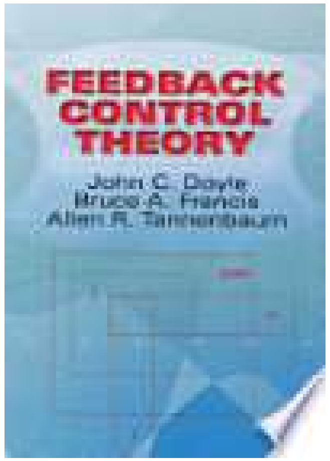
Feedback control systems for mechatronics and robotics

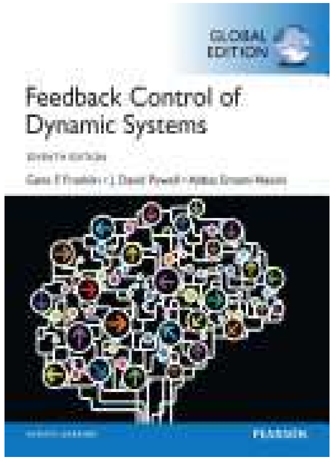
i. Introduction to feedback control systems
ii. Control-oriented modeling
iii. Dynamic system behavior in time and frequency domain
iv. Transfer function analysis and state-space modeling
v. Similarity forms, controllability, and observability of the systems
vi. Standard output feedback controllers
vii. State feedback controllers and prefilter extensions
viii. Stability analysis and robust control design
ix. Motion-, force- and impedance-control in mechatronics and robotics
x. Use of observers and estimators in feedback control systems

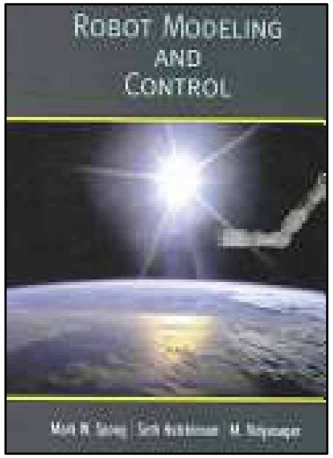
Michael Ruderman
michael.ruderman@uia.no

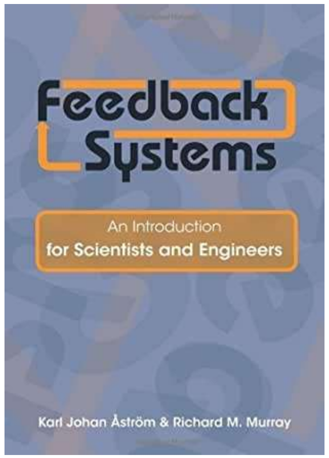
- # Recommended textbooks

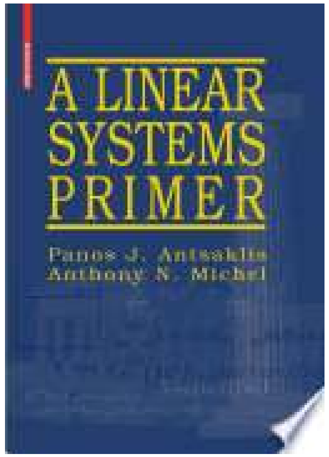
- ❑ Feedback control theory. J.C. Doyle, B.A. Francis, A.R. Tannenbaum. 2009, Dover
- ❑ Feedback control of dynamic systems. G.F. Franklin, J. Powell, A.F. Emami-Naeini. 2015, Pearson
- ❑ Robot Modeling and Control. M.W. Spong, S. Hutchinson, M. Vidyasagar. 2006, John Wiley & Sons
- ❑ Feedback systems: an introduction for scientists and engineers. K.J. Åström, R.M. Murray. 2010, Princeton
- ❑ A linear systems primer. P.J. Antsaklis, A.N. Michel. 2007, Birkhäuser











Feedback control systems for mechatronics and robotics

i. **Introduction to feedback control systems**
ii. Control-oriented modeling
iii. Dynamic system behavior in time and frequency domain
iv. Transfer function analysis and state-space modeling
v. Similarity forms, controllability, and observability of the systems
vi. Standard output feedback controllers
vii. State feedback controllers and prefilter extensions
viii. Stability analysis and robust control design
ix. Motion-, force- and impedance-control in mechatronics and robotics
x. Use of observers and estimators in feedback control systems

Michael Ruderman
michael.ruderman@uia.no

Introduction to feedback control systems

- Why do we need a *feedback*?

Funny Example 1

we want to take a bath (actually, what we want is to fill a water level)

with or without feedback?

Funny Example 1

we want to take a bath (actually, what we want is to fill water level), cont.

Let assume:

$u(t)$	rate at which the water flows into the bath (*what we can control*)
$h(t)$	instantaneous level of water in the bath (*what we are interested in*)
V, m	volume of water in the bath, and total mass (*current state*)
A, ρ	cross-section area of the bath, and density of the fluid (*i.e. water*)
H	desired level of the water (*which is our goal, i.e. what we want*)
K	"intensity" of controlling the water flow (*i.e. control gain*)

Simplified "physics" of filling the bath: $V(t) = A\,h(t)\,, \quad \dot{m} = \dfrac{dV(t)}{dt}\rho = u(t)$

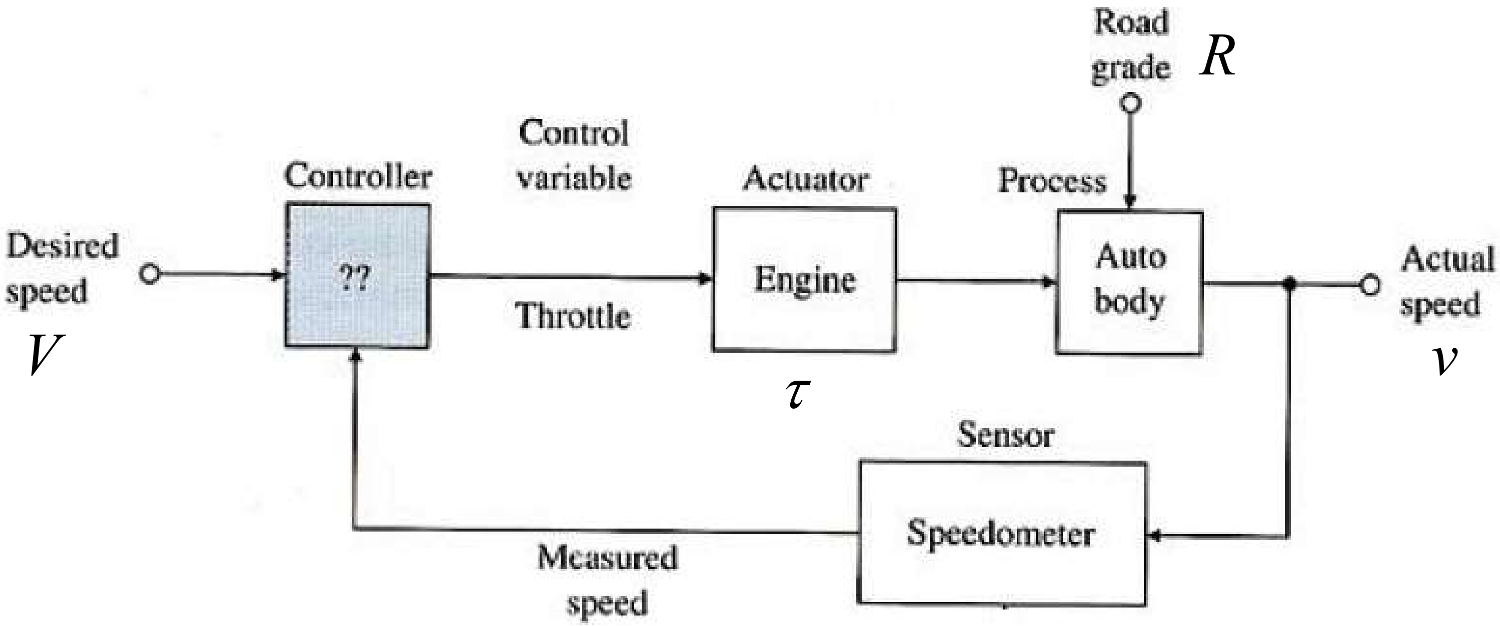
How we control inflow (i.e. adjust the inflow valve): $u(t) = K\,\underbrace{\big(H - h(t)\big)}_{\text{control error}}$

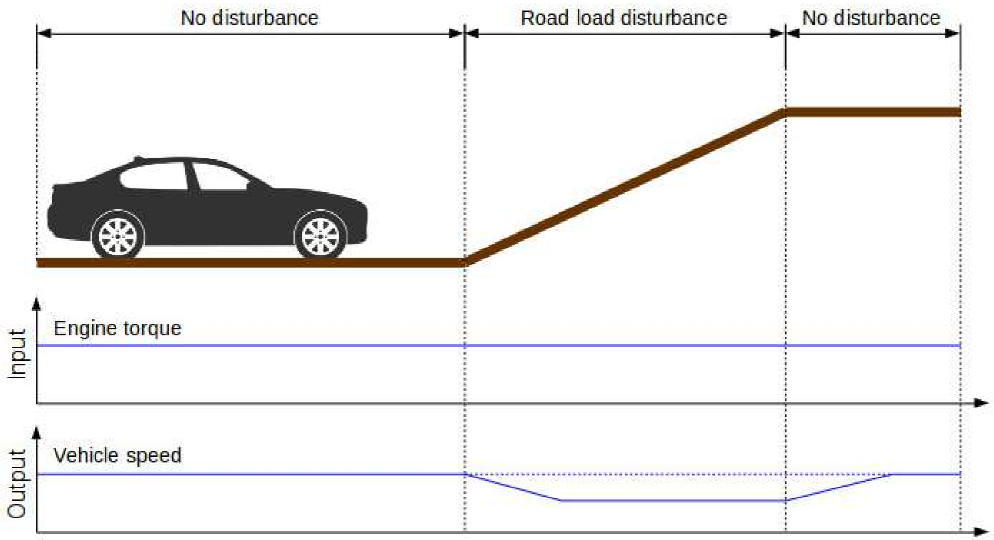
Resulted <u>controlled</u> (!) <u>dynamical</u> (!) behavior of the system:

$$A\rho\frac{dh(t)}{dt} = K\big(H - h(t)\big) = KH - Kh(t) \qquad \Rightarrow \qquad h(t) = \underbrace{H\left(1 - \exp\left(-\frac{K}{A\rho}t\right)\right)}_{\text{solution for output}}$$

Less funny Example 2

we want to ride a car (actually, what we want is to ride it without accidents)





Simplified "physics" of the car's motion

$$\sum F = m\frac{dv(t)}{dt} + b\,v(t) + R(t) = \tau(t)$$

Which feedback controller to use?

$$\tau(t) = ??\left(\underbrace{V(t)}_{desired} - \underbrace{v(t)}_{measured}\right)$$

Less funny Example 2

we want to ride a car (actually, what we want is to ride it without accidents), cont.

First trial: let's take a proportional (P) controller

$$\Rightarrow \quad \tau(t) = K_p\big(V(t) - v(t)\big) \quad \Rightarrow \quad m\frac{dv(t)}{dt} + \big(b + K_p\big)v(t) + R(t) = K_p V(t)$$

For steady-state (s.s.), we apply the final value theorem, set all derivatives to zero

$$\text{for } \frac{d}{dt} = 0 \quad \Rightarrow \quad \frac{b + K_p}{K_p}v + \frac{1}{K_p}R = V \quad \Rightarrow \quad e_{s.s.} = V - v = \frac{b}{K_p}v + \frac{1}{K_p}R \neq 0$$

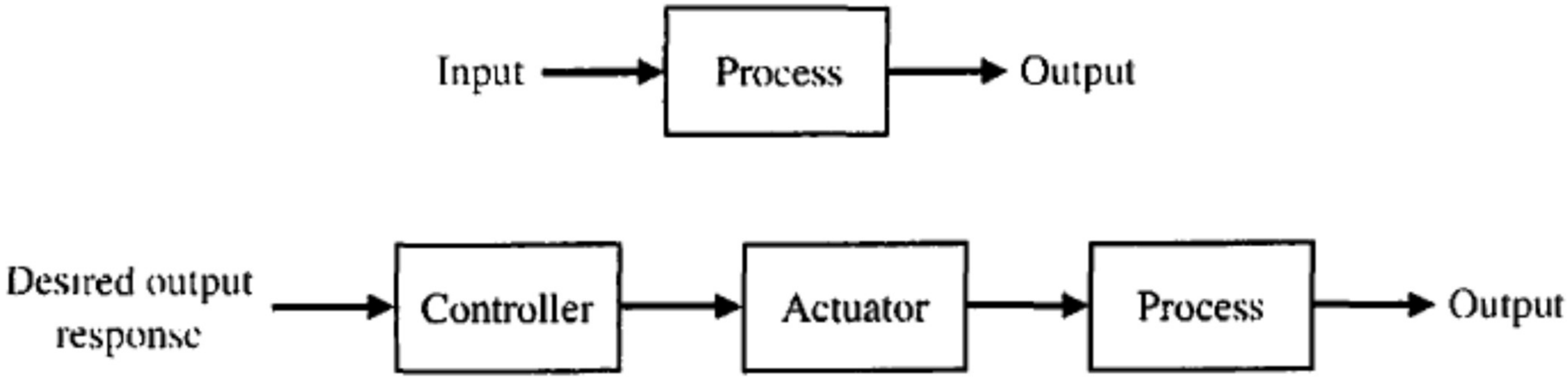
Second trial: let's take a proportional-integral (PI) controller

$$\Rightarrow \quad \tau(t) = K_p\big(V(t) - v(t)\big) + K_i\int\big(V(t) - v(t)\big)\,dt$$

Differentiate both sides of ODE and take the final values, if $V = \text{const}$, $R = \text{const}$

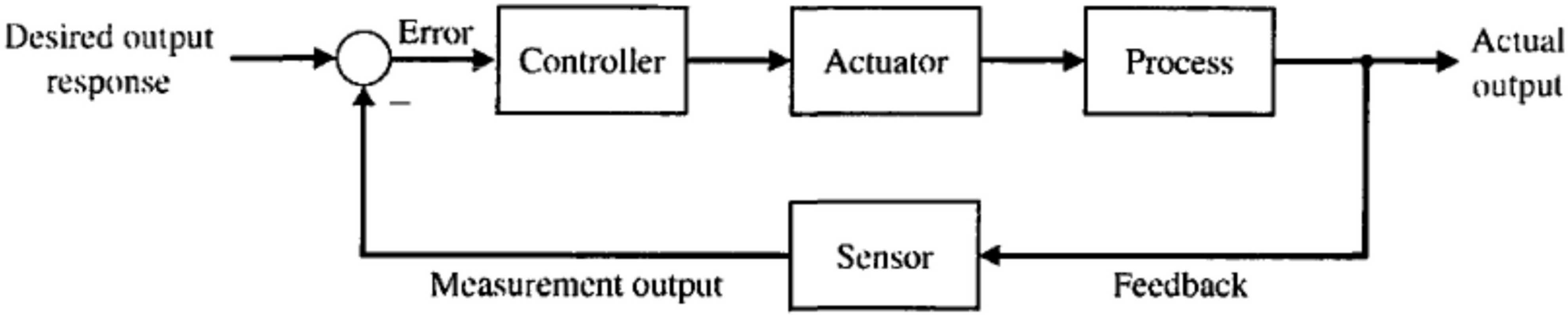
$$m\underbrace{\ddot{v}(t)}_{=0} + \big(b + K_p\big)\underbrace{\dot{v}(t)}_{=0} + K_i v(t) + \underbrace{\dot{R}(t)}_{=0} = K_p\underbrace{\dot{V}(t)}_{=0} + K_i V(t) \quad \Rightarrow \quad e_{s.s.} = V - v = 0$$

- Feedforward versus feedback control



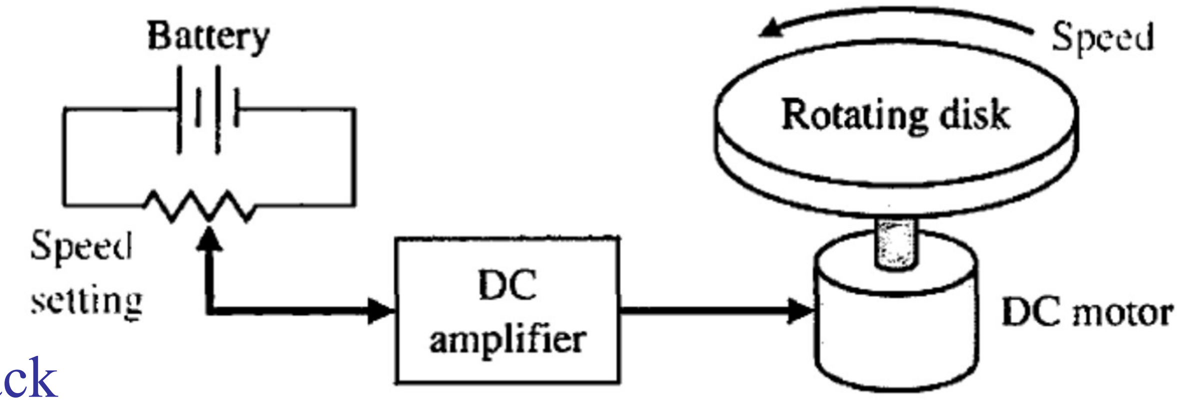
source: [1]

Now, let’s measure the output of interest and compare it with what we want!



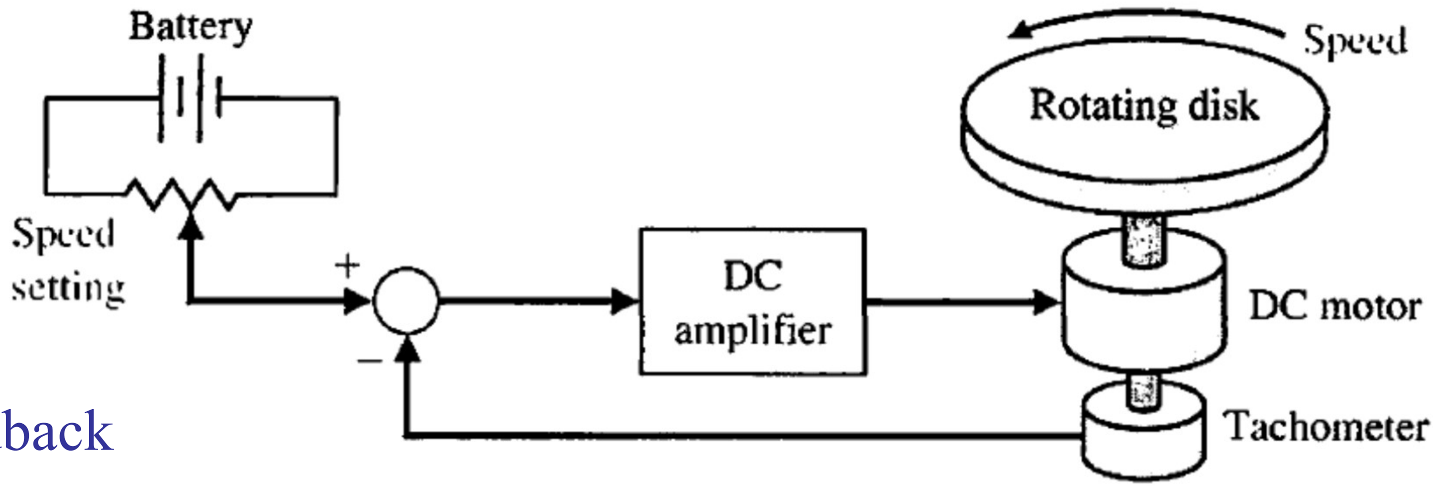
Example 3

we want a variable motor speed (actually, also accurate & load-independent)



without feedback

source: [1]



with feedback

- Feedback control is not new!

see [2]

Maybe the first industrial application of a feedback control system:

