \, a_i \in \mathbb{Z}\), then let

\[
S_2(x_1, x_2, x_3, x_5, x_6) = (x_1 + \alpha_1, \ x_2 + \alpha_2, \ x_3 + \alpha_3, \ x_5 + 6 \sum_{i=1}^{3} a_i x_i + 6\beta_3 + 2\beta_2, \ x_6 + x_5 + \beta_3 + \beta_2 + \beta_1)\quad (4.250)
\]
so that

\[ S_2^n(0) = (\alpha_1 n, \alpha_2 n, \alpha_3 n, 3\beta_3(n^2 + n) + 2\beta_2 n, \beta_3 n^3 + \beta_2 n^2 + \beta_1 n). \quad (4.251) \]

Both the transformations \( S_1 \) and \( S_2 \) are uniquely ergodic. If \( \beta_3 \) is rationally independent of \( 1, \alpha_1, \alpha_2, \alpha_3 \), let

\[ G = \{(x, x_4, x_5) \mid x \in G_{ud}(3), \ x_4, x_5 \in \mathbb{R}\} \quad (4.252) \]

and let \( \Gamma \) be the subgroup of elements having integer coordinates. Let

\[ \tilde{\beta} = \begin{pmatrix} 1 & \alpha_1 & 0 & - (\beta_3 + \beta_2 + \beta_1) \\ 1 & \alpha_2 & 0 & \alpha_3 \\ 1 & \end{pmatrix} \quad (4.253) \]

and define

\[ A_5(x) = \begin{pmatrix} 1 & x_1 & y_1 & z - x_5 \\ 1 & x_2 & y_2 & x_3 \\ 1 & \end{pmatrix}. \quad (4.254) \]

Then the transformation \((x, x_4, x_5) \mapsto (A_5 x, x_4, x_5)\) is an automorphism. Now define \( T \) on \( G/\Gamma \) by

\[ T(x, x_4, x_5) = (\tilde{\beta}, \beta_3, 6\beta_3 + 2\beta_2)(A_5 \circ Ax, x_4, x_5 + 6x_4), \quad (4.255) \]

where \( A \) is given by (4.240). \( T \) is an affine transformation and since the induced transformation on the maximal torus factor is the affine transformation \( S_1 \) restricted to \( K^5 \), \( T \) is uniquely ergodic. We have

\[ T^n(\Gamma) = \begin{pmatrix} 1 & \alpha_1 n & p_1(n) & p_3(n) - p(n) \\ 1 & \alpha_2 n & p_2(n) & \alpha_3 n \\ 1 & \end{pmatrix} \Gamma, \beta_3 n, 3\beta_3(n^2 + n) + 2\beta_2 n \quad (4.256) \]
and therefore,

\[ \theta_{14}(T^n(\Gamma, 0, 0) = \lfloor [\alpha_3 n|\alpha_2 n] \rfloor \alpha_1 n \]  

(4.257)

is uniformly distributed (mod 1).

If \( \beta_3 = a_0 + \sum_{i=1}^{3} a_i \alpha_i \) then let

\[ G = \{ (x, x_5) \mid x \in G_{ud}(3), \ x_5 \in \mathbb{R} \} \]  

(4.258)

and let \( \Gamma \) be the subgroup of elements having integer coordinates. Let

\[ T(x, x_5) = (\tilde{\beta}, 6\beta_3 + 2\beta_2)(A_5 \circ A_2, x_5 + 6 \sum_{i=1}^{3} a_i x_i). \]  

(4.259)

The rest of the proof follows as in the case when \( \beta_3 \) is rationally independent of \( 1, \alpha_1, \alpha_2, \alpha_3. \)

\[ \square \]

Recall from Example 3, Section 4.4 the group

\[ G_{AB} = \left\{ \begin{pmatrix} 1 & x & x^2 & \cdots & x^{k-1} & y_k \\ 1 & x & \cdots & x^{(k-1)!} & y_{k-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x & y_2 \\ 0 & 1 & y_1 \\ 1 & \end{pmatrix} \mid x, y_i \in \mathbb{R}, i = 1, \ldots, k \right\}, \]  