Watt’s flyball governor

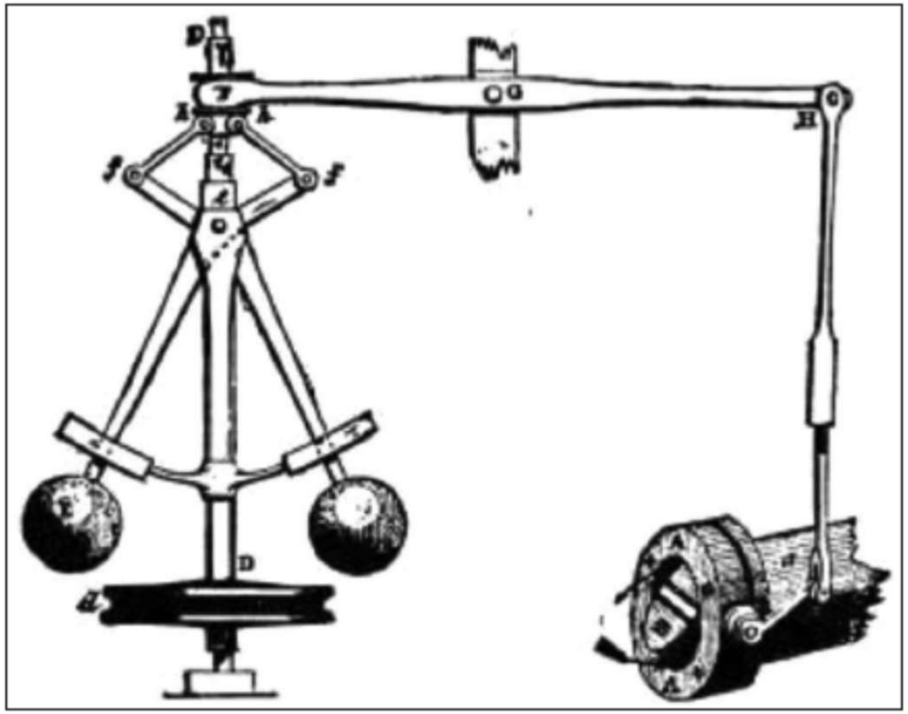
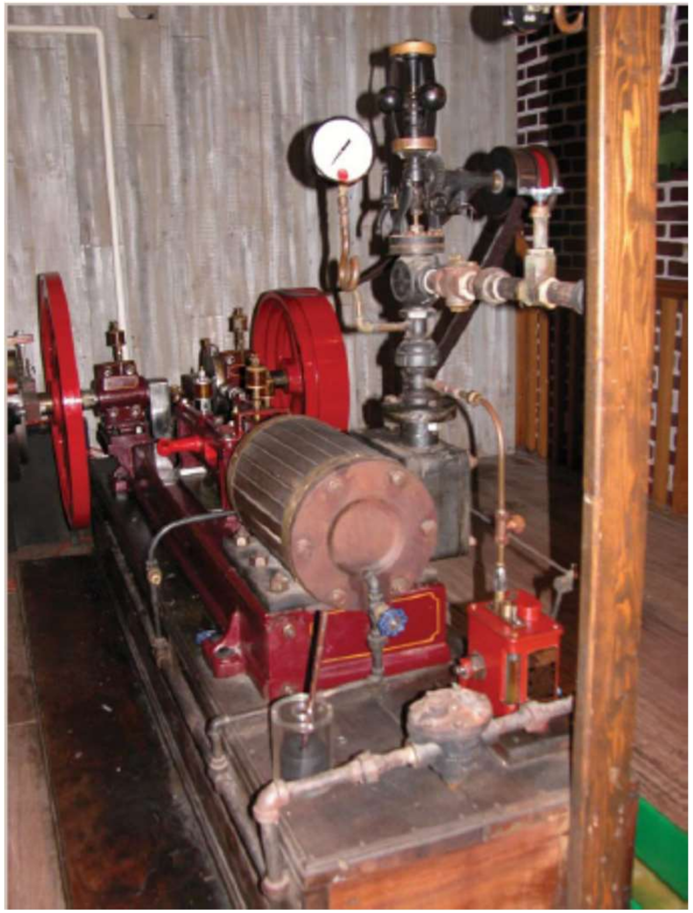
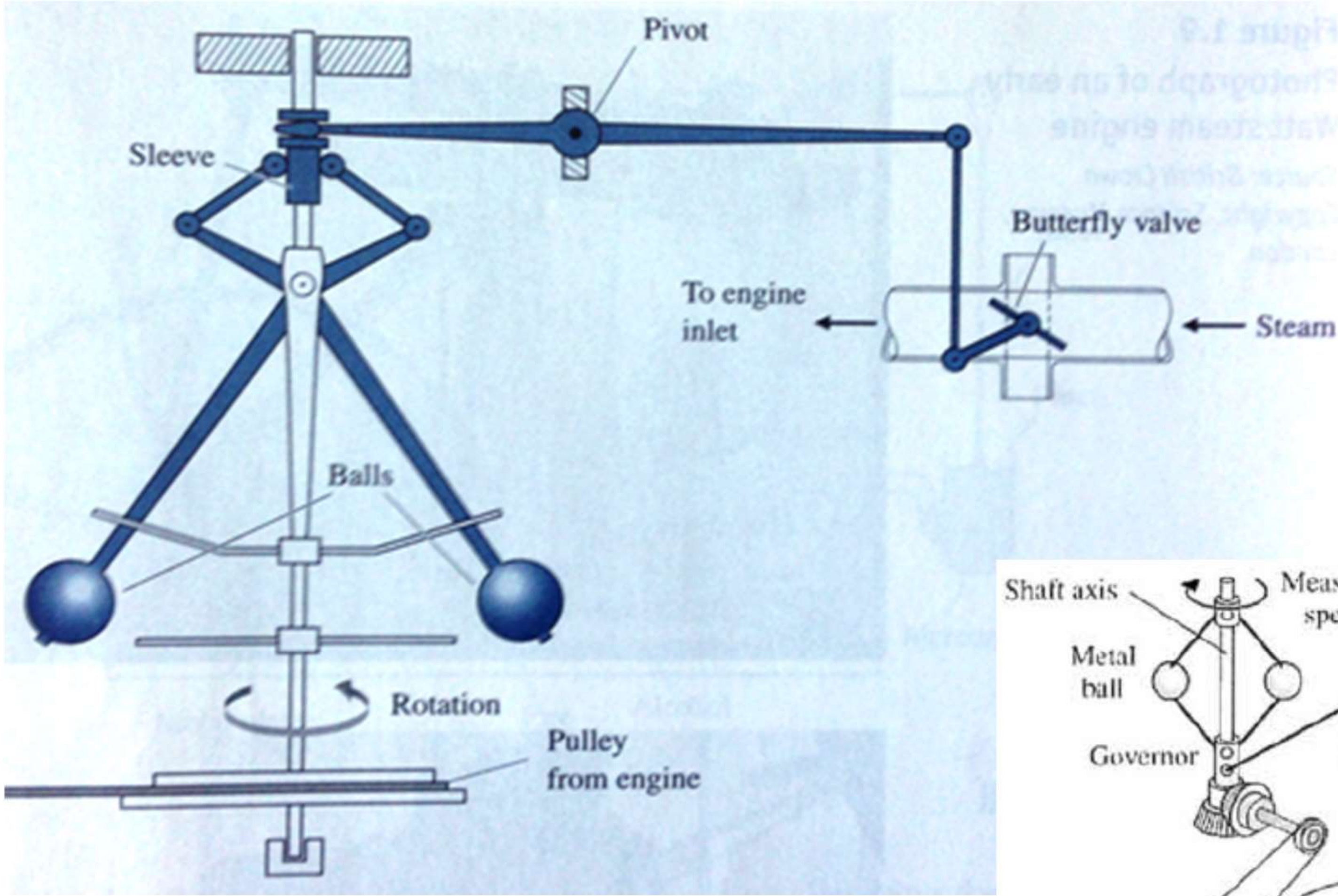


FIGURE 2 – The centrifugal governor was invented by James Watt

source: [3]





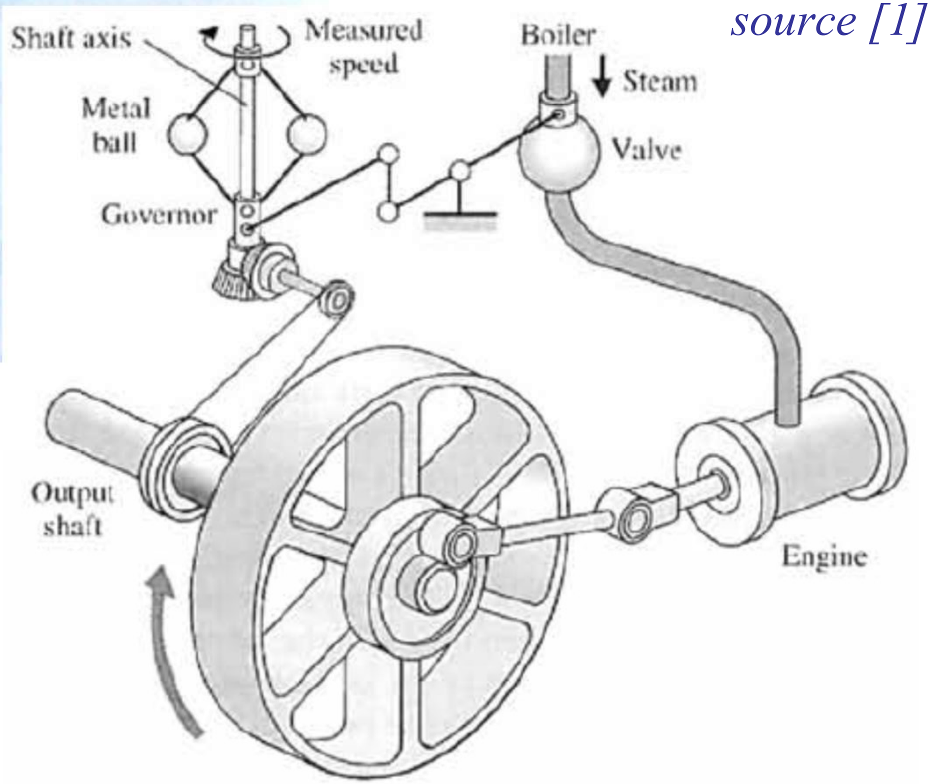
source [4]

The following communications were read :—

I. " On Governors." By J. Clerk Maxwell, M.A., F.R.SS.L. & E. Received Feb. 20, 1868.

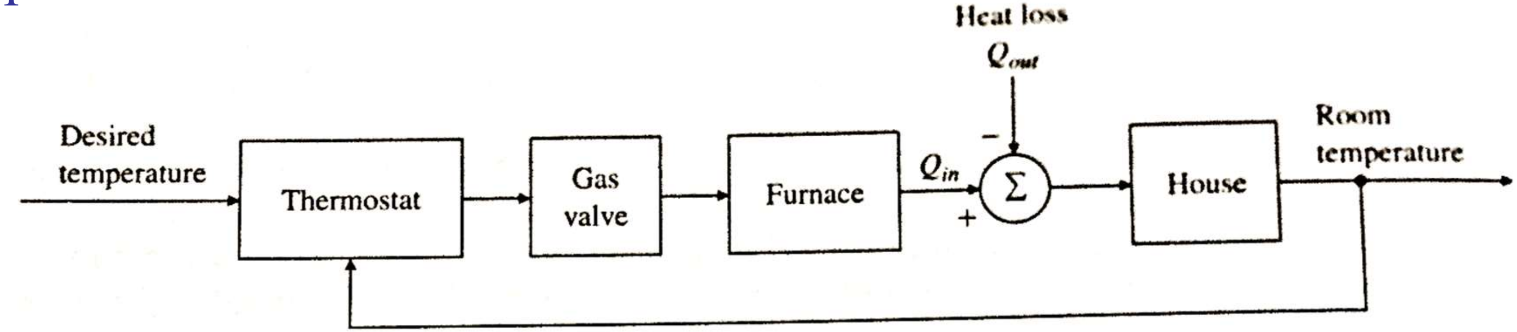
A Governor is a part of a machine by means of which the velocity of the machine is kept nearly uniform, notwithstanding variations in the driving-power or the resistance.

source [1]



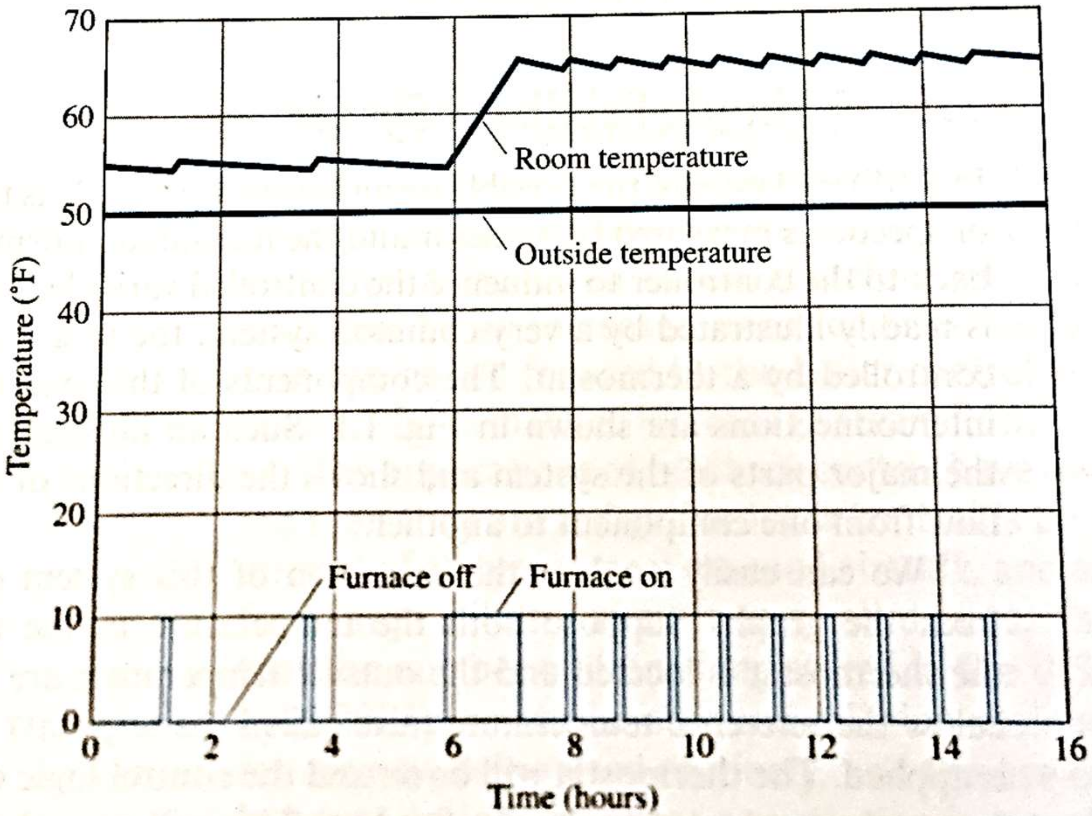
Simple (everyday) Example 4

room temperature control with thermostat

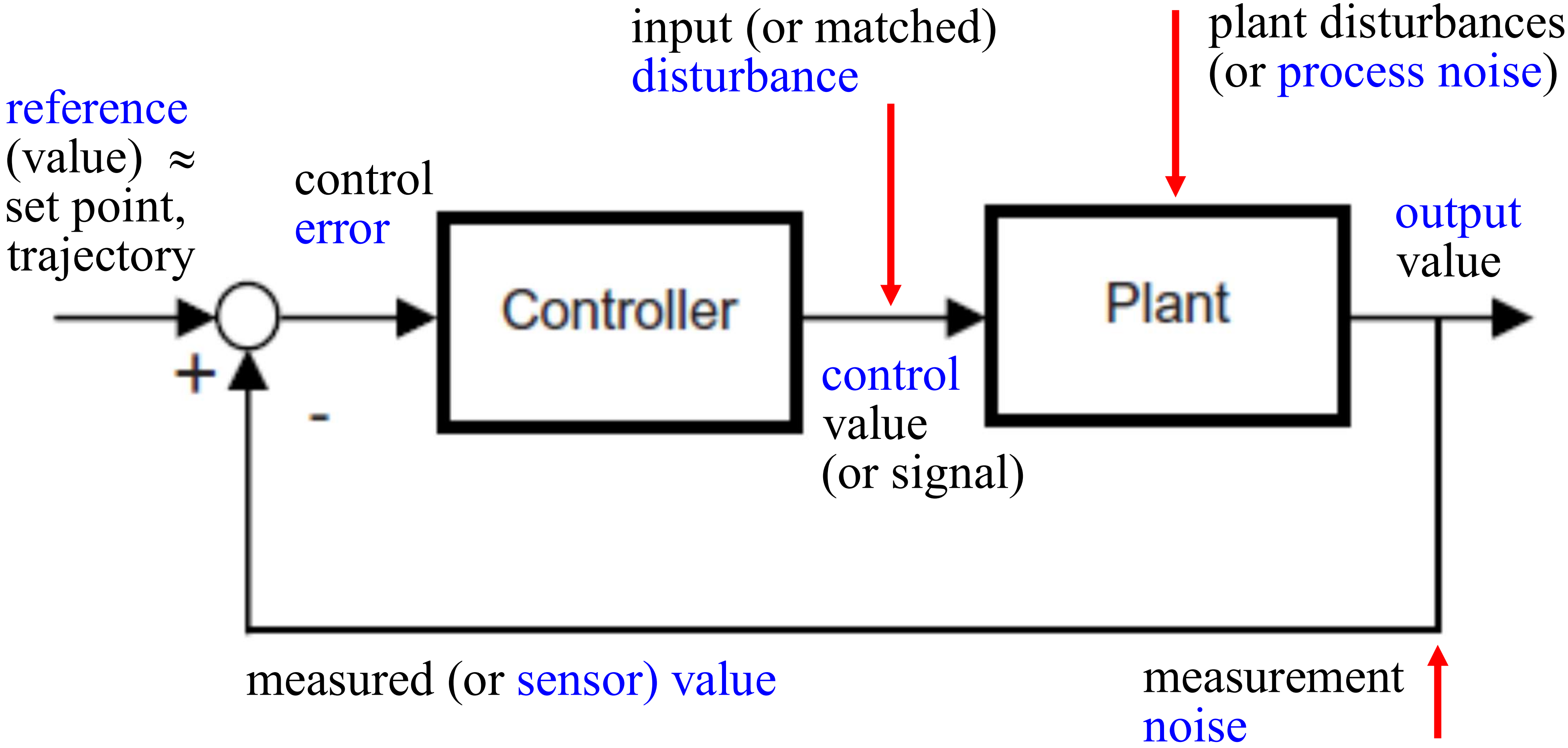


(a)

source: [4]

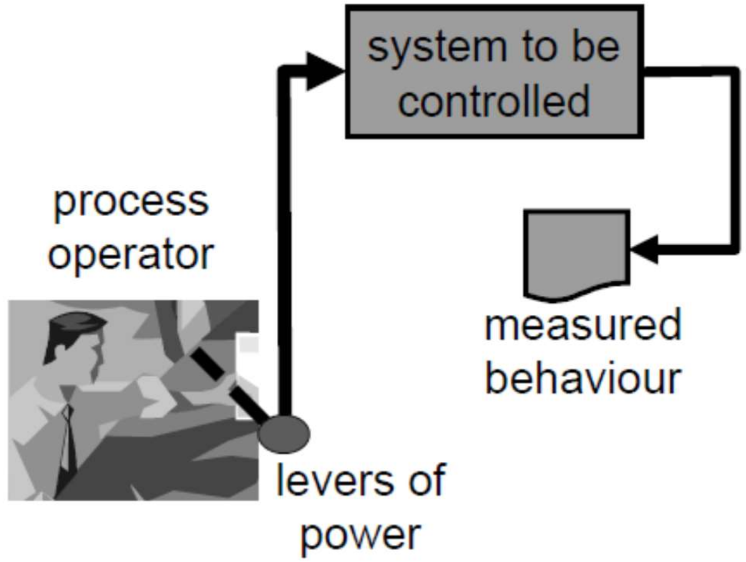


- Usual notations in feedback control systems

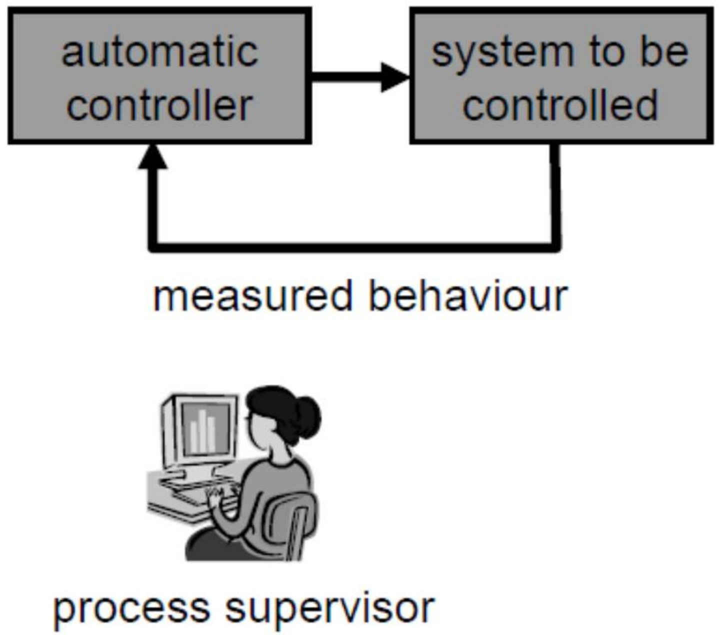


- Difference between manual & automatic (feedback) control

Manual Control



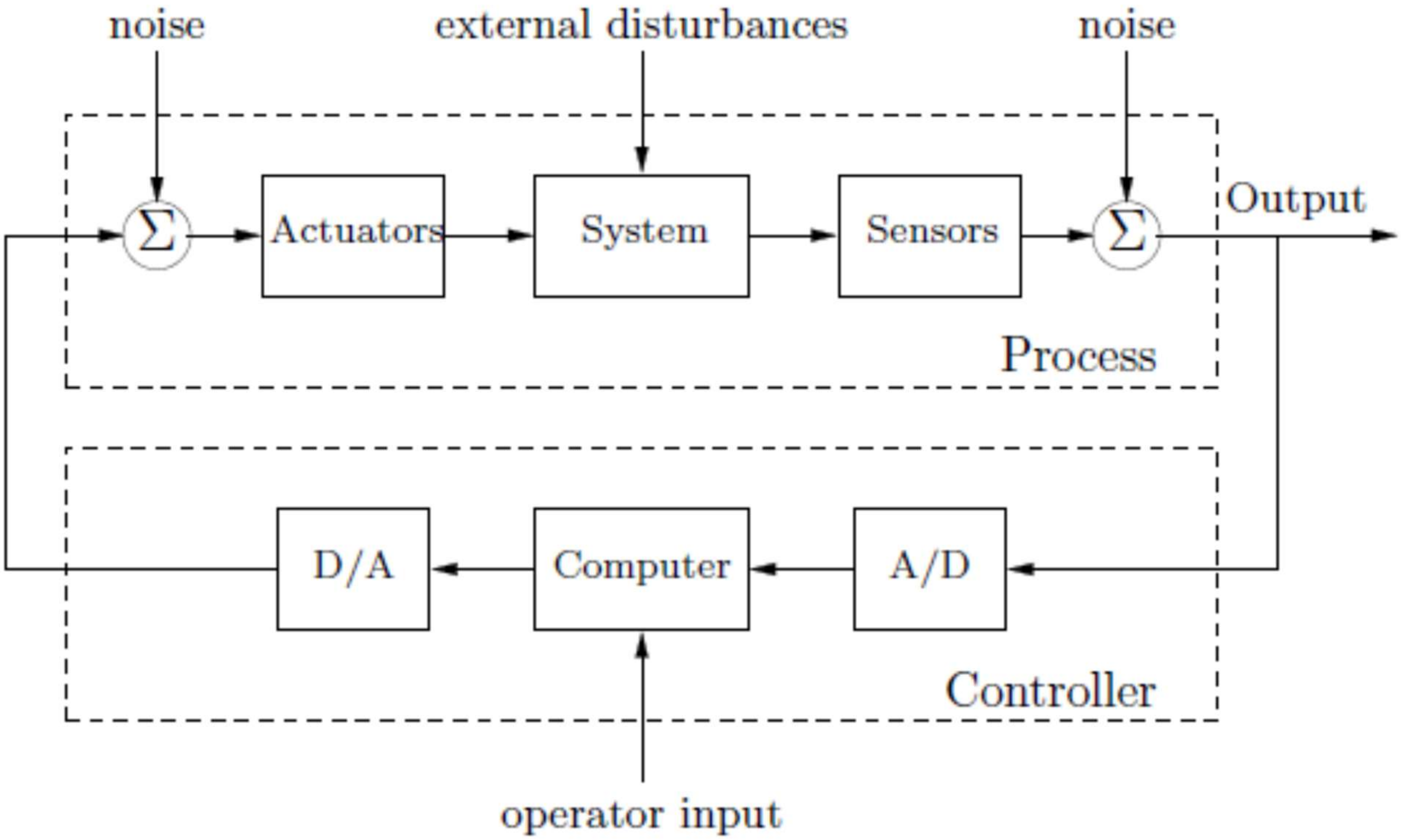
Automatic Control



For automatic control:

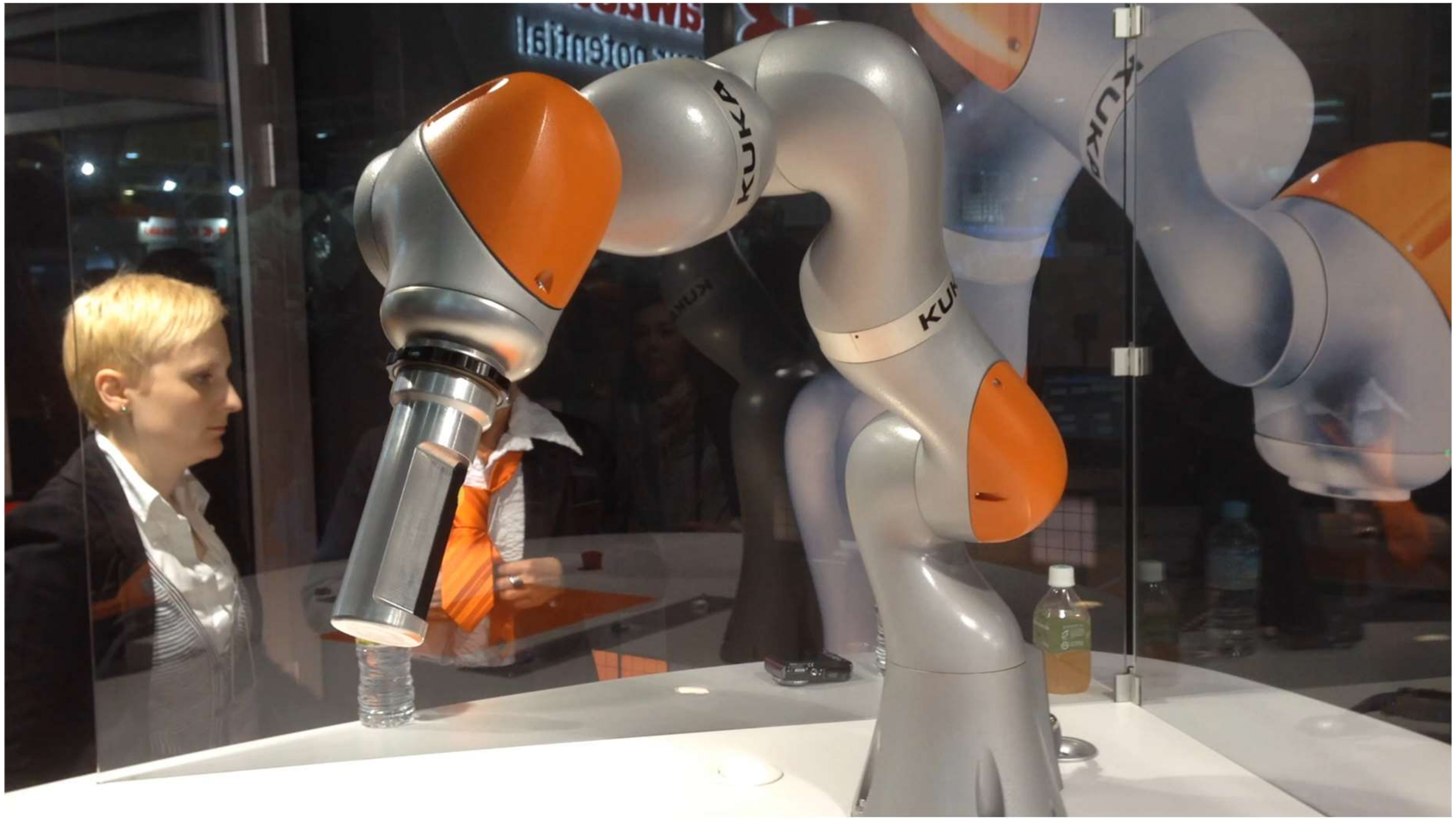
- measured behavior/output is a prerequisite for **feedback**
- control $\left\{\begin{array}{l}\text{strategy}\\ \text{law}\\ \text{function}\end{array}\right\}$ is a prerequisite for **automatic**

- Computer-/processor-controlled feedback system



We will next always assume $T_{sampling} \rightarrow 0 \quad \Rightarrow$ continuous time domain (t)

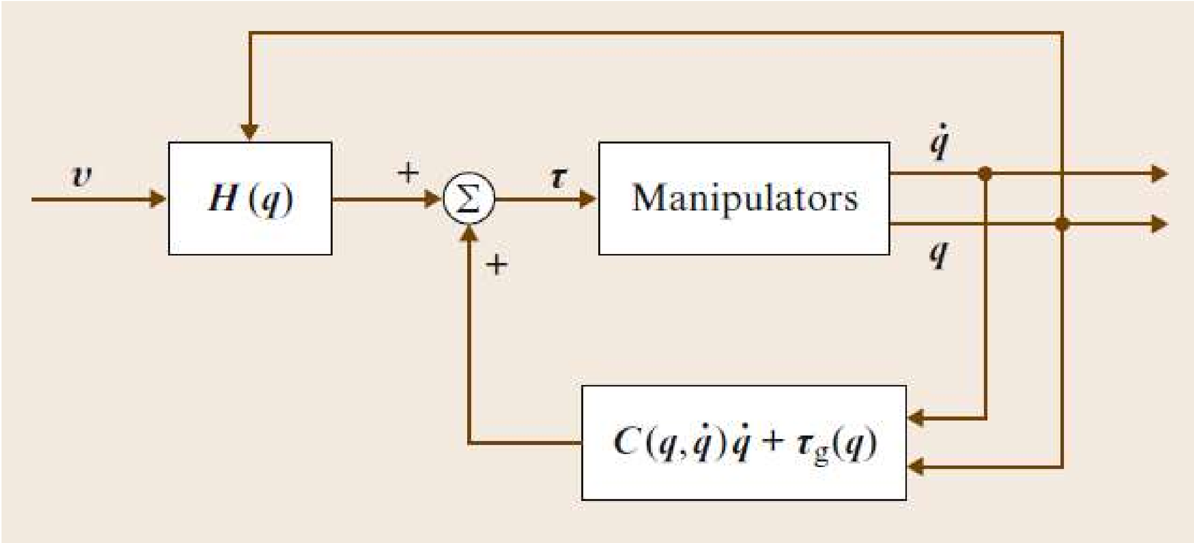
- Why control is as important for robotics and mechatronics?
  (some motivating movies)



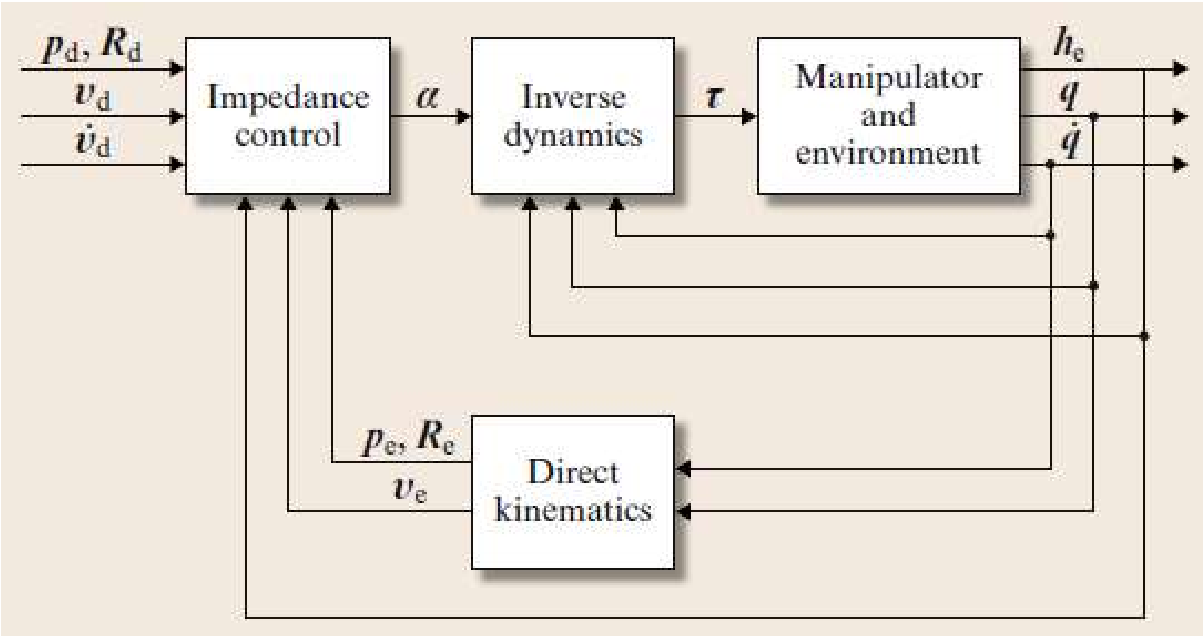
- Why control is as important for robotics and mechatronics? cont.
  (some motivating movies)

- Why control is as important for robotics and mechatronics? cont.
  (some motivating movies)

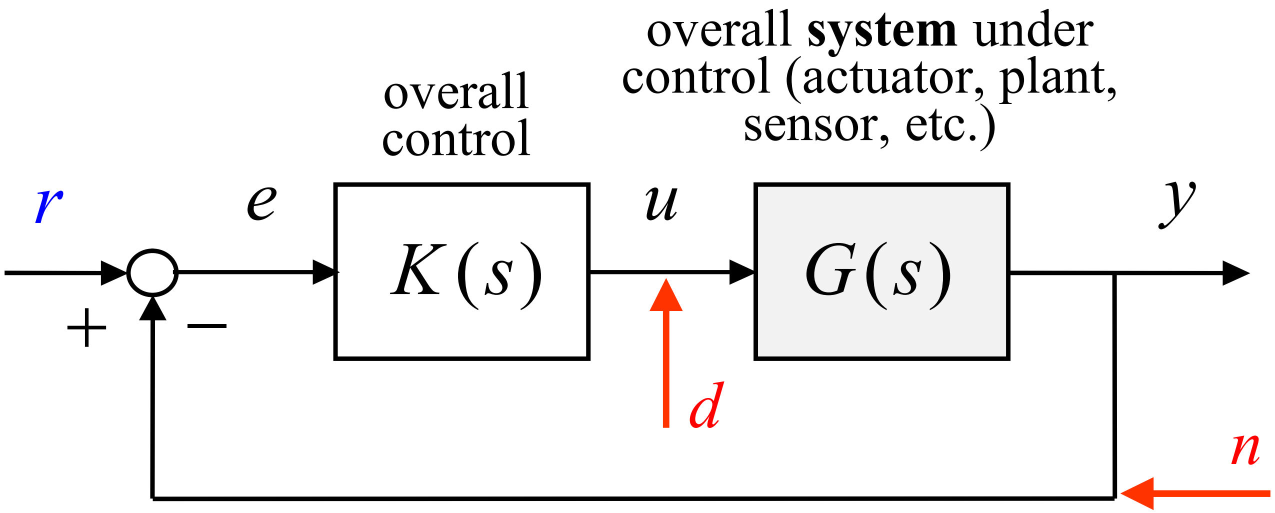
• Why control is as important for robotics and mechatronics? cont.



source: [5]



- Requirements and criteria for feedback control



– **Stability (!)**

Internal stability: if for all initial conditions and all bounded signals injected at any place in the system all states remain also bounded

– Sufficient reference tracking & disturbance rejection $y(s) \to r(s), \; \dfrac{y(s)}{d(s)} \to 0$

– Small (towards zero) residual (steady-state) error $\left| e(t) \right|_{t>T} < E_{\max}$

– Fast transient response, i.e. control bandwidth is large enough

– **Robustness (against uncertain parameters / unmodeled dynamics)**

- Why control stability is so important?

FEATURE

Respect the Unstable

The practical, physical (and sometimes dangerous) consequences of control must be respected, and the underlying principles must be clearly and well taught.

By Gunter Stein

Feedback control systems are all around us in modern technological life. They are at work in our homes, our cars, our factories, our transportation systems, our defense systems—everywhere we look. Certainly, one of the great achievements of the international controls research community is that the design principles for these systems are well developed and broadly understood by control engineers, so that the systems operate productively and safely in so many applications.

In this article, I want to talk about two trends that threaten to undermine this achievement. My objective is to heighten our awareness of these trends and hopefully bring about an appropriate response to them.

The first trend has to do with the applications themselves. Among the abundance of control systems operating today are increasing numbers of dangerous ones. Society trusts our technology. We are permitted to do things with automatic controls that cannot be done manually and that, if done improperly, can have dire consequences for property, the environment, and human life. Most, but not all, of these dangerous applications involve open-loop unstable plants with divergence rates violent enough to elude manual control. This characterization motivates the title of the article, and I will describe specific examples of such applications.

The second trend has been evident at our conferences, and certainly in our journals, over the years. This trend is the increasing worship of abstract mathematical results in control at the expense of more specific examinations of their practical, physical consequences. I will provide examples of this trend as well.

Gunter Stein's Bode Lecture

An understanding of fundamental limitations is an essential element in all engineering. Shannon's early results on channel capacity have always had center court in signal processing. Strangely, the early results of Bode were not accorded the same attention in control. It was therefore highly appropriate that the IEEE Control Systems Society created the Bode Lecture Award, an honor which also came with the duty of delivering a lecture. Gunter Stein gave the first Hendrik W. Bode Lecture at the IEEE Conference on Decision and Control in Tampa, Florida, in December 1989. In his lecture he focused on Bode's important observation that there are fundamental limitations on the achievable sensitivity function expressed by Bode's integral. Gunter has a unique position in the controls community because he combines the insight derived from a large number of industrial applications at Honeywell with long experience as an influential adjunct professor at the Massachusetts Institute of Technology from 1977 to 1996. In his lecture, Gunter also emphasized the importance of the interaction between instability and saturating actuators and the consequences of the fact that control is becoming increasingly mission critical.

After more than 13 years I still remember Gunter's superb lecture. I also remember comments from young control scientists who had been brought up on state-space theory who said: "I believed that controllability and observability were the only things that mattered." At Lund University we made Gunter's lecture a key part of all courses in control system design. Gunter was brought into the classroom via videotapes published by the IEEE Control Systems Society and the written lecture notes. It was a real drawback that the lecture was not available in more archival form. I am therefore delighted that IEEE Control Systems Magazine is publishing this article. I sincerely hope that this will be followed by a DVD version of the videotape. The lecture is like really good wine: it ages superbly.

—Karl J Åström, Professor Emeritus
Lund University, Lund, Sweden

12 IEEE Control Systems Magazine August 2003

source: [6]

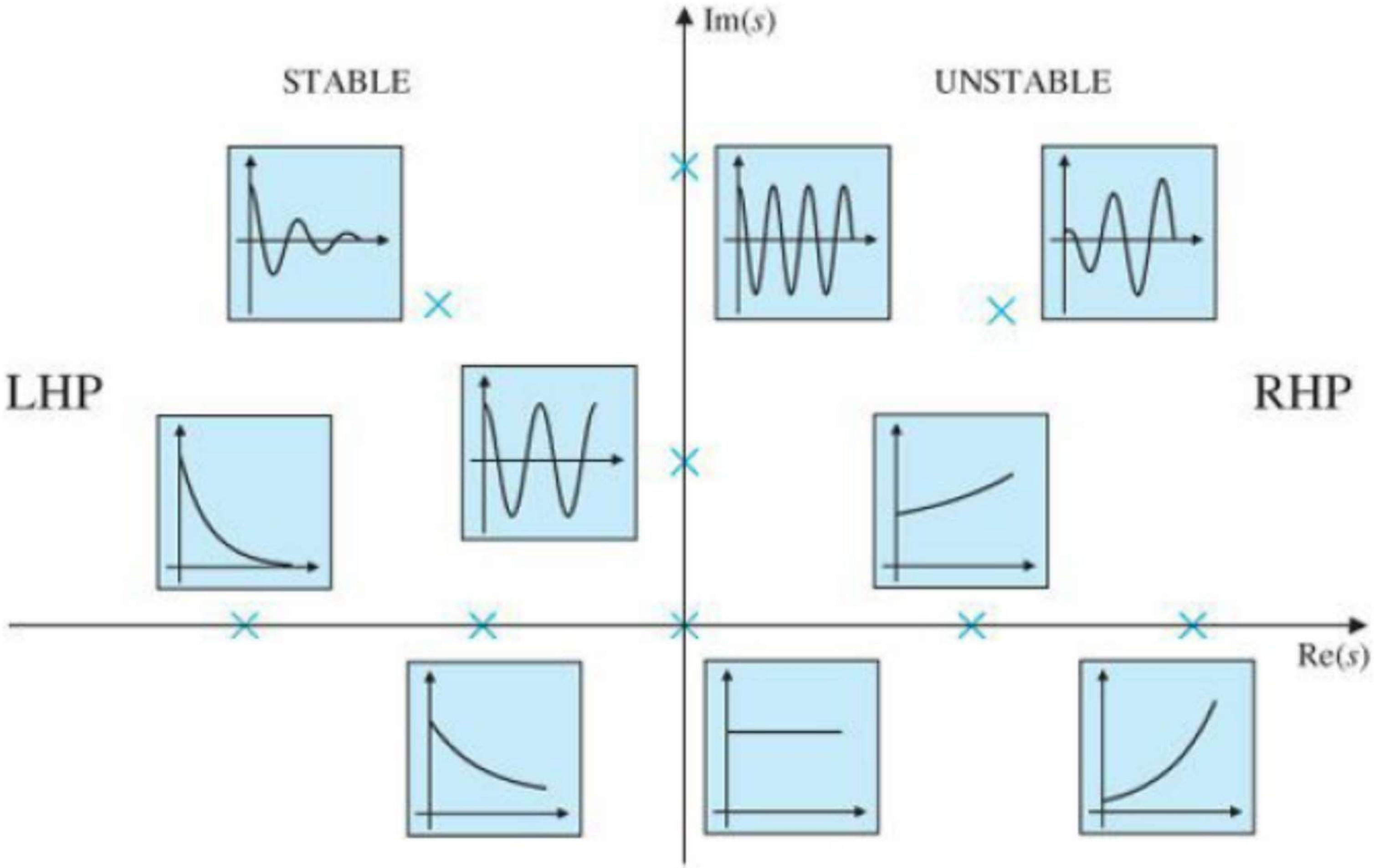
Figure 1. *Gripen JAS39 prototype accident on 2 February 1989. The pilot received only minor injuries.*

Figure 2. *Chernobyl nuclear power plant shortly after the accident on 26 April 1986.*

- Why control stability is so important? cont.

An LTI system is said to be stable if all the roots of the transfer function denominator polynomial have negative real parts (i.e., they are all in the left hand s-plane) and is unstable otherwise.

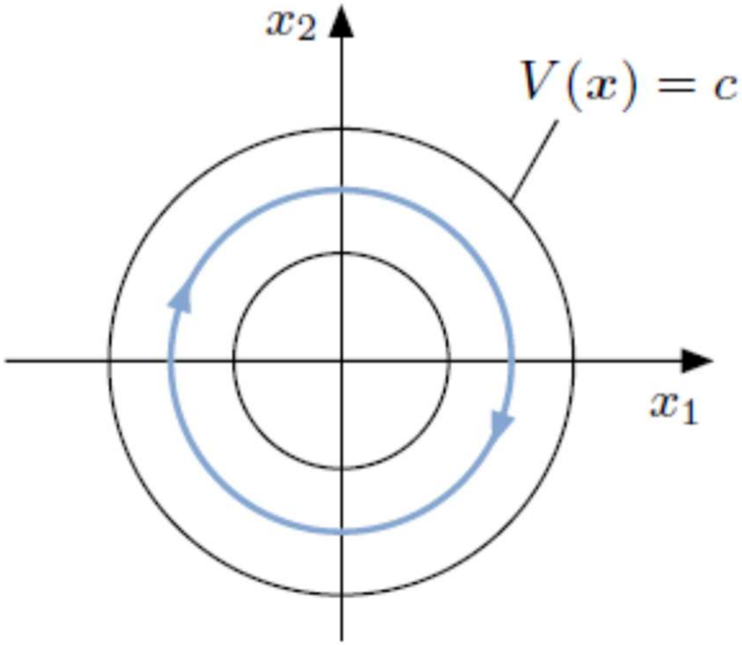
source: [4]

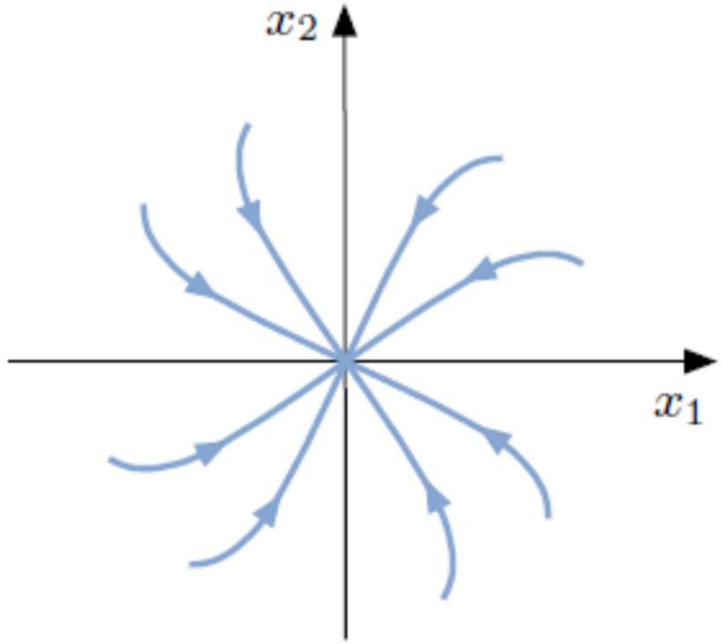


- Why control stability is so important? cont.

Stability in Lyapunov sense

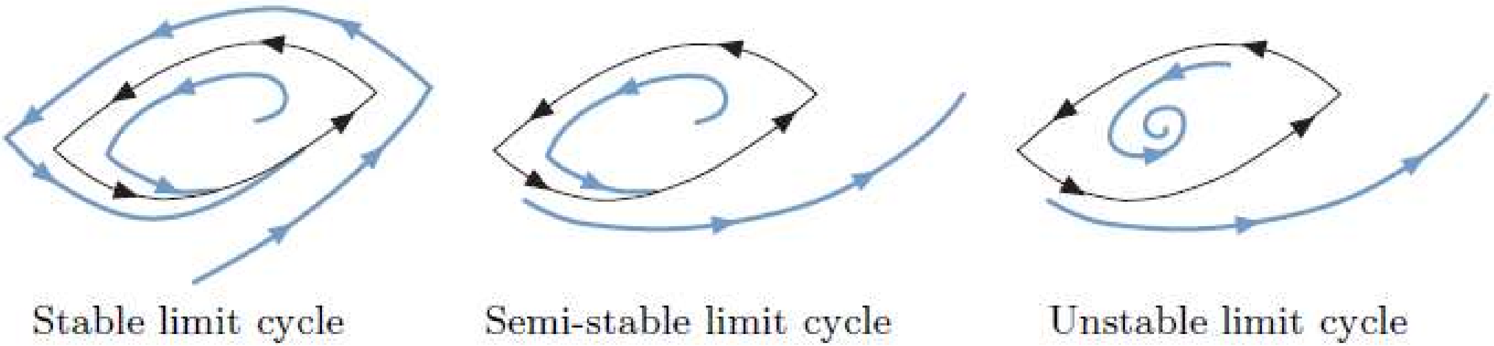
Asymptotic stability in Lyapunov sense





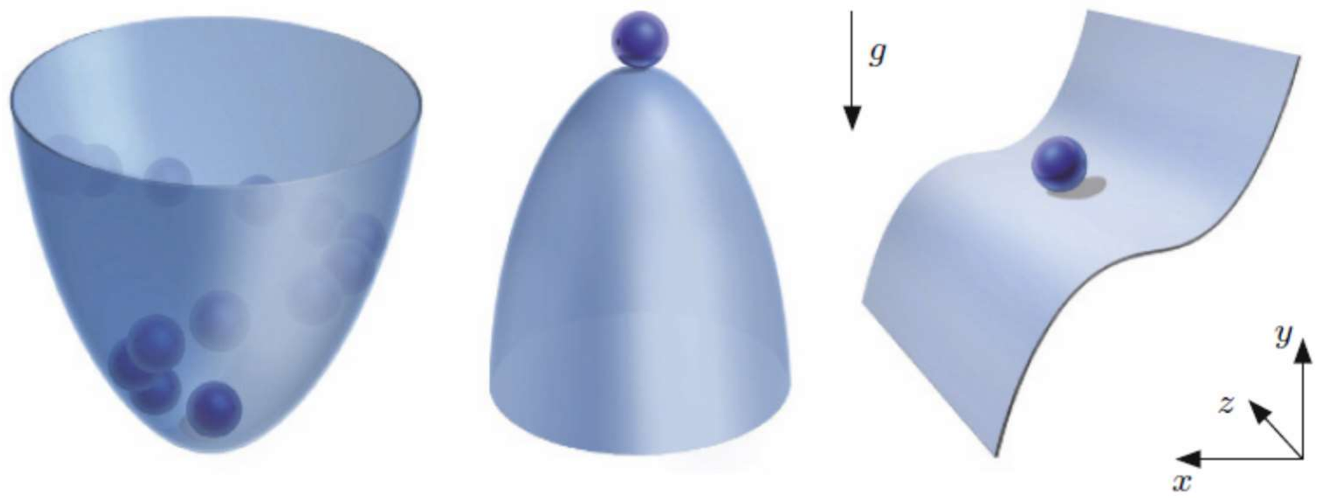
source: [7]

Also for the period solutions (i.e. trajectories), so-called limit cycles



- Why control stability is so important? cont.

  – Free ball in the gravity field

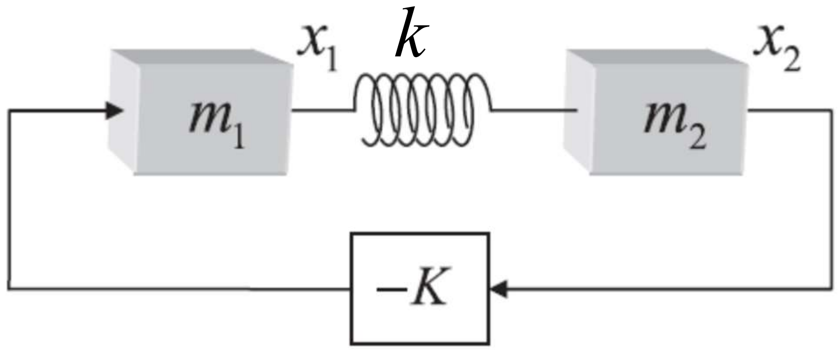


source: [7]

  – Ball with an energy source leaving the gravity field

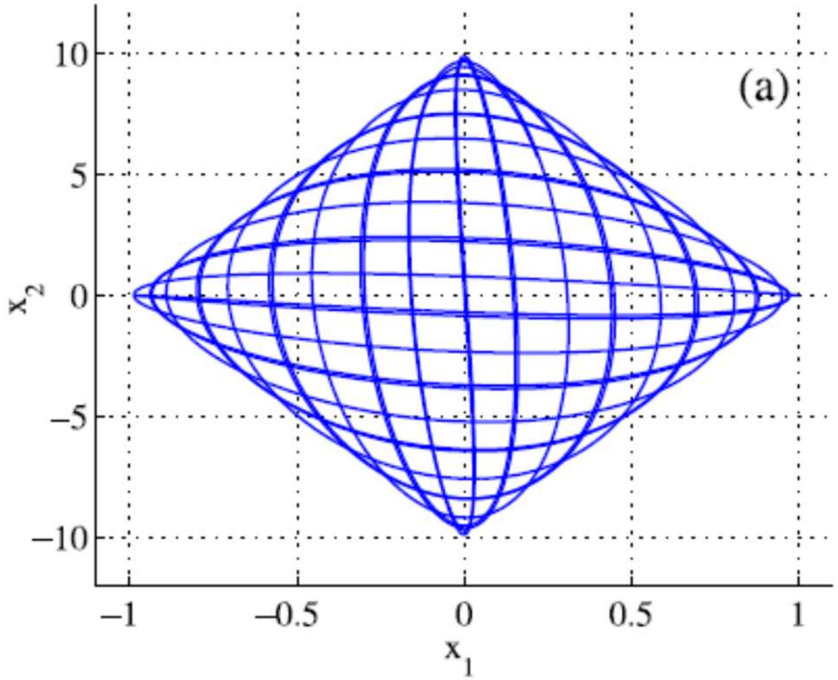
- Why control stability is so important? cont.

Example 5

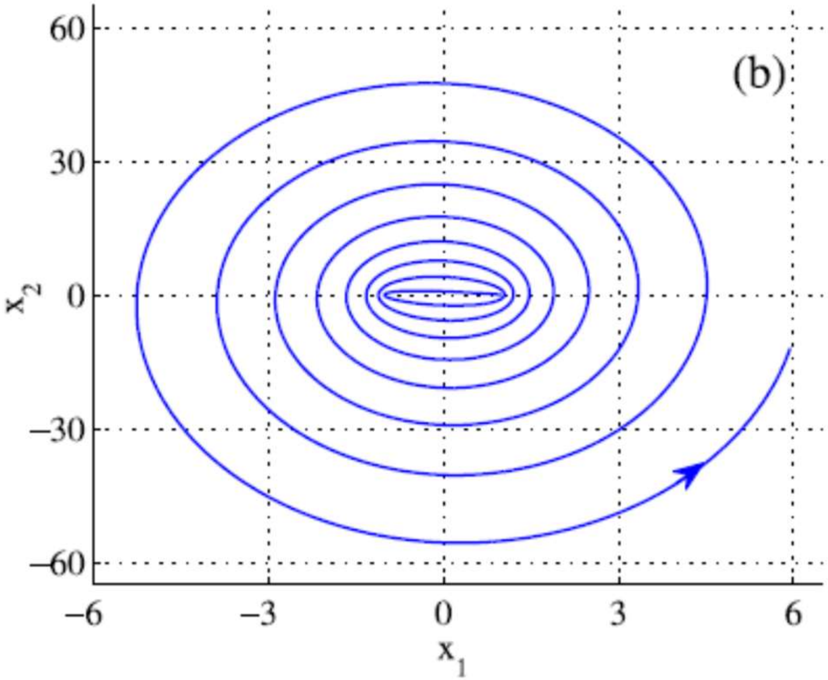


source: [8]

stable for $K <$ stiffness k



unstable for $K >$ stiffness k



• Issues in control system design and synthesis problem

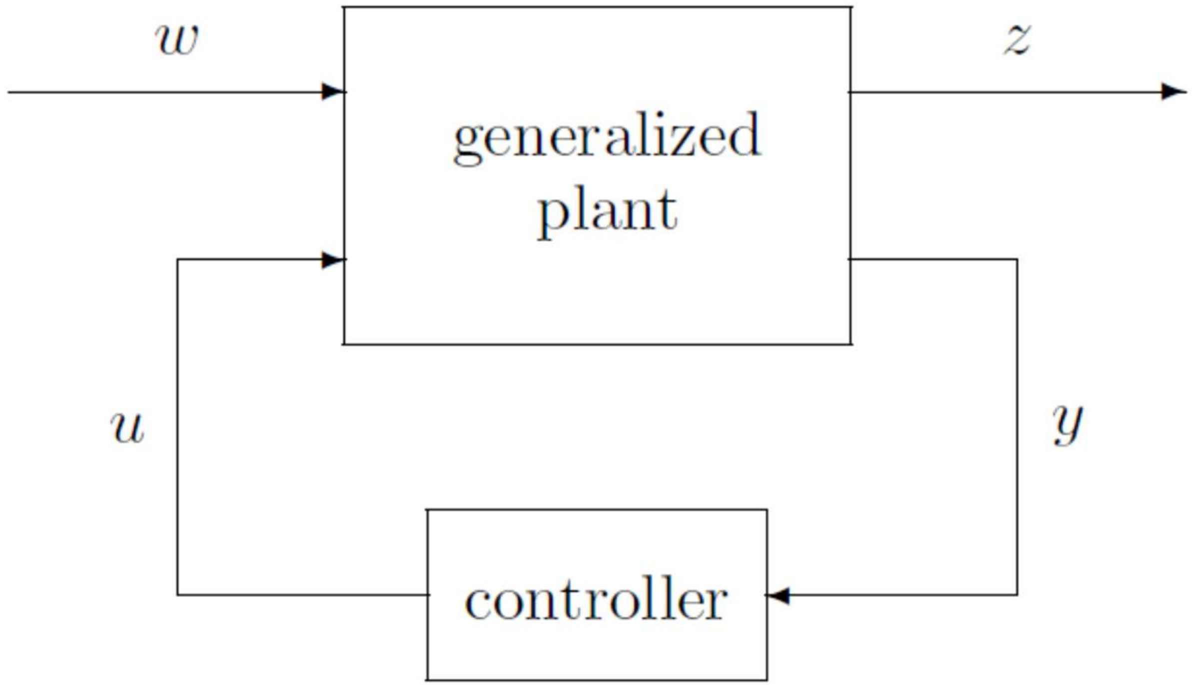
by Doyle, Francis, and Tannenbaum, 1992

The process of designing a control system generally involves many steps.
A typical scenario is as follows:

1. Study the system to be controlled and decide what types of sensors and actuators will be used and where they will be placed.
2. Model the resulting system to be controlled.
3. Simplify the model if necessary, so that it is tractable.
4. Analyze the resulting model; determine its properties.
5. Decide on performance specifications.
6. Decide on the type of controller to be used.
7. Design a controller to meet the specs, if possible; if not, modify the specs or generalize the type of controller sought.
8. Simulate the resulting controlled system, either on a computer or in a pilot plant.
9. Repeat from step 1 if necessary.
10. Choose hardware and software and implement the controller.
11. Tune the controller on-line if necessary.

source: [9]

- Issues in control system design and synthesis problem, cont.



source: [9]

The synthesis problem can be stated as follows: Given a set of generalized plants, a set of exogenous inputs w, and an upper bound on the size of z, design an implementable controller $y \rightarrow u$ to achieve this bound. How the size of z is to be measured (e.g., power or maximum amplitude) depends on the context.

- Practical (hand on) session 1

Consider Example 2 (from page 7)

For the given car's dynamics

$$m\frac{dv(t)}{dt}+bv(t)+R(t)=\tau(t)$$

assume the following parameter constants $m=1000, \quad b=400$

and the disturbance value $R(t)=\begin{cases} 0 & \text{for} \quad 0\le t\le 30 \ \text{sec} \\ 15000 & \text{for} \quad t>30 \ \text{sec} \end{cases}$

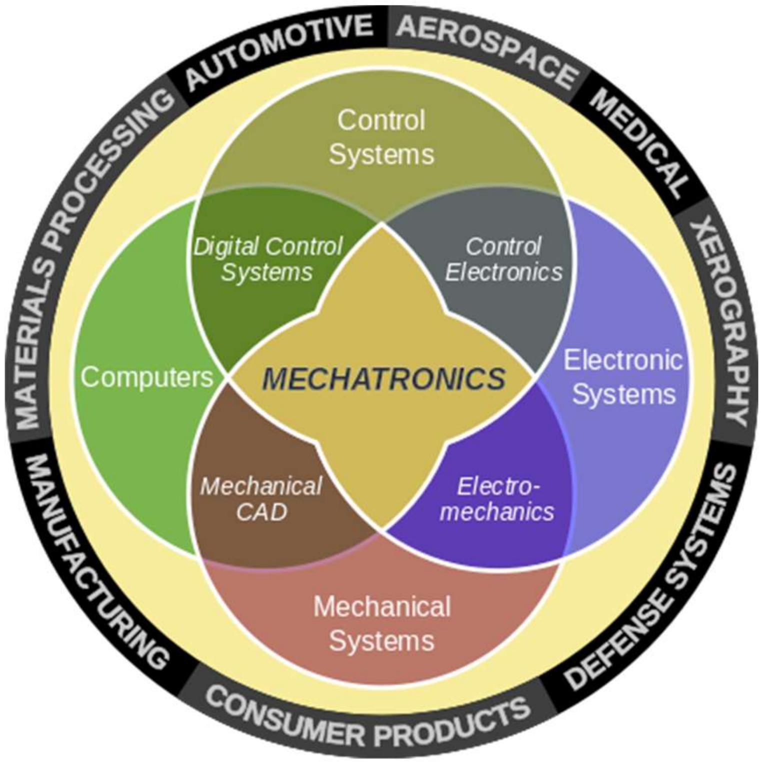
1. Design a simple PI velocity controller and implement the closed-loop control system (either in Matlab or Simulink). Show the step response.
2. Show what happens with a step response of the control system without the integral control part, i.e. K_i=0.
3. How will the step response without integral control part change, if there is no inclination road disturbance acting on the car, i.e. $R(t)$=0.

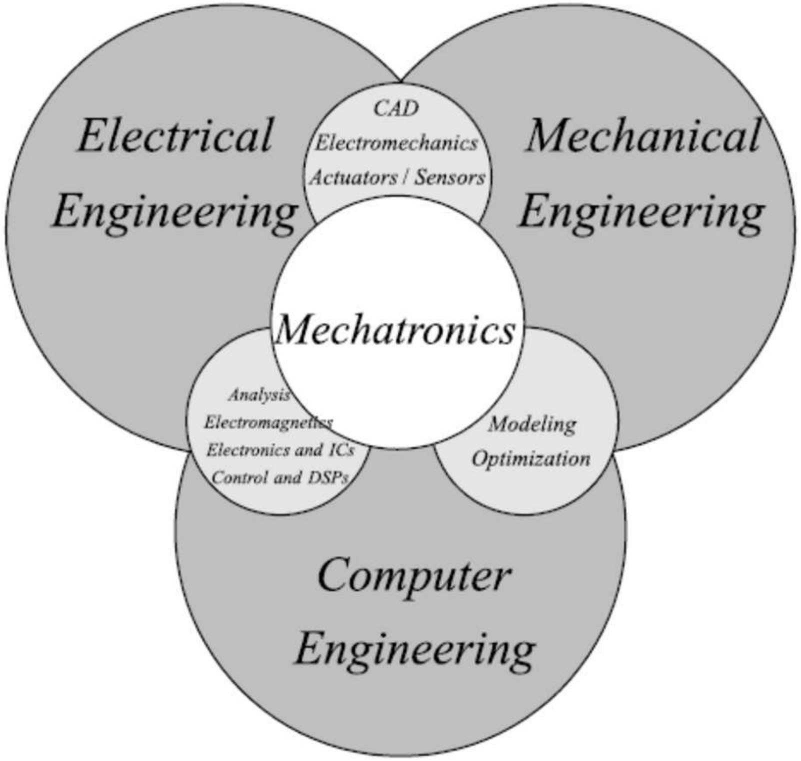
Feedback control systems for mechatronics and robotics

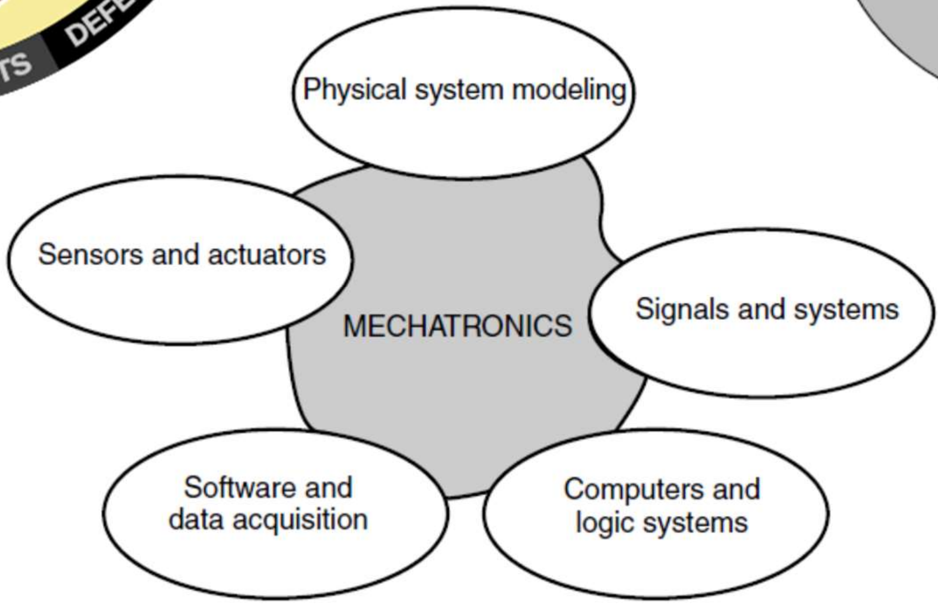
Michael Ruderman
michael.ruderman@uia.no