(4.260)

whose elements we denote by \( (x, y) \) or \( (x, y_1, \ldots, y_k) \), and which has the group operation (matrix multiplication)

\[ (x, y_1, \ldots, y_k)(x', y_1', \ldots, y_k') = \left( x + x', y_1 + y_1', \ldots, y_i + \sum_{j=0}^{i-1} \frac{x^j}{j!} y_{i-j}', \ldots, y_k + \sum_{j=0}^{k-1} \frac{x^j}{j!} y_{k-j}' \right). \]  

(4.261)

Also, we use \( \Gamma = \{ (m_0k!, m_1, \ldots, m_k) \mid m_i \in \mathbb{Z} \} \) as the uniform discrete subgroup of \( G_{AB}. \)
This group was used in [2, Chapter VIII] to show a result on diophantine approximation and in [8] in an attempt to show uniform distribution of sequences 
\[ q(n) = (\alpha k!)^{-1}\left(\{\alpha n\}^k + (-1)^{k+1}[\alpha n]^k\right), \quad n = 1, 2, 3, \ldots, \] (4.262)
where \(k\) is any integer greater than 1 and \(\alpha\) is an irrational number. Even though the proof in [8] is wrong, the result is true. For since \(q(n) \leftrightarrow \left[\alpha n\right]^k / \alpha k\), \(\alpha n\) and \(\alpha\) is irrational, \(q(n)\) is uniformly distributed (mod 1) by Theorem 3.3.1. By correcting the error in [8] we will show a similar result in Proposition 4.7.3.

Define
\[ \tilde{\alpha}(t) = \left(\alpha t, \lambda t, \frac{\alpha \lambda}{2!} t^2, \ldots, \frac{\alpha^{k-1} \lambda}{k!} t^k\right), \quad t \in \mathbb{R}, \] (4.263)
which we will show is a one-parameter subgroup of \(G_{AB}\). Let
\[ \tilde{\alpha}(t)\tilde{\alpha}(r) = \left(\alpha(t + r), y_1(t, r), \ldots, y_k(t, r)\right). \] (4.264)

By using (4.261), we have
\[ y_i(t, r) = \frac{\alpha^{i-1} \lambda}{i!} t^i + \sum_{j=0}^{i-1} \frac{(\alpha t)^j}{j!} \frac{\alpha^{i-j-1} \lambda}{(i-j)!} r^{i-j} \]
\[ = \frac{\alpha^{i-1} \lambda}{i!} \sum_{j=0}^{i} \frac{i!}{j!(i-j)!} t^j r^{i-j} \]
\[ = \frac{\alpha^{i-1} \lambda}{i!} (t + r)^i. \] (4.265)

So \(\tilde{\alpha}(t)\tilde{\alpha}(r) = \tilde{\alpha}(t + r)\) for all \(t, r \in \mathbb{R}\).

Since \(G_{AB}^1 = \{(0, 0, y_2, \ldots, y_k) \mid y_i \in \mathbb{R}\}\), a rotation by
\[ \tilde{\alpha}(1) = \left(\alpha, \lambda, \frac{\alpha \lambda}{2!}, \ldots, \frac{\alpha^{k-1} \lambda}{k!}\right) \] (4.266)
on either $G_{AB}/\Gamma$ or $\Gamma\backslash G_{AB}$ is uniquely ergodic if and only if $1, \alpha, \lambda$ are rationally independent.

Remark: Here is where Brezin made a mistake. He claims by referring to §5 of [1], that the sequence $\tilde{\alpha}(n), n = 1, 2, \ldots$, is uniformly distributed in $\Gamma\backslash G_{AB}$, i.e. that the rotation by $\tilde{\alpha}(1)$ is uniquely ergodic, if and only if the real numbers $\alpha$ and $\lambda$ are linearly independent over the rational numbers. However, this condition is true only for the flow $\tilde{\alpha}(t), t \in \mathbb{R}$, and not for $\tilde{\alpha}(n), n \in \mathbb{N}$. This is easy to see using Brezin’s choice of $\alpha$ and $\lambda$, $\alpha$ irrational and $\lambda = 1$. For then

$$\Gamma\tilde{\alpha}(n) = \Gamma(\alpha n, \lambda n, \frac{\alpha^2}{2!} n^2, \ldots, \frac{\alpha^{k-1}}{k!} n^k)$$

which is even far from being dense in $\Gamma\backslash G_{AB}$ since the second coordinate is $n$.