Control-oriented modeling

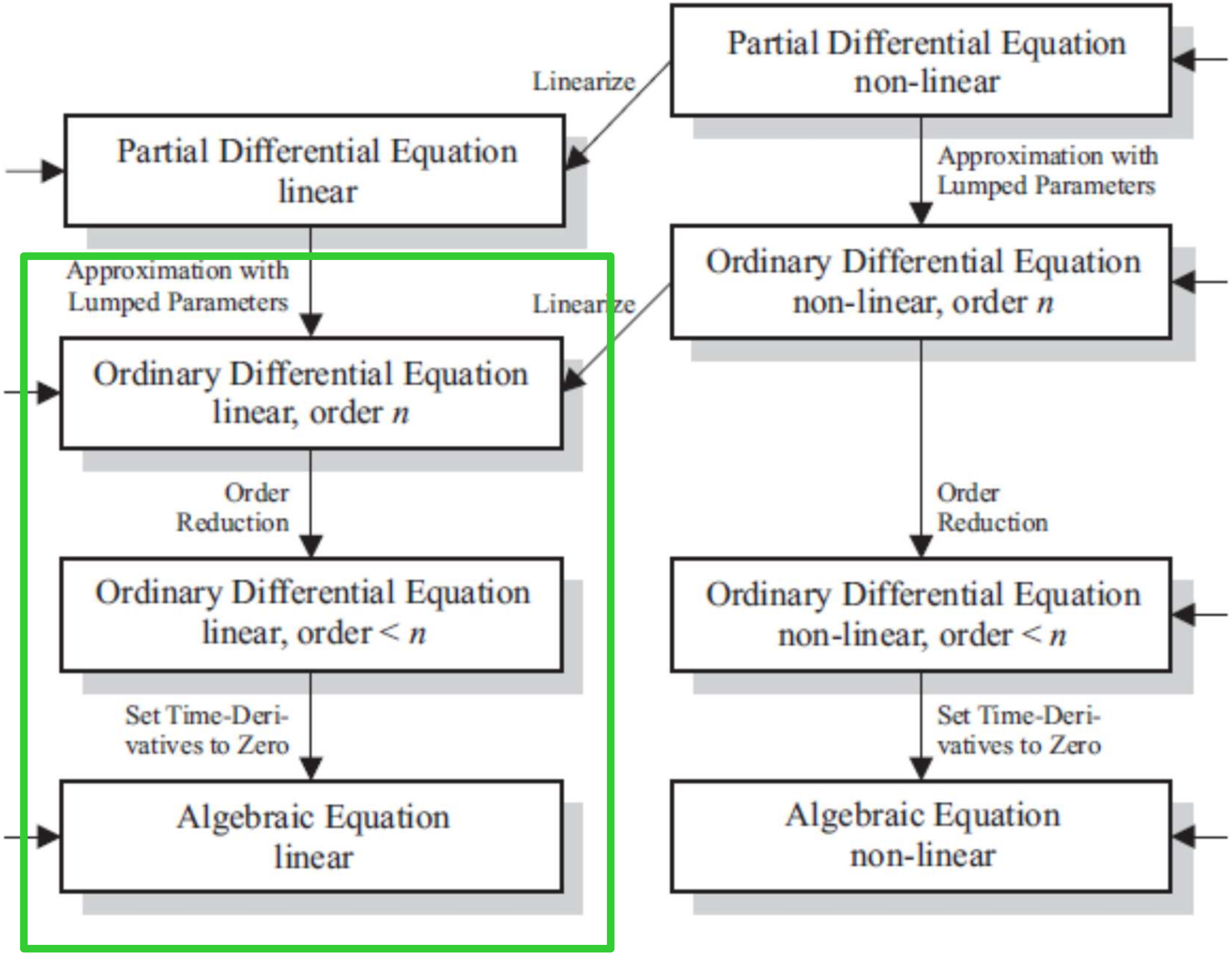
- Domain of knowledge in mechatronics and robotics







- Which modeling approach for dynamic systems?

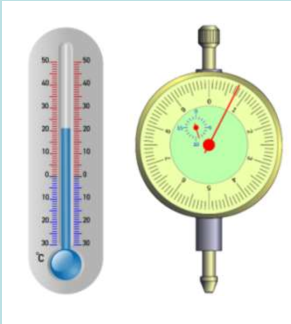


source: [10]

- Which level of detail do we need?

Physical (white-box) models

$$\nabla \cdot \mathbf{D} = \rho$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$



\+ detailed (prime) system knowledge
\- requires domain-specific theories
\- not easy to handle (part. diff. equations)

Mathematical models

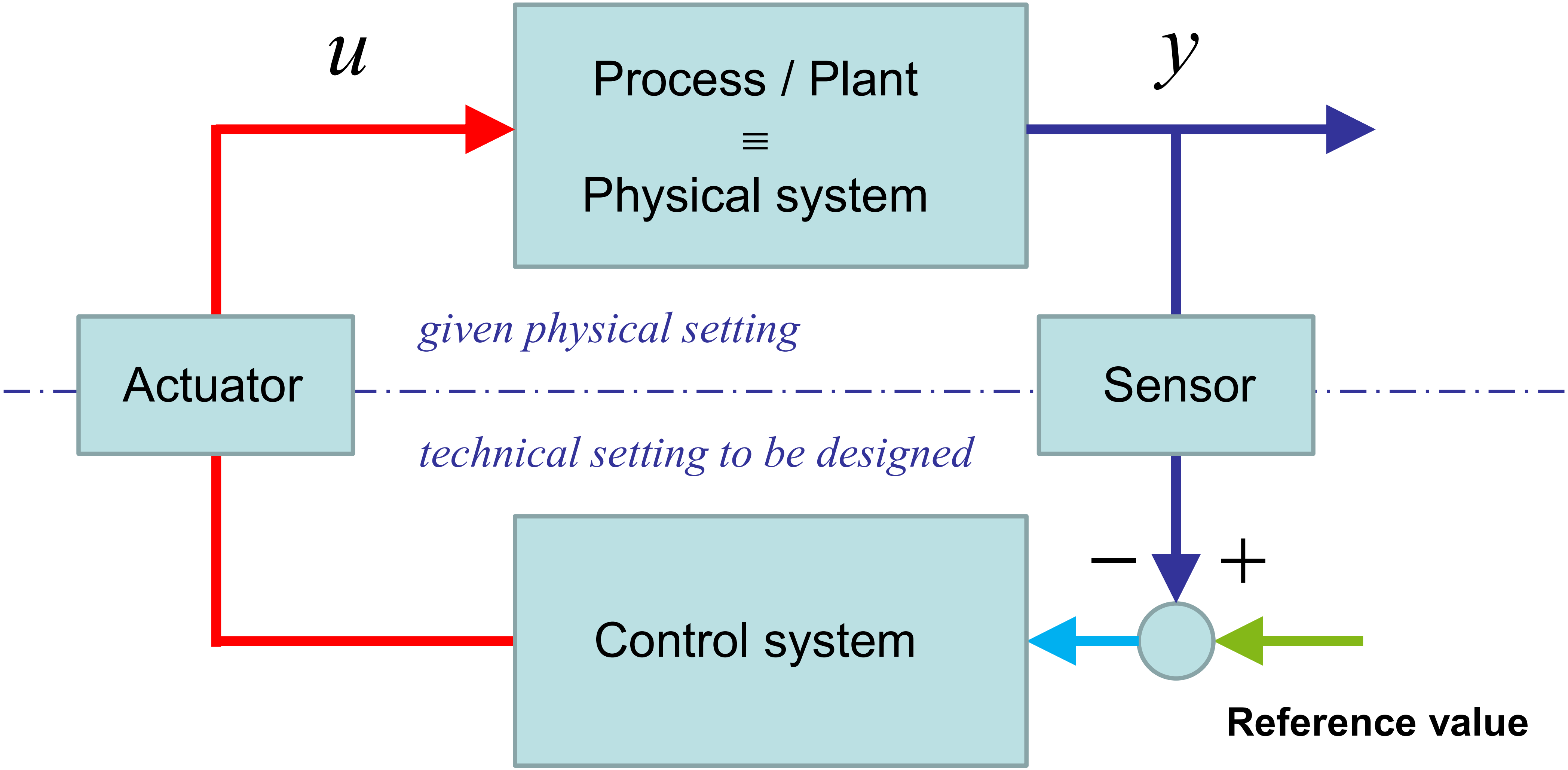
$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$y^{(n)} + a_{n-1} y^{(n-1)} + \ldots$$

\+ universal mathematical formalism
\- several (vague) a-priory assumptions
\- no direct physical interpretation

- “Mixture” from both for a control-oriented modeling

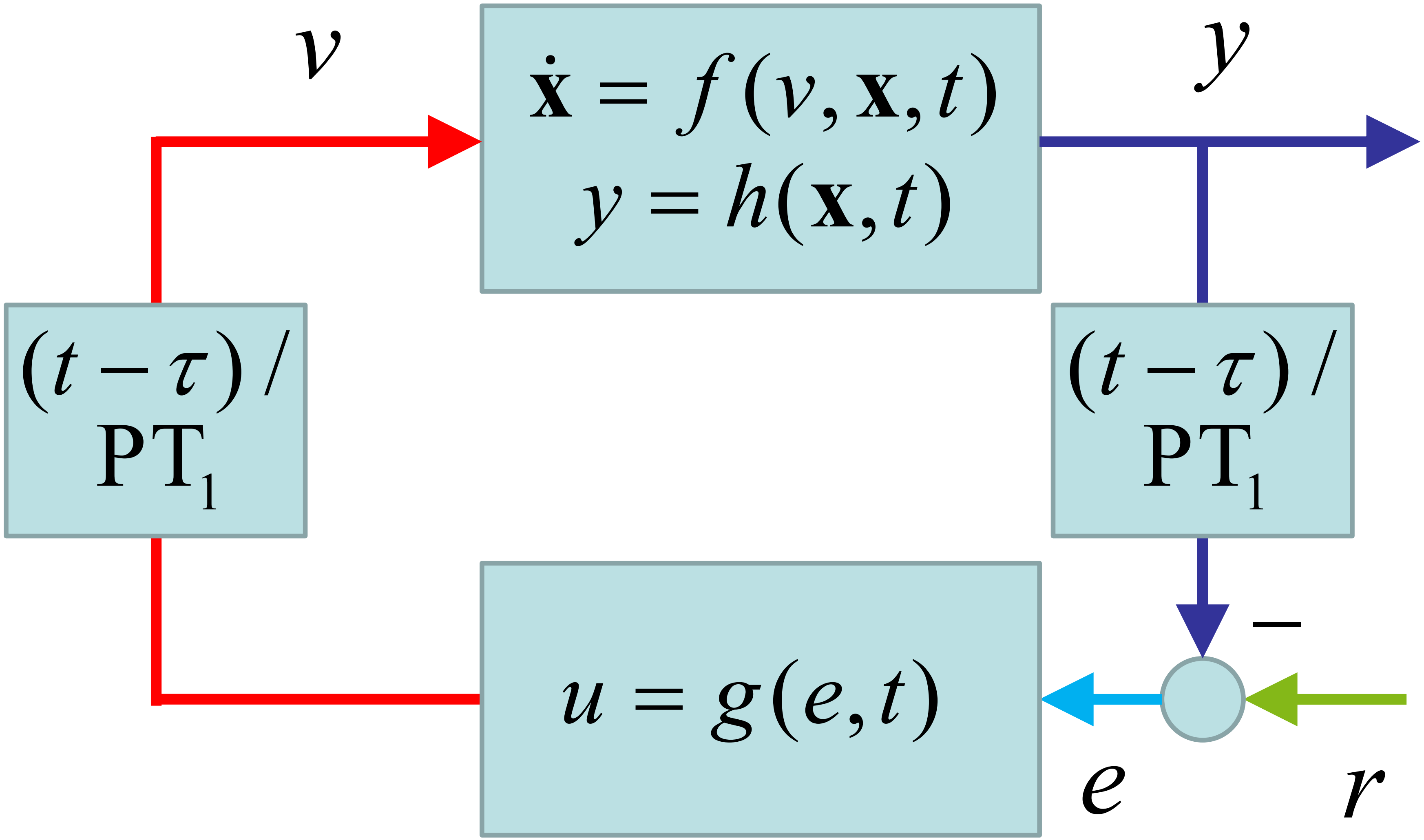
i. Use general and domain-specific knowledge to derive the system <u>structure</u>
ii. Use decomposition into <u>sub-systems</u> to detect forward and feedback <u>couplings</u>
iii. Use measurements (much as possible) to evaluate <u>dynamics</u> and <u>steady-states</u>

- How do we model input-output systems in the loop?



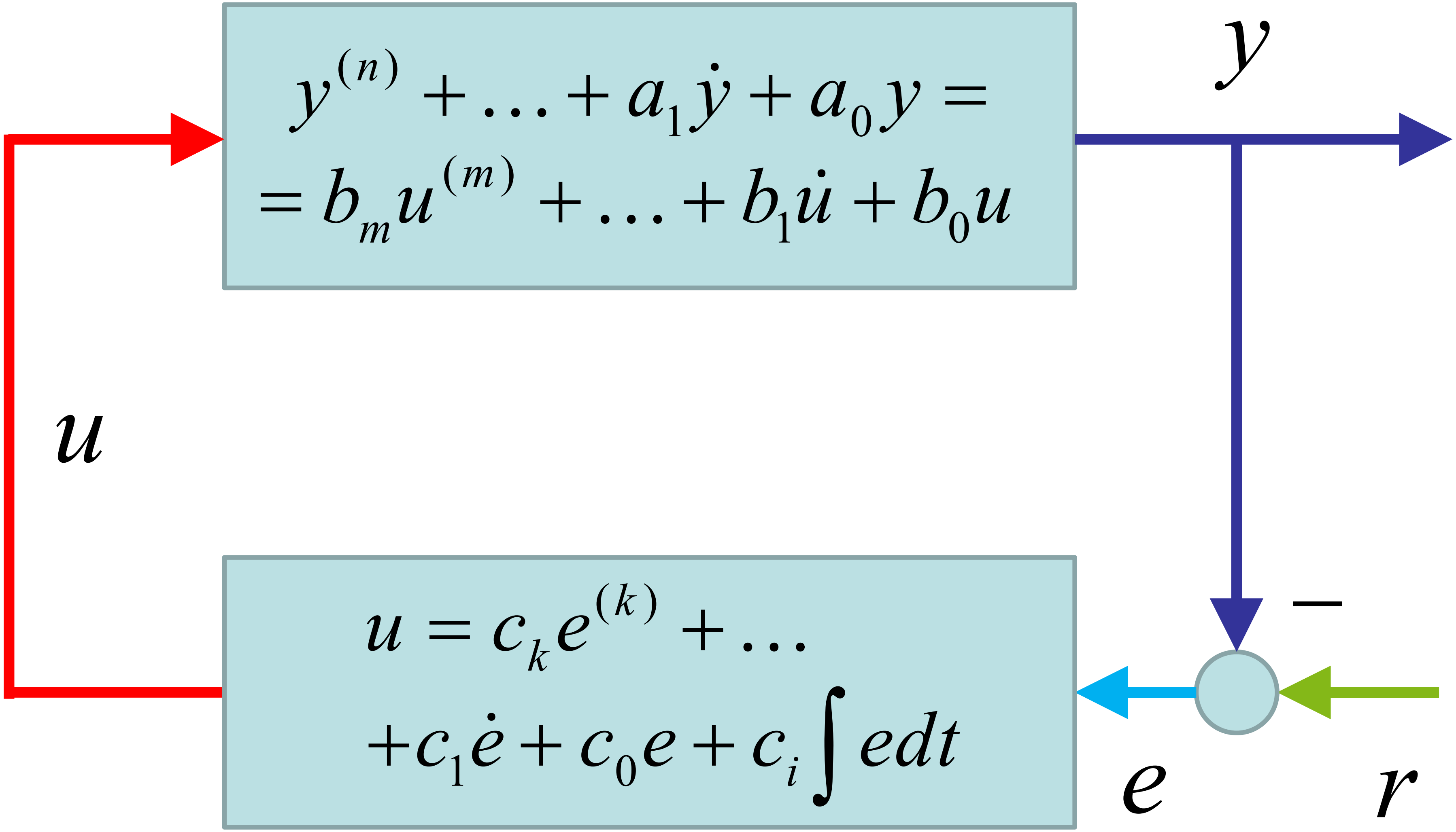
$\Rightarrow$ Representation of the system elements by the transfer-blocks

- How do we model input-output systems in the loop? cont.



$\Rightarrow$ Generic view of dynamics: with nonlinear equations and delays

- How do we model input-output systems in the loop? cont.



$\Rightarrow$ Generic view of linear dynamics: with standard ODEs

- How to perform linearization?

Case 1: output $y(t)$ is a function of only one input $u(t)$

$$y(t) = f(u)$$

Taylors series expansion about the point $\bar{u}, \bar{y}$

$$y = f(\bar{u}) + \left.\frac{df}{du}\right|_{u=\bar{u}} (u - \bar{u}) + \frac{1}{2!} \left.\frac{d^2 f}{du^2}\right|_{u=\bar{u}} (u - \bar{u})^2 + \ldots.$$

If $u - \bar{u}$ is small, we ignore higher order derivative terms

$$y = \bar{y} + k \cdot (u - \bar{u}) \; ; \text{ where } \; \bar{y} = f(\bar{u}) \; \text{ and } k = \left.\frac{df}{du}\right|_{u=\bar{u}}$$

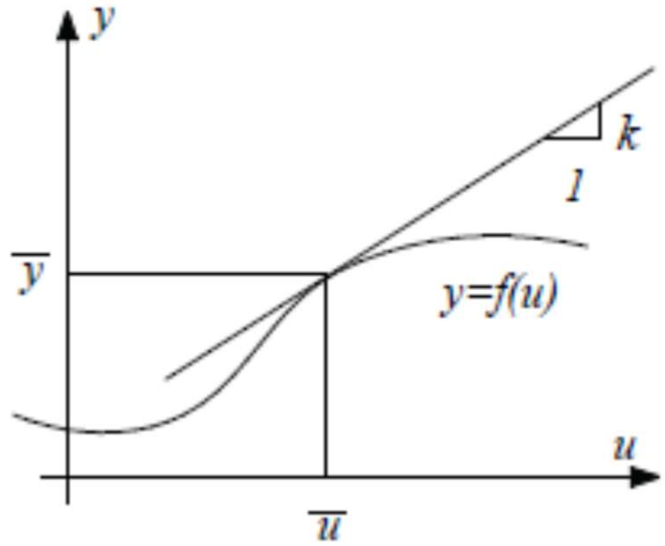
- How to perform linearization? cont.

$$y - \bar{y} = k \cdot (u - \bar{u})$$

$$\Downarrow$$

$$y = k \cdot u$$

(Linear model)



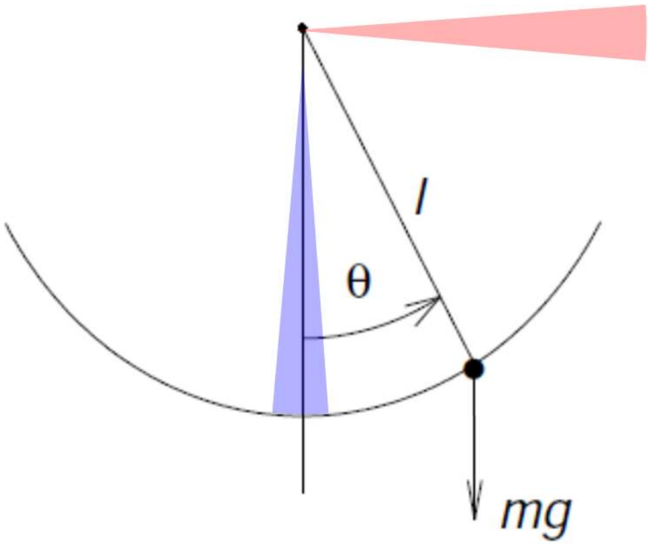
Case 2: output $y(t)$ is a function of multiple inputs

$$y = f(\bar{u}_1, \bar{u}_2) + \left.\frac{\partial f}{\partial u_1}\right|_{\substack{u_1 = \bar{u}_1 \\ u_2 = \bar{u}_2}} (u_1 - \bar{u}_1) + \left.\frac{\partial f}{\partial u_2}\right|_{\substack{u_1 = \bar{u}_1 \\ u_2 = \bar{u}_2}} (u_2 - \bar{u}_2)$$

- How to perform linearization? cont.

Illustrative Example 6

(gravity pendulum) *source: [11]*



$$ml\ddot{\theta} = -mg\sin\theta - kl\dot{\theta}$$

Linearizing around working point $\theta_w = 0$ (rad):

$$\sin\theta\big|_{\theta_w=0} \approx \theta \quad \Rightarrow$$

$$a_2\ddot{\theta} + a_1\dot{\theta} + a_0\theta = 0 \quad \text{with} \quad a_2 = ml,\ a_1 = kl,\ a_0 = mg$$

Linearizing around working point $\theta_w = \pi/2$ (rad):

$$\sin\theta\big|_{\theta_w=\pi/2} \approx 1 \quad \Rightarrow$$

$$a_2\ddot{\theta} + a_1\dot{\theta} = -\text{const} \quad \text{with} \quad a_2 = ml,\ a_1 = kl,\ \text{const} \equiv mg \equiv disturbance$$

- State-space model via linearization

  - Generic class of nonlinear dynamic systems Σ (here SISO)

$$\dot{x} = f(x,u),$$
$$y = g(x,u), \qquad u, y \in \mathbb{R}^1,\ x \in \mathbb{R}^n.$$

  - Use Taylor expansion for linearization

$$\dot{x} = x_0 + x\left(\frac{\partial f}{\partial x}\right)_{x_0,u_0} + u_0 + u\left(\frac{\partial f}{\partial u}\right)_{x_0,u_0} + f_{h.o.t.}$$
$$y = x_0 + x\left(\frac{\partial g}{\partial x}\right)_{x_0,u_0} + u_0 + u\left(\frac{\partial g}{\partial u}\right)_{x_0,u_0} + g_{h.o.t.}$$

with working point movable into origin: $x_0 = 0,\ u_0 = 0$, and higher order terms ($h.o.t.$) which are neglected

  - Matrix/vector form, i.e. Jacobian matrices

$$\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \cdots & \partial f_1/\partial x_n \\ \partial f_2/\partial x_1 & \ddots & \ddots & \partial f_2/\partial x_n \\ \vdots & \ddots & \ddots & \vdots \\ \partial f_n/\partial x_1 & \partial f_n/\partial x_2 & \cdots & \partial f_n/\partial x_n \end{pmatrix} \qquad \mathbf{B} = \frac{\partial \mathbf{f}}{\partial u} = \begin{pmatrix} \partial f_1/\partial u \\ \partial f_2/\partial u \\ \vdots \\ \partial f_n/\partial u \end{pmatrix}$$

- State-space model via linearization, cont.

$$\mathbf{C}^T = \frac{\partial g}{\partial \mathbf{x}} = \begin{pmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \cdots & \frac{\partial g}{\partial x_n} \end{pmatrix}, \quad \mathbf{D} = \frac{\partial g}{\partial u}; \quad \text{if } \Sigma \text{ is stictly proper } \Rightarrow \mathbf{D} = 0$$

- General form of state-space model for MIMO systems

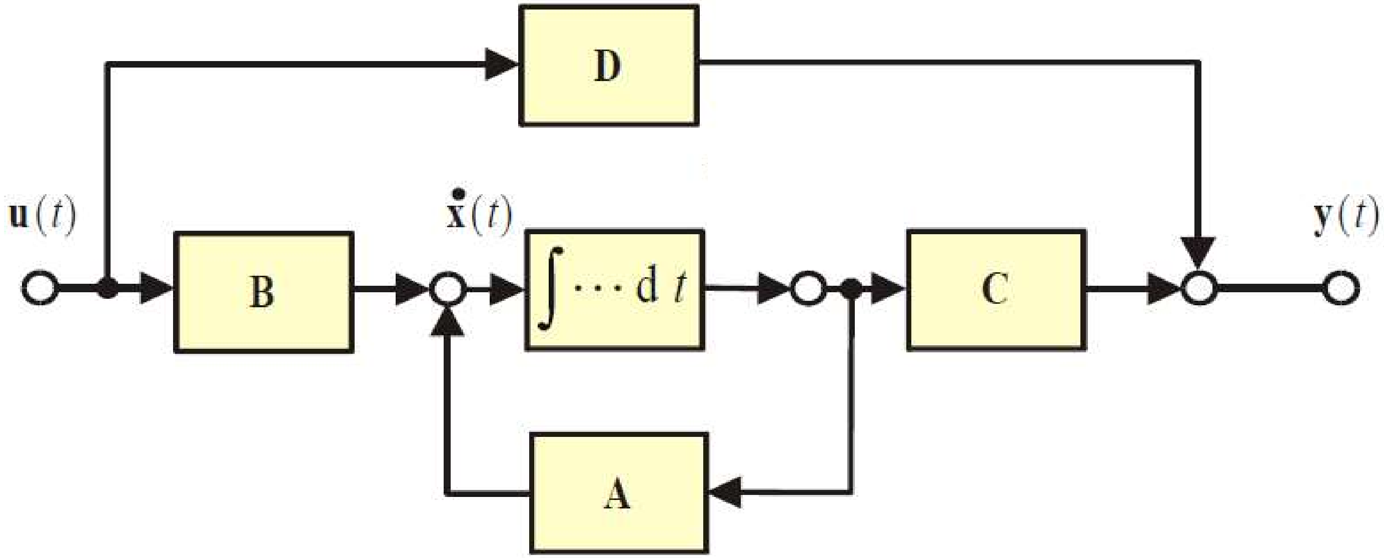
$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

$\mathbf{x}(t) \in \mathbb{R}^n$: vector of system states

$\mathbf{u}(t) \in \mathbb{R}^r$: vector of input values

$\mathbf{y}(t) \in \mathbb{R}^m$: vector of output values

$\mathbf{A} \in \mathbb{R}^{n\times n}, \mathbf{B} \in \mathbb{R}^{n\times r}, \mathbf{C} \in \mathbb{R}^{m\times n}, \mathbf{D} \in \mathbb{R}^{m\times r}$: matrices with system parameters



- Linearization analysis for ‘small-signals’

$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$ Linear state-space model (SISO) to be derived

$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u)$ Generalized nonlinear system (SISO)

First, determine equilibrium values (in the operation point) $\mathbf{x}_o, u_o$ such that

$$\dot{\mathbf{x}}_o = 0 = \mathbf{f}(\mathbf{x}_o, u_o)$$

Then, consider small-signal perturbations from that equilibrium

$$\mathbf{x} = \mathbf{x}_o + \delta\mathbf{x}, \quad u = u_o + \delta u$$

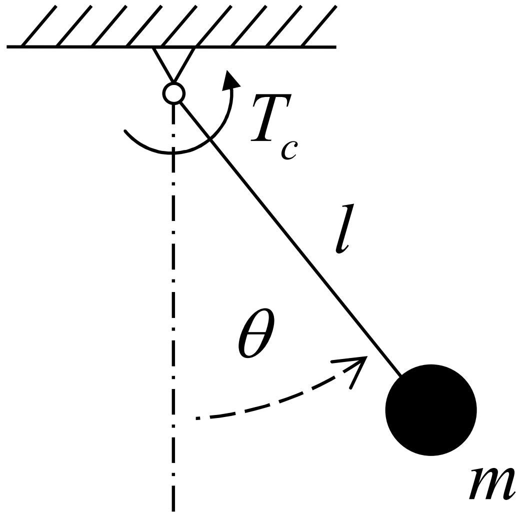
Then, the dynamics is approximated by

$$\dot{\mathbf{x}}_o + \delta\dot{\mathbf{x}} \cong \mathbf{f}(\mathbf{x}_o, u_o) + \mathbf{A}\delta\mathbf{x} + \mathbf{B}\delta u \qquad \mathbf{A} = \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right]_{\mathbf{x}_o, u_o} \qquad \mathbf{B} = \left[\frac{\partial \mathbf{f}}{\partial u}\right]_{\mathbf{x}_o, u_o}$$

Linearized dynamics around the equilibrium point

$$\delta\dot{\mathbf{x}} = \mathbf{A}\delta\mathbf{x} + \mathbf{B}\delta u$$

- **Example 7**: linearization of actuated pendulum



$$\ddot{\theta} + \frac{g}{l}\sin\theta = \frac{T_c}{ml^2}$$: equation of motion (without damping)

Introducing $\left[x_1, x_2\right]^T = \left[\theta, \dot{\theta}\right]^T$ and $\omega_0 = \sqrt{\frac{g}{l}}$, $u = \frac{T_c}{ml^2}$

$$\Rightarrow \qquad \dot{\mathbf{x}} = \begin{bmatrix} x_2 \\ -\omega_0^2 \sin x_1 + u \end{bmatrix} = \begin{bmatrix} f_1(\mathbf{x}, u) \\ f_2(\mathbf{x}, u) \end{bmatrix} = \mathbf{f}(\mathbf{x}, u)$$

Equilibrium with zero input (i.e. $u_0=0$)

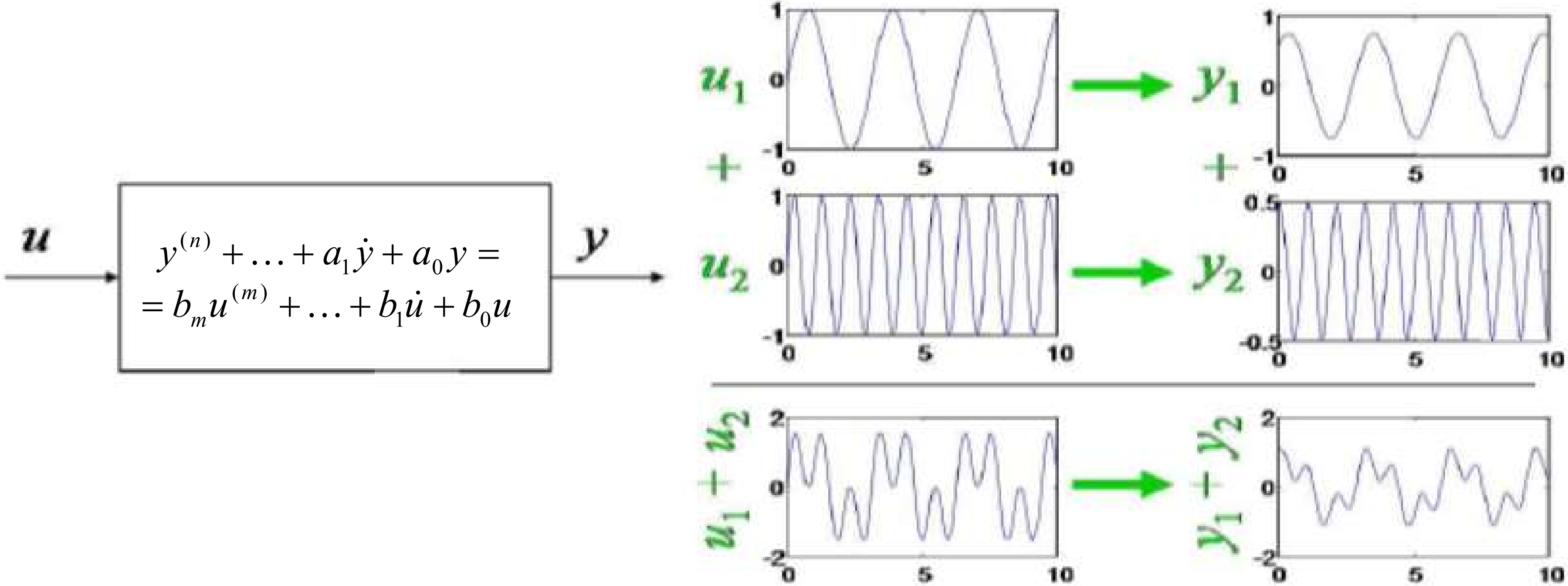
$$\dot{x}_1 = \dot{\theta} = 0, \qquad \dot{x}_2 = \ddot{\theta} = -\frac{g}{l}\sin\theta = 0 \quad \Rightarrow \quad \mathbf{x}_0 = \left[\theta_0 \;\; 0\right]^T, \quad u_0 = 0, \qquad \theta_0 = 0, \pi$$

Linearized dynamics (at 0 and π)

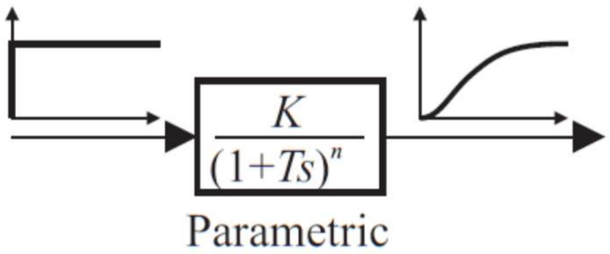
$$\mathbf{A} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{\mathbf{x}_0, u_0} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 \cos\theta_0 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix}_{\mathbf{x}_0, u_0} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

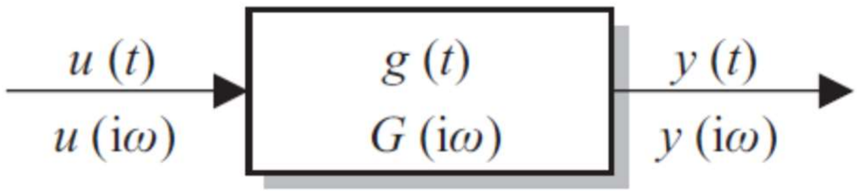
- Advantage of linear time invariant (LTI) systems identification

  - Superposition principle in LTI input-output systems



  - Tunable parameters from the response in both, time- and frequency-domain

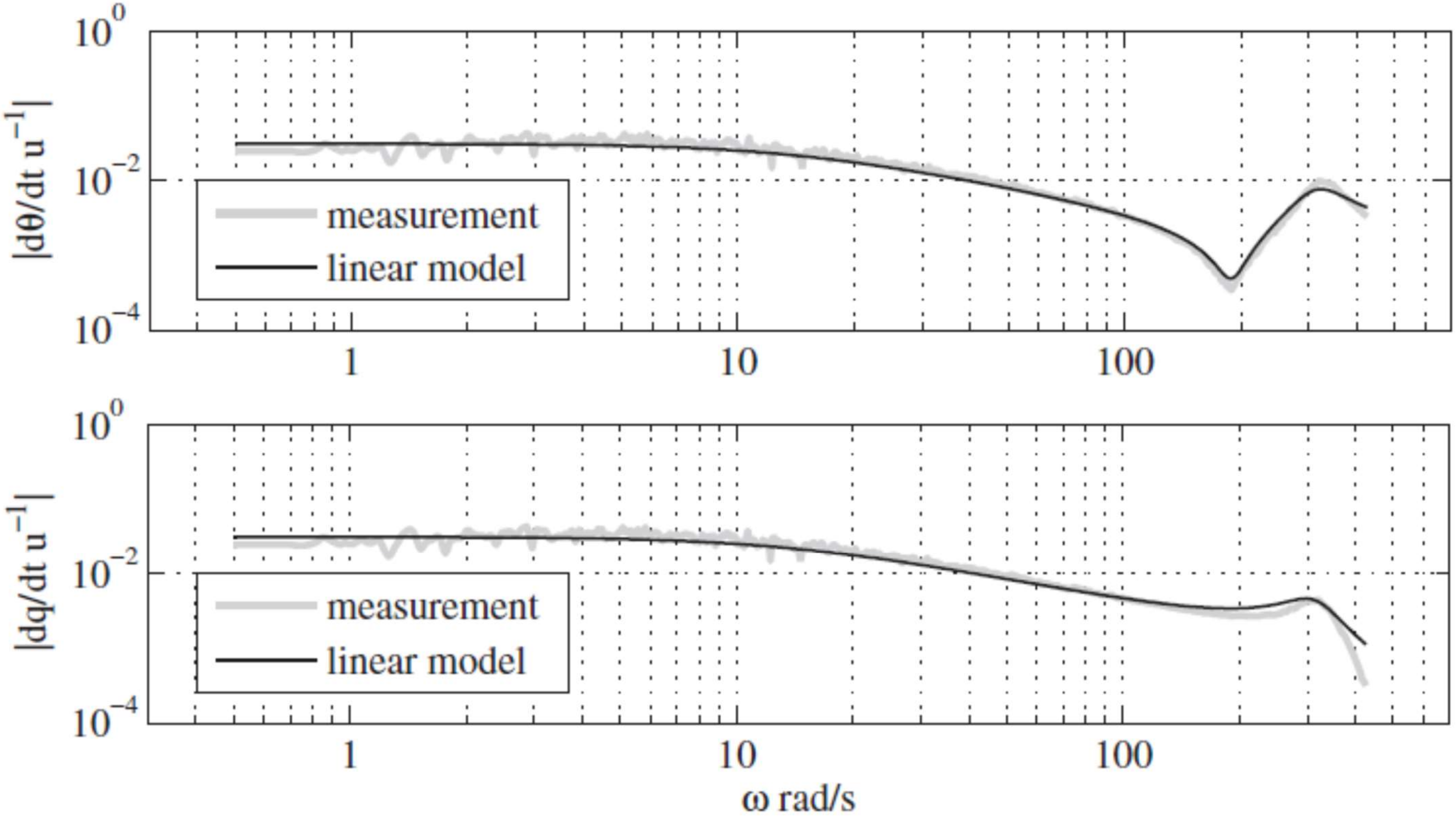




- Example of real applications: joint (q and θ) with elasticities

$$
\begin{aligned}
m\ddot{q} + d\dot{q} + K(q-\theta) + G(\dot{q}-\dot{\theta}) &= u \\
M\ddot{\theta} + D\dot{\theta} - K(q-\theta) - G(\dot{q}-\dot{\theta}) &= 0
\end{aligned}
$$

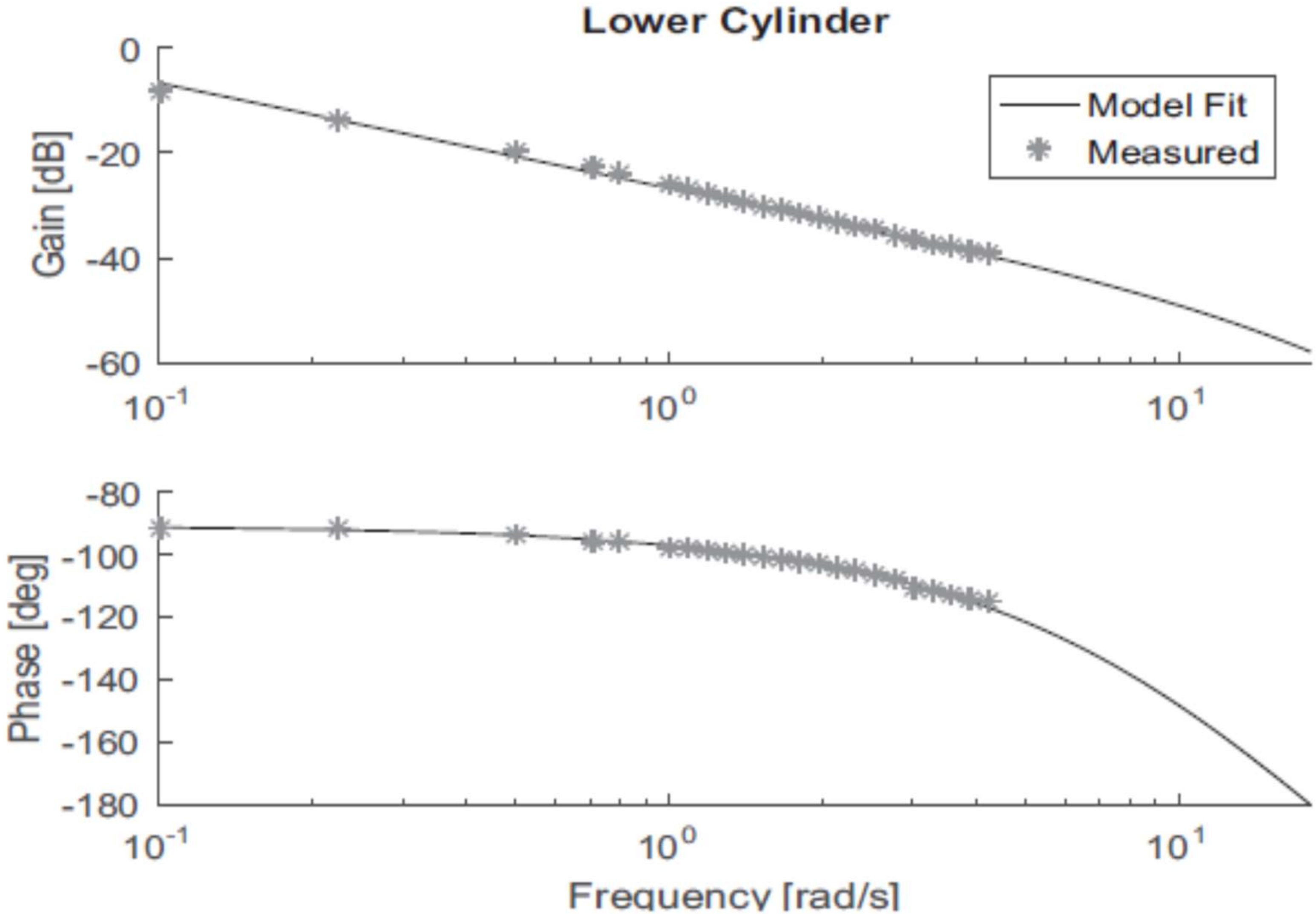
source: [12]



- Example of real applications: hydraulic cylinder of a crane

$$G(s) = \frac{1}{s} \cdot \frac{K \cdot {\omega_n}^2}{s^2 + 2 \cdot \zeta \cdot \omega_n \cdot s + {\omega_n}^2}$$