The sequence in the fundamental domain $F = \{(x, y_1, \ldots, y_k) | x \in [0, k!], y_i \in [0, 1)\}$ corresponding to $\tilde{\alpha}(n)\Gamma = (\alpha n, \lambda n, \frac{\alpha^2}{2!} n^2, \ldots, \frac{\alpha^{k-1}}{k!} n^k)\Gamma$ is

$$\big((an - \lfloor \frac{an}{k!} \rfloor, \{a_1(n)\}, \ldots, \{a_k(n)\})\big),$$

where

$$a_1(n) = \lambda n,$$

$$a_2(n) = \frac{\alpha \lambda}{2!} n^2 - \lfloor n \lambda \rfloor \alpha n$$

$$a_3(n) = \frac{\alpha^2 \lambda}{3!} n^3 - \lfloor n \lambda \rfloor \frac{\alpha^2}{2!} n^2 - \bigg(\frac{\alpha \lambda}{2!} n^2 - \lfloor n \lambda \rfloor \alpha n\bigg) \alpha n$$

and in general

$$a_i(n) = \frac{\alpha^{i-1} \lambda}{i!} n^i - \sum_{j=1}^{i-1} \alpha_j(n) \frac{\alpha^{i-j}}{(i-j)!} n^{i-j}, i = 2, \ldots, k.$$  

Since the rotation by $\tilde{\alpha}\Gamma$ is uniquely ergodic, we get that $(a_1(n), \ldots, a_k(n))$ is uniformly distributed (mod 1).
Let us now use right cosets $\Gamma x$.

**Proposition 4.7.3** Let $1, \alpha, \alpha \beta$ be rationally independent. The generalized polynomial $[\alpha n]^k \beta$, $k \geq 1$, is uniformly distributed (mod 1) and is generated by a rotation $\Gamma \tilde{\alpha}$ of the nilmanifold $\Gamma \backslash G_{AB}$, where $G_{AB}$ is the group (4.260).

**Proof:** Let $\lambda = (-1)^{k+1} (k!)^2 - k \alpha \beta$ and $\alpha_k = k! \alpha$. Then $1, \alpha_k, \lambda$ are rationally independent. So the rotation by

\[
\tilde{\alpha} = \Gamma \left( \alpha_k, \lambda, \frac{\alpha_k \lambda}{2!}, \ldots, \frac{\alpha_k^{k-1} \lambda}{k!} \right)
\]

(4.272) is uniquely ergodic. We have already shown that

\[
\tilde{\alpha}(n) = \left( \alpha_k n, \lambda n, \frac{\alpha_k \lambda}{2!} n^2, \ldots, \frac{\alpha_k^{k-1} \lambda}{k!} n^k \right).
\]

(4.273)

If $\alpha_k n + k! m = (\alpha n + m) k! \in [0, k!]$, then $m = -[\alpha n]$. Let

\[
b(n) = \left( \{\alpha n\} k!, b_1(n), \ldots, b_k(n) \right)
\]

be the sequence in the fundamental domain $F$ corresponding to $\Gamma \tilde{\alpha}(n)$. By using (4.261) with $x = -[\alpha n] k!$, $y_i = -b_i(n)$ and $y'_i = \frac{(ak!)^{i-1} \lambda}{\alpha} n^i$, $i = 1, \ldots, k$, we have

\[
b_k(n) = \sum_{j=0}^{k-1} \frac{(-[\alpha n] k!)^j (ak!)^{k-j-1} \lambda}{(k-j)!} n^{k-j} \quad \text{(mod 1)}
\]

\[
= \frac{\lambda}{\alpha} (k!)^{k-2} \sum_{j=0}^{k-1} \frac{k!}{(k-j)! j!} (-[\alpha n])^j (\alpha n)^{k-j} \quad \text{(mod 1)}
\]

\[
= \frac{\lambda}{\alpha} (k!)^{k-2} \left( \sum_{j=0}^{k} \binom{k}{j} (-[\alpha n])^j (\alpha n)^{k-j} - (-1)^k [\alpha n]^k \right)
\]

\[
= \frac{\lambda}{\alpha} (k!)^{k-2} \left( (\alpha n - [\alpha n])^k + (-1)^{k+1} [\alpha n]^k \right)
\]

\[
= \frac{\lambda}{\alpha} (k!)^{k-2} \left( \{\alpha n\}^k + (-1)^{k+1} [\alpha n]^k \right)
\]

\[
= \beta (-1)^{k+1} \{\alpha n\}^k + [\alpha n]^k \beta.
\]

(4.274)
Since \((an, [an]^k \beta + \beta(-1)^{k+1}\{an\})\) is uniformly distributed (mod 1), it follows that \([an]^k \beta\) is uniformly distributed (mod 1).

\[\square\]

**Remark:** Even though we have used the same nilmanifold as Brezin used in [8], our method is not completely the same as the one found in [8]. Brezin made use of a class of continuous functions on \(\Gamma \setminus G_{AB}\). In order to discuss his method we will first describe the group \(G_{AB}\) as it is found in [8]. Let \(X_1, \ldots, X_k\) denote the standard basis for \(\mathbb{R}^k\), and define a linear map \(T : \mathbb{R}^k \rightarrow \mathbb{R}^k\) by setting \(TX_i = X_{i+1}\) for \(1 \leq i < k\), and \(TX_k = 0\). Define a semidirect product \(\mathbb{R}^k \cdot \mathbb{R}\) by setting
\[
(y, x)(y', x') = (y + \exp(xT)y', x + x')
\]
for all \(y, y' \in \mathbb{R}^k\) and all \(x, x' \in \mathbb{R}\), where \(\exp(xT)\) means \(\sum_{j=0}^{k-1} \frac{x^j}{j!}T^j\). Note that \(\exp(xT)y' = \left(\sum_{j=0}^{k-1} \frac{x^j}{j!}y'_{i-j}\right)_{i=1}^k\). So \(\mathbb{R}^k \cdot \mathbb{R} = G_{AB}\) and \(\Gamma\) is the subgroup \(\{(m, k!) | m \in \mathbb{Z}^k, l \in \mathbb{Z}\}\) of \(\mathbb{R}^k \cdot \mathbb{R}\). Let \(M = \Gamma \setminus (\mathbb{R}^k \cdot \mathbb{R})\). Now, for each \(m \in \mathbb{Z}^k\), define ([9, §3]) a “homogenizing” map \(\Theta^m : L^2(\mathbb{R}) \rightarrow L^2(M, \mu)\) by
\[
(\Theta^m f)(y, x) = \sum_{l=-\infty}^{\infty} \left(\exp 2\pi i \langle m, \exp(k!T)y \rangle\right) f(x + k!l).
\]
By [9, Theorem 3.1.], \(\Theta^m\) takes \(C^\infty(\mathbb{R})\) into \(C^\infty(M)\). This gives a class of continuous functions on \(M = \Gamma \setminus G_{AB}\). Taking \(f \in C^\infty_0([0, k!])\), we get
\[
\Theta^m f(y, x) = f\left(\left\{\frac{x}{k!}\right\}k!\right) \exp 2\pi i \langle m, \exp(k!T)y \rangle
= f\left(\left\{\frac{x}{k!}\right\}k!\right) \exp 2\pi i \langle m, \left(\sum_{j=0}^{i-1} \frac{(-1)^j}{j!}y_{i-j}\right)_{i=1}^k\rangle
= f(\phi_{12}(y, x)) \exp 2\pi i \langle m, (\phi_{i,k+1}(y, x))\rangle,
\]
where $\phi_{ij}$ are the functions defined in Example 3, Section 4.4. Recall from Section 4.4 that we used the function $g(y, x) = \exp 2\pi i \langle m, (\phi_{i,k+1}(y, x)) \rangle$ on $\Gamma \backslash G_{AB}$, which is continuous outside a set of measure 0, to show that a sequence in the fundamental domain of $\Gamma \backslash G_{AB}$ is uniformly distributed (mod 1). Note that $\int_M \Theta^m f \, d\mu = \int_M g \, d\mu = 0$ if $m \neq 0$.