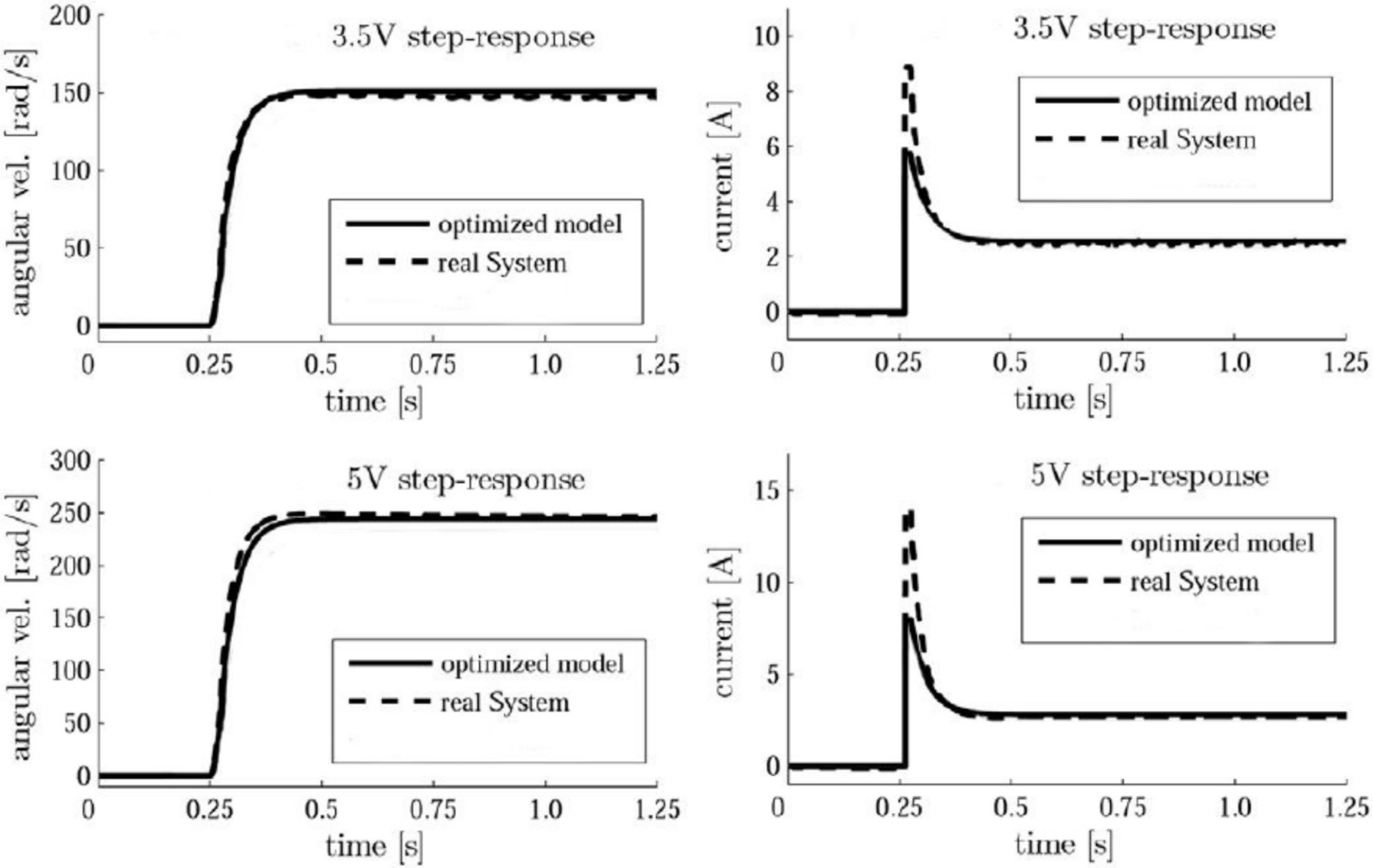
source: [13]



- Example of real applications: DC motor velocity and current

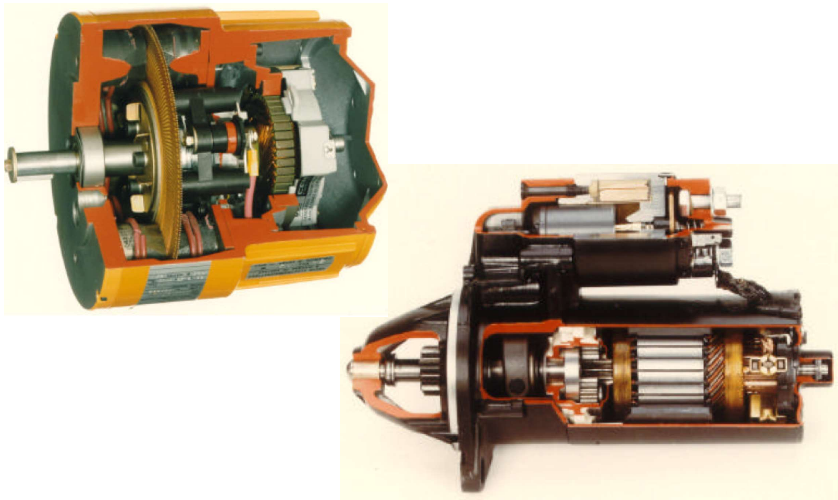
$$\dot{\mathbf{x}} = \begin{bmatrix} -\frac{R_m}{L} & -\frac{K_e}{L} \\ \frac{K_m}{J} & -\frac{K_d}{J} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ -\frac{F_c}{J} \end{bmatrix} \mathrm{sgn}(x_2)$$

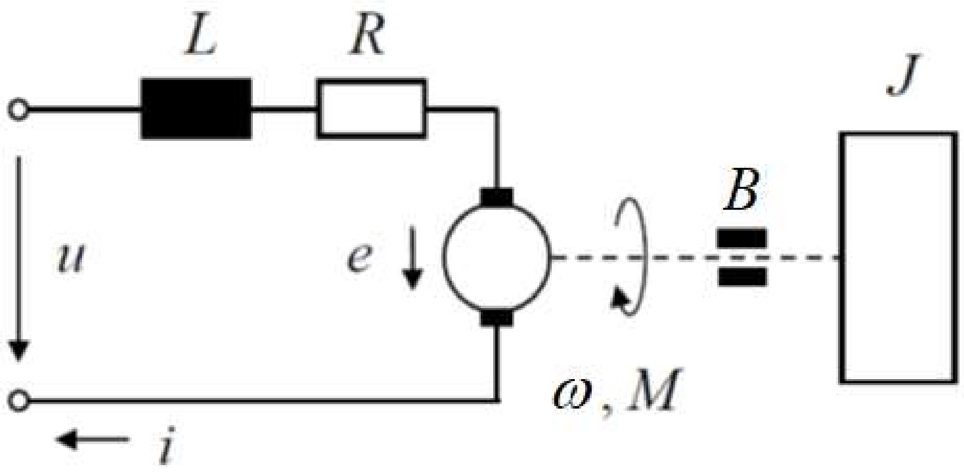
source: [14]



- Dynamic system of 2nd order

Direct current (DC) motor



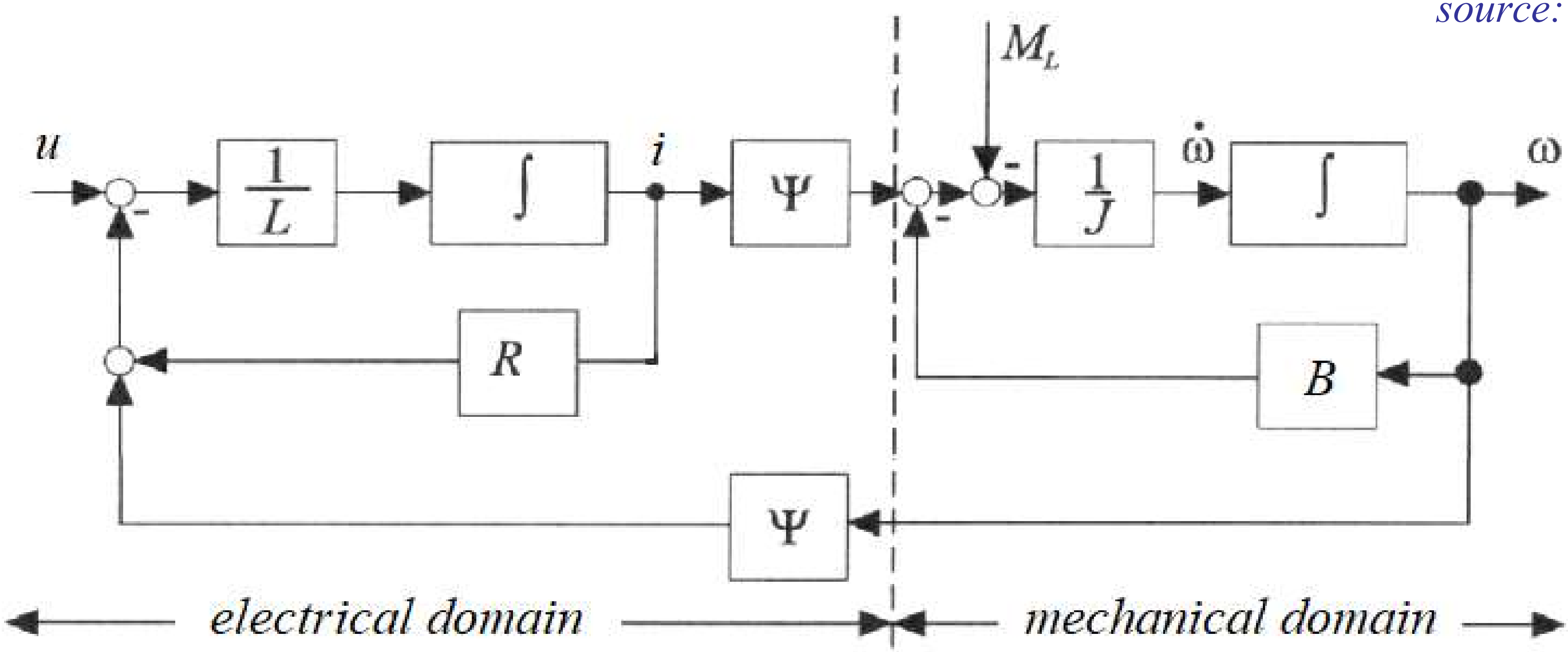


Linear dynamic system model

$$\left.\begin{aligned} L\frac{di}{dt}+R\,i &= u-\underbrace{\Psi\,\omega}_{e} \\ J\frac{d\omega}{dt}+B\omega &= \Psi\,i-M_L \end{aligned}\right\} \Rightarrow \frac{\omega(s)}{u(s)}=\frac{\Psi}{LJ\,s^2+\left(JR+BL\right)s+BR+\Psi^2}$$

- Dynamic system of 2nd order, cont.

source: [15]

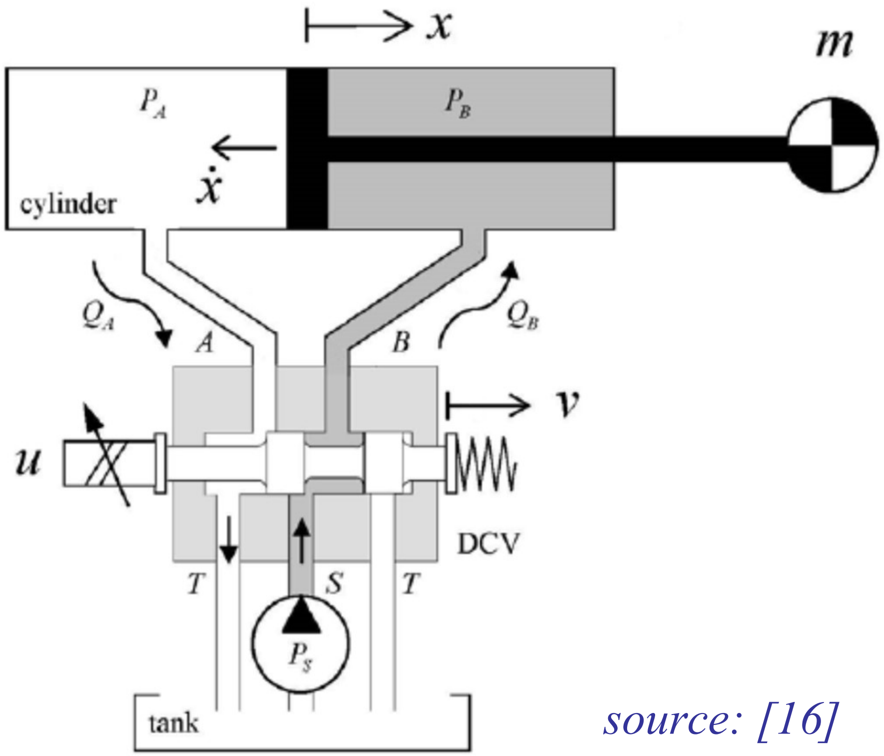


Relevant assumptions to be made:

i. No power-electronics, correspondingly no PWM and no circuits dynamics
ii. Constant, correspondingly uniform, magneto-mechanical coupling (Ψ)
iii. Linear viscous damping only (i.e. no torque ripples, no Coulomb friction)

- Dynamic system of 3rd order

Valve controlled hydraulic cylinder



source: [16]

Static input nonlinearity of the valve

$z = h(v)$

Orifice equations of the valve

$$Q_A(z) = \begin{cases} zK\sqrt{P_S - P_A}, & \text{for } z > 0, \\ zK\sqrt{P_A - P_T}, & \text{for } z < 0, \\ 0, & \text{otherwise;} \end{cases}$$

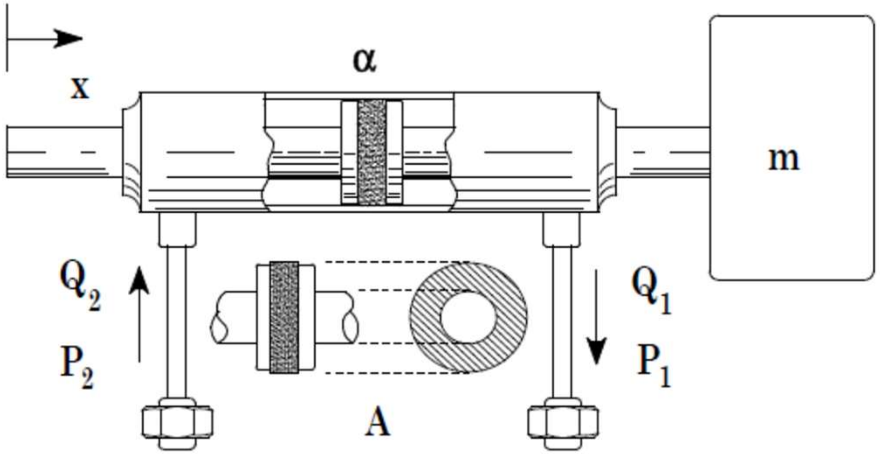
$$Q_B(z) = \begin{cases} -zK\sqrt{P_B - P_T}, & \text{for } z > 0, \\ -zK\sqrt{P_S - P_B}, & \text{for } z < 0. \\ 0, & \text{otherwise.} \end{cases}$$

Continuity equations of fluid in hydraulic circuits

$$\dot{P}_A = \frac{E}{V_A}\Big(Q_A - A_A\dot{x} - C_L(P_A - P_B)\Big)$$

$$\dot{P}_B = \frac{E}{V_B}\Big(Q_B + A_B\dot{x} - C_L(P_B - P_A)\Big)$$

- Dynamic system of 3rd order, cont.



Linearized **lumped** equations of the hydraulic system (here *v*=*z*)

$$\begin{cases} Q(s)+k_{qp}\cdot P(s)=k_q\cdot v(s) & \textit{Valve equation} \\ \dfrac{V_0}{4\beta}\cdot s\cdot P(s)+A\cdot s\cdot x(s)=Q(s) & \textit{Flow-pressure equation } (\beta{=}E) \\ m\cdot s^2\cdot x(s)+\alpha\cdot s\cdot x(s)=A\cdot P(s) & \textit{Motion equation} \end{cases}$$

Transfer function from the valve spool position to the cylinder piston stroke

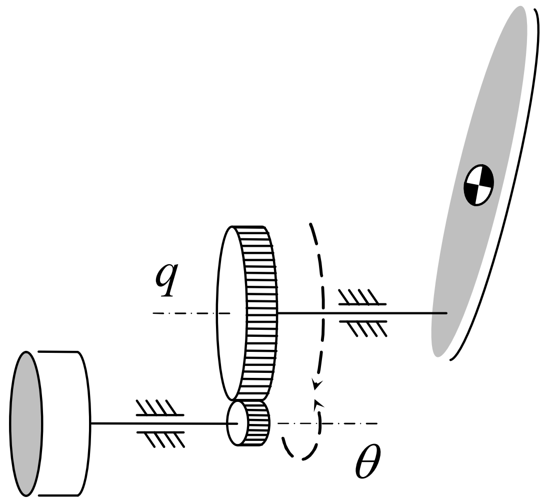
$$\frac{x(s)}{v(s)}=\frac{4A\cdot k_q\cdot\beta}{m\cdot V_0\cdot s^3+\left(V_0\cdot\alpha+4k_{qp}\cdot\beta\cdot m\right)\cdot s^2+\left(4A^2\cdot\beta+4k_{qp}\cdot\alpha\cdot\beta\right)\cdot s}$$

- Dynamic system of 4^{th} order

Flexible joint in robotics and mechatronics

– After neglecting (or compensating) for the residual plant's dynamics

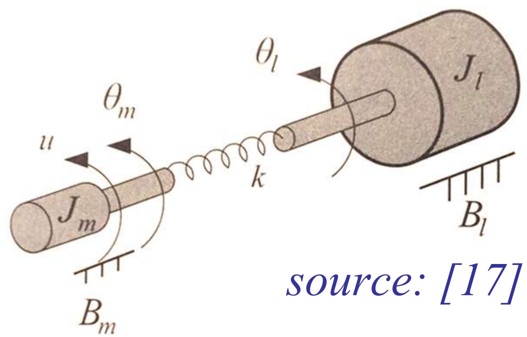
→ **two-mass (-inertia) system** connected by a gear transmission



– If neglecting elasticities, i.e. $q = \theta$

→ **single (lumped) mass and damping**

$$(m + M)\ddot{q} + (d + D)\dot{q} = u$$

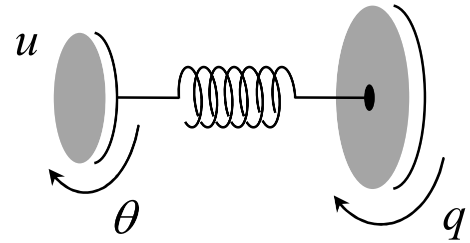


source: [17]

– If significant elasticities, e.g. flexible parts in gear (like for example in harmonic drives)

→ **additional internal dynamics**

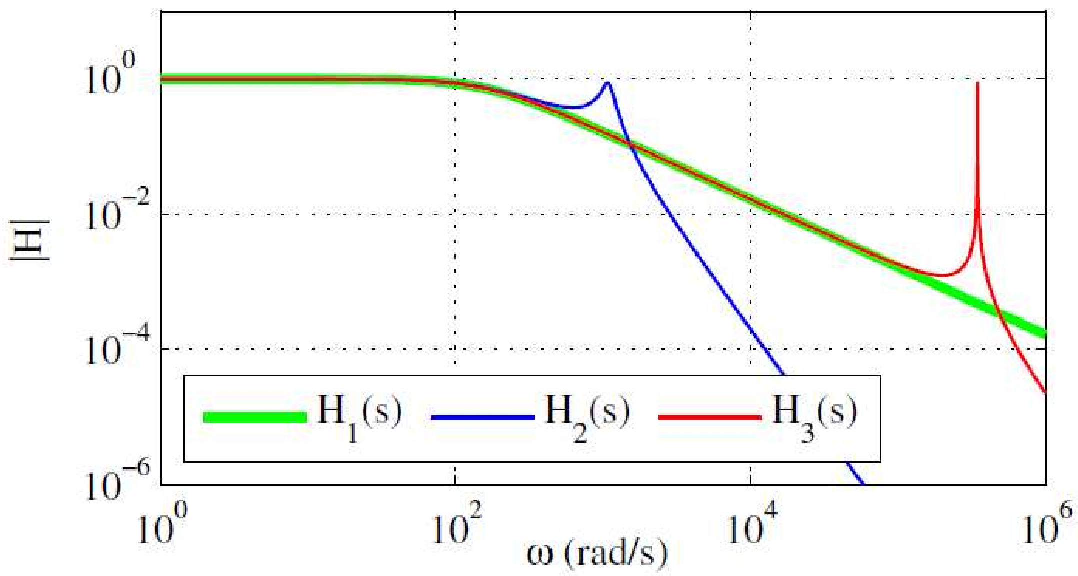
$$m\ddot{\theta} + d\dot{\theta} + K(\theta - q) = u$$
$$M\ddot{q} + D\dot{q} - K(\theta - q) = 0$$



- Dynamic system of 4$^{\text{th}}$ order, cont.