Let $m = (0, \ldots, 0, m) \in \mathbb{Z}^k$. Then

$$\Theta^m f(\tilde{\alpha}(n)) = f(\{\alpha n\} k!) \exp 2\pi i mb_k(n)$$

so that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\{\alpha n\} k!) \exp 2\pi i mb_k(n) = 0.$$ (4.281)

This is what Brezin does in [8] (except that he takes $f \in C_0^\infty([0, 1])$, which is another, however minor, error in his proof). To complete the proof, he approximates the function 1 on $[0, 1]$ by $C^\infty$ functions.
CHAPTER V

Conclusion

The main theme of this dissertation has been uniform distribution of generalized polynomials. However, in Chapter IV we also studied properties like unique ergodicity, (point) distality and almost automorphy of generalized polynomials. There are still many open questions concerning unique ergodicity and (point) distality. In Table 1 and 2 we give an overview of what we know about some simple generalized polynomials. Their distribution properties were established in Chapter III.

5.1 Open problems

In Chapter IV we showed how to construct uniquely ergodic dynamical systems $(G/\Gamma, T)$ corresponding to some classes of generalized polynomials $q(n)$ in such a way that $e^{2\pi i q(n)} = g(T^n(\Gamma))$ for some function $g$ on $G/\Gamma$ which is continuous outside a set of measure 0 (in which case we say that $q(n)$ is generated dynamically). However, we have been able to do this only for certain classes of generalized polynomials. Here are some problems which remained unresolved.

(1) We do not know how to generate dynamically simple generalized polynomials
Table 1: Properties of some generalized polynomials, I.

<table>
<thead>
<tr>
<th>Generalized polynomial</th>
<th>Distribution mod 1</th>
<th>Known features</th>
<th>Generating dynamical system</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha n^l, l \geq 1$</td>
<td>$\alpha \in \mathbb{Q}$: not dense $\alpha \notin \mathbb{Q}$: ud (mod 1)</td>
<td>distal uniquely ergodic almost periodic if $l = 1$ (see [17])</td>
<td>$\alpha$ irrational: affine transf. on $K^l$ (rotation on $K$ if $l = 1$)</td>
</tr>
<tr>
<td>$[\alpha n^l] \beta, l \geq 1$</td>
<td>$\beta \in \mathbb{Q}$: not dense $\beta \notin \mathbb{Q}$ and $\beta = a + \frac{b}{n}$, $a, b \in \mathbb{Q}$: dense, not ud(mod1) $1, \alpha, \alpha \beta$ rat. indep.: ud (mod 1)</td>
<td>uniquely ergodic alm.point distal (l=1) Besicovitch a.p. if $l = 1$</td>
<td>$1, \alpha, \alpha \beta$ rat. indep.: affine trans. on $K^{2l}$ (rotation on $K^2$ if $l = 1$)</td>
</tr>
<tr>
<td>$[\alpha n^l] \beta, l \geq 2$</td>
<td>$\beta \in \mathbb{Q}$: not dense $\beta \notin \mathbb{Q}$: ud (mod1)</td>
<td>uniquely ergodic point distal (but not point distal)</td>
<td>$1, \alpha, \alpha \beta$ rat. indep.: rotation on $\Gamma \backslash G_{AB}$ (see Prop. 4.7.3)</td>
</tr>
<tr>
<td>$[an + \frac{1}{l}] \beta, l \geq 1$</td>
<td>Same as for $[\alpha n^l] \beta$</td>
<td>uniquely ergodic alm.autom. if $l = 1$ Besicovitch if $l = 1$</td>
<td>Same as for $[\alpha n^l] \beta$</td>
</tr>
<tr>
<td>$[\alpha_1 n][\alpha_2 n] \beta$</td>
<td>$\beta \in \mathbb{Q}$: not dense $\frac{\alpha_1}{\alpha_2} = \sqrt{c}, \beta = a + b\sqrt{c} \notin \mathbb{Q}$: dense, not ud(mod1) otherwise: ud(mod1)</td>
<td>uniquely ergodic $[\alpha_1 n][\alpha_2 n] \beta \equiv [\alpha_1 n][\alpha_2 \beta n] + [\alpha_2 n][\alpha_1 \beta n] - \alpha_1 \alpha_2 \beta n^2 \pmod{1}$</td>
<td>rotation on $N_k/\Gamma_k$ or aff.trans. on $N_k/\Gamma_k \times K^d$, $k \leq 2$ (see Sec. 4.5)</td>
</tr>
<tr>
<td>$\prod_{i=1}^{l} [\alpha_i n] \beta, l \geq 3$</td>
<td>$\beta \in \mathbb{Q}$: not dense $\beta \notin \mathbb{Q}$: ud (mod1)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

ud(mod1)=uniformly distributed (mod 1) 
aff.trans.=affine transformation 
rat.ind.=rationally independent 
alm.autom.=almost automorphic 
Besicovitch=Besicovitch almost periodic 

For the results about unique ergodicity, ((almost) point) distality, almost automorphy and Besicovitch almost periodicity, see Section 4.2. 

**Remark:** As mentioned in the introduction, $[\alpha n^l] \beta, l > 1$, $\beta$ irrational, is uniformly distributed (mod 1) by a result of Peres, [31].
### Table 2: Properties of some generalized polynomials, II.

<table>
<thead>
<tr>
<th>Generalized polynomial</th>
<th>Distribution mod 1</th>
<th>Known features</th>
<th>Generating dynamical system</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[\alpha n] \beta n$</td>
<td>$\beta \in \mathbb{Q}$: not dense  $\beta \notin \mathbb{Q}$, $\beta = a + b\alpha$, $a, b \in \mathbb{Q}$, $\alpha^2 \in \mathbb{Q}$: dense, not ud(mod1) $\alpha^2 \notin \mathbb{Q}$ and $\beta$ rat. ind. of 1, $\alpha$ or $\alpha^2 \notin \mathbb{Q}$ and $\beta \notin \mathbb{Q}$: ud (mod1)</td>
<td>uniquely ergodic $2[\alpha n] \alpha n \equiv \alpha^2 n^2 - {\alpha n}^2$ (mod 1)</td>
<td>$\beta = a + b\alpha$, $a, b \in \mathbb{Z}$ and $\alpha^2 \notin \mathbb{Q}$: affine trans. on $K^2$ 1, $\alpha, \beta$ rat. indep.: aff. trans. on $N_1/\Gamma_1 \times K^d$ (see Thm. 4.5.2)</td>
</tr>
<tr>
<td>$[\alpha n^2] \beta n^2$</td>
<td>Same as for $[\alpha n] \beta n$</td>
<td>$2[\alpha n^2] \alpha n^2 = \alpha^2 n^4 - {\alpha n^2}^2$ (mod 1)</td>
<td>$\beta = b\alpha$, $\alpha^2 \notin \mathbb{Q}$: affine trans. on $K^4$</td>
</tr>
<tr>
<td>$[\alpha n^l] \beta n^k$, $l \neq k$</td>
<td>$\beta \in \mathbb{Q}$: not dense  $\beta \notin \mathbb{Q}$: ud (mod1)</td>
<td></td>
<td>$\alpha = \beta$ and $l = 2, k = 1$ or $l = 1, k = 2$: aff. trans. on $N_1/\Gamma_1 \times K^d$ (see Prop. 4.6.4)</td>
</tr>
<tr>
<td>$[\alpha_1 n] [\alpha_2 n] \beta n$</td>
<td>$\beta \in \mathbb{Q}$: not dense  $\alpha_1 \in \mathbb{Q}$, $\alpha_1^2 \notin \mathbb{Q}$, $\beta = a + b\alpha_1 \notin \mathbb{Q}$: dense, not ud(mod1) otherwise: ud(mod1)</td>
<td>$3[\alpha n] \alpha n \equiv \alpha^3 n^3 - {\alpha n}^3$ (mod 1)</td>
<td>$1, \alpha_1, \alpha_2, \beta$ rat. ind., and some other conditions, see Prop. 4.7.2: affine trans. on $G_{ud}(3)/\Gamma \times K^d$</td>
</tr>
</tbody>
</table>

ud(mod1)=uniformly distributed (mod 1)  
aff.trans.=affine transformation  
rat.ind.=rationally independent
like \([\alpha n^l] \beta n^k, l \neq k\), or \([\alpha n^l] \beta n^l\) if \(\beta \neq a + b\alpha, a, b \in \mathbb{Q}\). Remark, however, that we have shown how to generate certain sums of such generalized polynomials (see Theorem 4.6.3).

(2) For some special choices of \(\alpha, \beta, \gamma\), we showed in Proposition 4.7.2 that \([\alpha n] \beta n \gamma n\) can be generated dynamically. Can this be extended to \([\cdots [\alpha_1 n] \alpha_2 n \cdots ] \alpha_k n\) for any \(k \in \mathbb{N}\) and any \(\alpha_i, i = 1, \ldots, k\)?

(3) We have seen that \([\alpha n]^k \beta\) can be generated by a rotation on a nilmanifold if \(1, \alpha, \alpha \beta\) are rationally independent (Proposition 4.7.3). Furthermore, any generalized polynomial of form \([\alpha n][\beta n] \gamma\) can be generated dynamically because \([\alpha n][\beta n] \gamma \sim [\alpha n] \beta \gamma n + [\beta n] \alpha \gamma n - \alpha \beta \gamma n^2\). However, it is quite unclear how to generate \([\alpha_1 n][\alpha_2 n] \cdots [\alpha_k n] \beta\) for \(k > 3\) if it does not reduce to \([\alpha n]^k \beta\).

Neither the van der Corput method nor the ”dynamical method” is good for showing that a generalized polynomial is not uniformly distributed (mod 1). It would be helpful if we could extend the van der Corput’s method to also give a criterion for when a generalized polynomial is not uniformly distributed (mod 1). For example, is the following conjecture true?

\textbf{Conjecture 2} Let \(q(n)\) be a generalized polynomial with only irrational coefficients and of degree at least 2. Suppose that \(q(n)\) has no term of form \([\cdots [q_1(n)] \lambda_1] \cdots ] \lambda_k, k \geq 1\), or \([q_1(n)] \alpha n\), where \(q_1(n)\) is a generalized polynomial for which \(q_1(n) \neq [q_1(n)]\). Then \(q(n)\) is uniformly distributed (mod 1) if and only if \(q^h(n) = q(n + h) - \ldots\)
q(n) is uniformly distributed \((\text{mod } 1)\).

Note that this is true for non-linear polynomials with irrational coefficients.

A set \(R \subset \mathbb{N}\) is a set of recurrence if for any measure preserving system \((X, \mathcal{B}, \mu, T)\) and any \(A \in \mathcal{B}\), \(\mu(A) > 0\), there exists \(r \in R\) such that \(\mu(A \cap T^{-r}A) > 0\).

\(\text{(5.1)}\)

For example, if \(p(x)\) is a polynomial which takes integer values on integers, then \(p(\mathbb{N}) = \{p(n) \mid n \in \mathbb{N}\}\) is a set of recurrence if and only if \(p(\mathbb{N})\) is divisible by any \(k \in \mathbb{N}\). The following theorem is from [13, p.22].

**Theorem 5.1.1 (Uniformity of sets of recurrence)** Let \(R\) be a set of recurrence and let \(a > 0\). Then there exist constants \(N(a)\) and \(\epsilon(a) > 0\) so that given any measure preserving system \((X, \mathcal{B}, \mu, T)\) and any \(A \in \mathcal{B}\) with \(\mu(A) \geq a\), there exists \(r \in R, r \leq N(a)\) such that \(\mu(A \cap T^{-r}A) \geq \epsilon(a)\).

Since \(p(\mathbb{N})\) is a set of recurrence if \(p(n)\) is a polynomial with integer coefficients and without a constant term, it follows by using uniformity of sets of recurrence that for a big class of generalized polynomials \(q(n)\) for which \(q(n) = [q(n)]\), the sets \(q(\mathbb{N})\) are sets of recurrence. This class contains all \(q(n) = [q_1(n)]\) where \(q_1(n)\) has independent coefficients and has no constant terms. However, not all generalized polynomials give rise to sets of recurrence in this way. For example, since \(\left\lfloor 2\alpha n \frac{1}{\alpha} \right\rfloor = \left\lfloor 2n - \{2\alpha n\} \frac{1}{\alpha} \right\rfloor = 2n - 1\) if \(\alpha > 1\), the set \(\{\left\lfloor 2\alpha n \frac{1}{\alpha} \right\rfloor \mid n \in \mathbb{N}\}, \alpha > 1\), is not a set of recurrence. On the other hand, it is unknown to us if \(\{\left\lfloor \sqrt{2}n \sqrt{2} \right\rfloor \mid n \in \mathbb{Z}\}\) and \(\{\left\lfloor \sqrt{2}n \sqrt{2} \right\rfloor \mid n \in \mathbb{Z}\}\)
are sets of recurrence. Both can be shown to be divisible by any $k \in \mathbb{N}$ which is a necessary condition for being a set of recurrence (a consequence of [17, Theorem 3.18]).

We will end this dissertation by listing some more questions and open problems.

1. Can any uniformly distributed generalized polynomial be shown to be uniformly distributed (mod 1) by (a version of) van der Corput’s method, as done in Chapter III for some classes of generalized polynomials?

2. What are necessary and sufficient conditions for $\prod_{i=1}^{k} [p_i(n)]^\gamma$, $p_i(n)$ a polynomial, to be uniformly distributed (mod 1)? We showed in Theorem 3.3.1 that $\prod_{i=1}^{k} [\alpha_i n]^\gamma$, $k > 2$, is uniformly distributed (mod 1) iff $\gamma$ is irrational. However, any generalized polynomial of form $[\alpha n]^\frac{1}{\alpha}$ or $[\alpha n][\alpha \sqrt{cn}]\sqrt{c}$ fails to be uniformly distributed (mod 1) (see Proposition 3.1.3). For the same reasons $[\alpha n^l]^\frac{1}{\alpha}$ and $[\alpha n^l][\alpha \sqrt{cn^l}]\sqrt{c}$ are not uniformly distributed (mod 1). Moreover, generalized polynomials which reduce to (see remark preceding Proposition 3.1.3) a generalized polynomials of form $[\sqrt{cn^l}]\sqrt{cn^l}$ are not uniformly distributed (mod 1) (see (3) below). We conjecture, however, that as long as $\gamma$ is irrational and $\prod_{i=1}^{k} [p_i(n)]^\gamma$ does not reduce to one of the above mentioned forms, it is uniformly distributed (mod 1).

3. What are necessary and sufficient conditions for $\cdots [\alpha_1 n]^{\alpha_2 n} \cdots]^{\alpha_k n}$ to be uniformly distributed (mod 1)? We make the following conjecture.

**Conjecture 3** $\cdots [\alpha_1 n]^{\alpha_2 n} \cdots]^{\alpha_k n}$, $\alpha_k \not\in \mathbb{Q}$, is uniformly distributed (mod 1)
if and only if it does not reduce to (see remark preceding Proposition 3.1.3) a generalized polynomial of type

\[ b \left[ \cdots \left[ \alpha^{[n^l]} \alpha^{n_l^l} \cdots \right] \alpha^{n_l^l} \right], \ (r - 1 \text{ brackets}), \quad (5.2) \]

where \( b \in \mathbb{Q}, \ \alpha^r \in \mathbb{Q} \) and \( rl = k \).

We are able to prove by using identities for \( \cdots \left[ \alpha^{[a]} \alpha \cdots \right] a, \ (r - 1 \text{ brackets}) \) when \( r \leq 4 \), that \( (5.2) \) are not uniformly distributed \( \text{(mod 1)} \) when \( r \leq 4 \). We suppose that there are similar identities for \( r > 4 \) as well. The criterion of Conjecture 2 above would be of great help in proving Conjecture 3.

4. Are all uniformly distributed generalized polynomials well-distributed also? Since the methods we have used to show uniform distribution give well-distribution as well, all generalized polynomials which we have shown to be uniformly distributed \( \text{(mod 1)} \) are also well-distributed. We would think this is true for any uniformly distributed generalized polynomial.

5. In Section 4.2 we discussed distality properties of generalized polynomials (see Definitions 4.2.4 and 4.2.3 for the different notions of distality). The following questions are still unanswered. Is every generalized polynomial at least almost point distal? Which generalized polynomials are point distal? Are any generalized polynomials other than the polynomials distal? Confer the discussion after Proposition 4.2.3.

6. Is every generalized polynomial uniquely ergodic? See discussion around Conjecture 1 at the end of Section 4.2.
7. Can any uniformly distributed generalized polynomial be generated dynamically by a uniquely ergodic transformation on a nilmanifold? If yes, are they all affine? If no, what other systems can generate them?
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