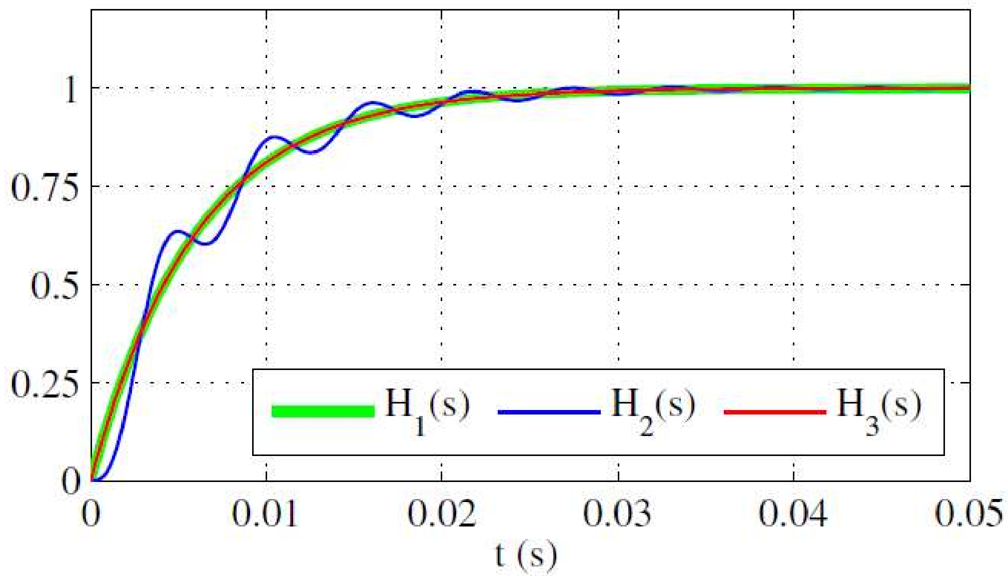
  – Neglecting elasticity (first-order system)

$$H_1(s) = \frac{\dot{q}(s) = \dot{\theta}(s)}{U(s)}$$



  – With detectable elasticity (low stiffness)

$$H_2(s) = \left.\frac{\dot{q}(s)}{U(s)}\right|_{K>m,M}$$



  – With “hidden” elasticity (high stiffness)

$$H_3(s) = \left.\frac{\dot{q}(s)}{U(s)}\right|_{K\gg m,M}$$

• Practical (hand on) session 2

Consider DC motor (from page 49)

Assume the following parameter values for the DC motor:

$$R = 1, \quad L = 0.0002, \quad \Psi = 0.04,$$
$$B = 0.0001, \quad J = 0.00005$$

1. Implement the numerical model of DC motor in Simulink.
2. Implement the numerical model of DC motor in MATLAB.
3. Compare the step response of both (from 1. and 2.) implemented models.
4. Can the parameters of DC motor be selected so that the response is oscillatory?

Feedback control systems for mechatronics and robotics

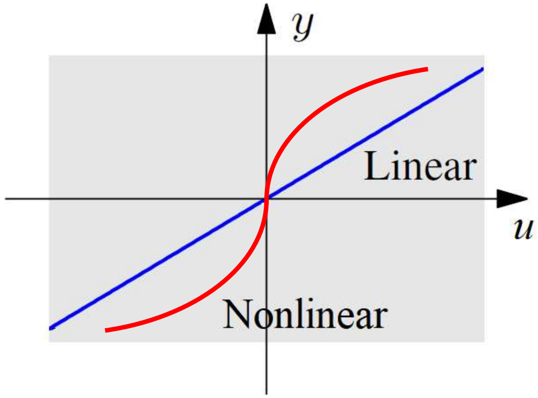
Michael Ruderman
michael.ruderman@uia.no

Dynamic system behavior in time and frequency domain

- Why do we need differential equations in control?

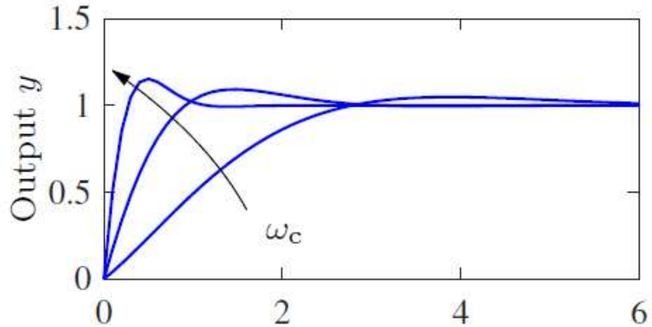
$$u \longrightarrow \boxed{y = f(u)} \longrightarrow y$$

Static input-output function (no time evolution)



$$u(t) \longrightarrow \boxed{\begin{aligned} & y^{(n)} + \ldots + a_1 \dot{y} + a_0 y = \\ & b_m u^{(m)} + \ldots + b_1 \dot{u} + b_0 u \end{aligned}} \longrightarrow y(t)$$

Dynamic (i.e. time-dependent) input-output function with a transient phase and steady-state



- Consider Linear Time-Invariant (LTI) systems

  - Given a general dynamic model as differential equation (i.e. ODE)

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t)$$
$$= \quad b_m \frac{d^m u(t)}{dt^m} + b_{m-1}\frac{d^{m-1} u(t)}{dt^{m-1}} + \cdots + b_1 \frac{du(t)}{dt} + b_0 u(t)$$

  - Corresponding transfer function (in Laplace domain)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

All (!) dynamic LTI-systems can be described by means of the ordinary differential equations (ODEs) of the order n, where $n \in \mathbb{N}^+$

- Mostly, we are dealing with proper and causal systems

$r = n - m$	Relative degree	Number of poles minus number of zeros; determines whether a system is strictly proper, biproper or improper
$r > 0$	Strictly proper	The system has more poles than zeros; it is causal and therefore implementable, it has an improper inverse and zero high-frequency gain
$r = 0$	Biproper	The system has equal number of poles and zeros; it is implementable, has a biproper inverse and has a feed-through term, i.e., a non-zero and finite high-frequency gain
$r < 0$	Improper	The system has more zeros than poles; it is not causal, cannot be implemented, has a strictly proper inverse and has infinite high-frequency gain.

- If the right-hand-side of ODE is zero, then we have a **free system** (i.e. only the own dynamics ≡ natural response)

$$y^{(n)} + \ldots + a_1 \dot{y} + a_0 y = 0$$

homogeneous ODE

- Otherwise, it includes also externally excited (i.e. input-driven) dynamics

$$y^{(n)} + \ldots + a_1 \dot{y} + a_0 y = \underbrace{b_m u^{(m)} + \ldots + b_1 \dot{u} + b_0 u}_{\text{external part of system dynamics}}$$

honmogeneous ODE

- Linear homogeneous 1st-order differential equation

$$\dot{y}(t) + ay(t) = 0 \quad \Rightarrow \quad \dot{y}(t) = -ay(t)$$

We are interested in some function, whose time derivative is equal to the function itself multiplied by some constant

Take the <u>Euler function</u> (exponent)

$$z(t) = e^{\alpha t} z_0 \quad \Rightarrow \quad \dot{z}(t) = \alpha \underbrace{e^{\alpha t} z_0}_{z(t)} = \alpha z(t)$$

Thus, the function $y = e^{-P(t)}$ has that property: $\dot{y} = \underbrace{-\dot{P}(t)}_{-a} e^{-P(t)} = -ay$

Since P is the so-called **antiderivartive** (i.e. integral) of a $\Rightarrow$

$\Rightarrow$ $y = e^{-P(t)} = e^{-\int a dt} = e^{-at}$ is a **solution** of 1st-order ODE

- Linear homogeneous 1st-order differential equation, cont.

  - **General solution** (independent of initial conditions) yields from the fact:

if P is antiderivative of a, i.e. $P(t) = \int a dt,$ then any other $P(t)+K$, with

K to be a constant, is also an antiderivative of a, because of $P(t) = \int a dt + K$

$\Rightarrow \quad y = e^{-\int a dt + K} = \underbrace{e^{K}}_{\equiv C} e^{-at} = Ce^{-at}$ is also a **solution**

Since C can be any constant $\Rightarrow \quad y = Ce^{-at}$ is called **general solution**

  - General solution is also used to solve **initial value problem** of $y = Ce^{-at}$

If, for the given ODE it is known $y(t=0) = y_0$, then

$y(0) = Ce^{-a \cdot 0} \quad \Rightarrow \quad C = y_0 \quad \Rightarrow \quad y(t) = y_0 e^{-at}$: **particular solution** (for given initial value)

- Linear homogeneous 1st-order differential equation, cont.

Example 8 (solve an initial value problem)

$$t\dot{y} + 2y = 0, \quad \text{with } \; y(1) = 5, \; t > 0$$

$$\Rightarrow \quad \dot{y} + \frac{2}{t}y = 0, \quad y(1) = 5. \quad \Rightarrow \; \text{exponential term is } \; a = \frac{2}{t}$$

$$P(t) = \int \frac{2}{t}dt = 2\ln|t| = \ln|t|^2 = \ln t^2$$

Then, the <u>general solution</u> is

$$y(t) = Ce^{-P(t)} = Ce^{-\ln t^2} = Ct^{-2}$$

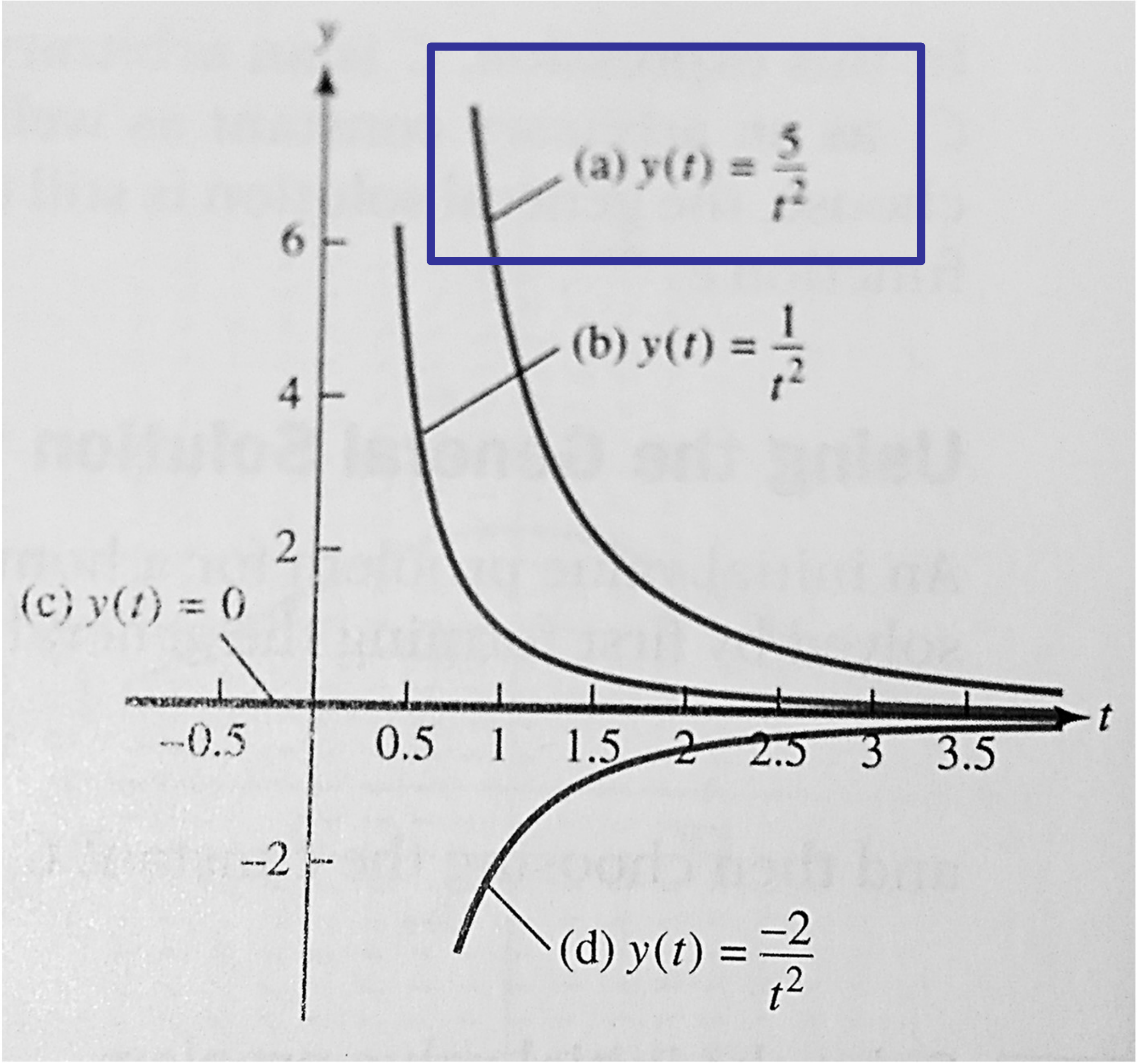
For the initial value $y(1)=5 \; \Rightarrow \; C=5$

Thus, the <u>particular solution</u> is $\quad y(t) = 5\,t^{-2} = \dfrac{5}{t^2}$

- Linear homogeneous 1st-order differential equation, cont.

Example 8 (solve an initial value problem), cont.

source: [18]



- Linear non-homogeneous 1st-order differential equation

$$\dot{y}(t) + ay(t) = bu(t), \quad y(0) = y_0 \tag{i}$$

First, consider the solution of (corresponding) homogeneous equation, i.e. u=0

$$\dot{y}(t) = -ay(t), \quad y(0) = y_0 \tag{ii}$$

Substituting the <u>general solution</u>

$$y(t) = ke^{\lambda t}, \quad \dot{y}(t) = k\lambda e^{\lambda t} \tag{iii}$$

into (ii) and, then, evaluating the initial value results in

$$k\lambda e^{\lambda t} = -ake^{\lambda t} \quad \Rightarrow \quad k\lambda = -ak \quad \Rightarrow \quad \lambda = -a$$

$$y(0) = ke^{-a\cdot 0} = y_0 \quad \Rightarrow \quad k = y_0$$

Thus, the <u>homogeneous solution</u> of (ii) is

$$\Rightarrow \quad y(t) = ke^{-at} = y_0 e^{-at} \tag{iv}$$

- Linear non-homogeneous 1st-order differential equation, cont.

Now, taking the time derivative (by chain rule) of the general solution (iii)

$$\dot{y}(t) = \dot{k} \cdot e^{-at} - k \cdot a \cdot e^{-at}$$

and substituting $y(t)$ and $\dot{y}(t)$ into (i) results in

$$\dot{k} e^{-at} - k a e^{-at} + k a e^{-at} = bu$$

$$\Rightarrow \quad \dot{k} e^{-at} = bu$$

$$\Rightarrow \quad \dot{k} = e^{at} bu \tag{v}$$

Integrating the left- and right-hand-side results in

$$\int^{t} \dot{k} \, d\tau = k(t) - k(0) = \int^{t} e^{a \cdot \tau} bu(\tau) d\tau$$

Then, solving it with respect to $k(t)$ results in

$$k(t) = \int^{t} e^{a \cdot \tau} bu(\tau) d\tau + k(0) \tag{vi}$$

- Linear non-homogeneous 1st-order differential equation, cont.

Now, substituting (vi) into homogeneous solution (iv) results in

$$y(t) = k(0)\cdot e^{-at} + e^{-at}\int_0^t e^{a\tau} bu(\tau)d\tau =$$

$$= \underbrace{k(0)}_{y_0}\cdot e^{-at} + \int_0^t e^{-a(t-\tau)} bu(\tau)d\tau$$

: General solution of the non-homogeneous ODE

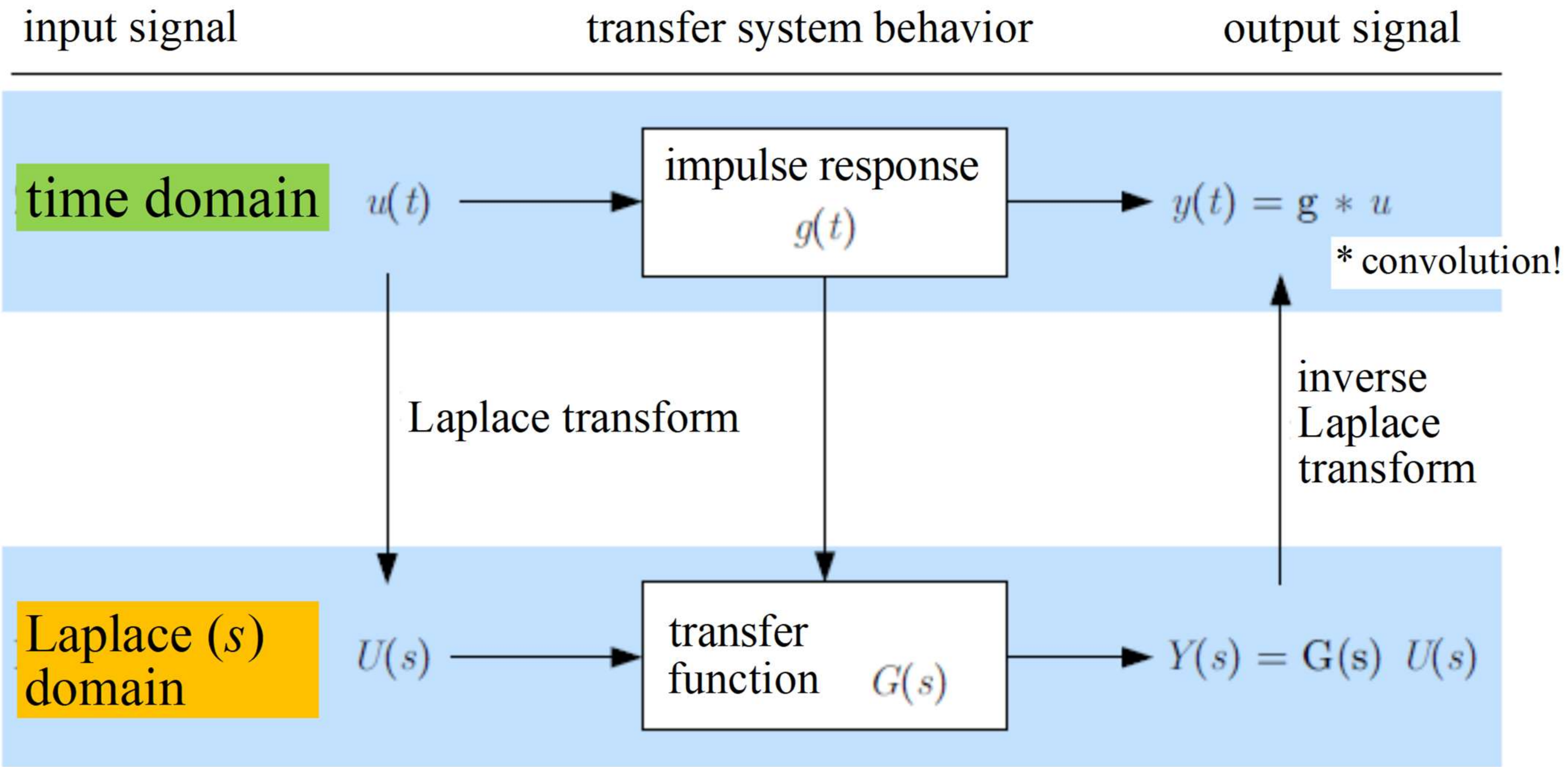
General solution of ODE is a superposition of own and excited dynamics

$$y(t) = \underbrace{y_0 e^{-at}}_{\text{homogeneous solution (own dynamics)}} + \underbrace{\int_0^t e^{-a(t-\tau)} bu(\tau)d\tau}_{\text{particular solution (externally excited dynamics)}} \qquad \text{(vii)}$$

- Laplace transform and its inverse

$$\mathcal{L}\left[y(t)\right] = Y(s) = \int_{0^-}^{\infty} e^{-st} y(t) dt \qquad \mathcal{L}^{-1}\left[y(s)\right] = y(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} e^{st} Y(s) ds$$

- Why do we use it?



- Laplace transformation: from time-domain into the complex s-domain Laplace domain (and also back-transformation)

Linear differential equations $\Rightarrow$ Laplace transformation

complex Laplace variable

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

$$s = \sigma + j\omega$$

	$f(t)$	$F(s)$
1	unit impulse $\delta(t)$	1
2	unit step $1(t)$	$\frac{1}{s}$
3	unit ramp t	$\frac{1}{s^2}$
4	e^{-at}	$\frac{1}{s+a}$
5	te^{-at}	$\frac{1}{(s+a)^2}$
6	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
7	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
8	$t^n \quad (n = 1,2,3,\dots)$	$\frac{n!}{s^{n+1}}$
9	$t^n e^{-at} \quad (n = 1,2,3,\dots)$	$\frac{n!}{(s+a)^{n+1}}$

10	$\frac{1}{b-a}(e^{-at}-e^{-bt})$	$\frac{1}{(s+a)(s+b)}$
11	$\frac{1}{b-a}(be^{-bt}-ae^{-at})$	$\frac{s}{(s+a)(s+b)}$
12	$\frac{1}{ab}\left[1+\frac{1}{a-b}(be^{-at}-ae^{-bt})\right]$	$\frac{1}{s(s+a)(s+b)}$
13	$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega}$
14	$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega}$
15	$\frac{1}{a^2}(at-1+e^{-at})$	$\frac{1}{s^2(s+a)}$
16	$\frac{\omega_n}{\sqrt{1-\varsigma^2}}e^{-\varsigma\omega_n t}\sin\omega_n\sqrt{1-\varsigma^2}\,t$	$\frac{\omega_n^2}{s^2+2\varsigma\omega_n s+\omega_n^2}$
17	$\frac{-1}{\sqrt{1-\varsigma^2}}e^{-\varsigma\omega_n t}\sin(\omega_n\sqrt{1-\varsigma^2}\,t-\phi)$ $\phi=\tan^{-1}\frac{\sqrt{1-\varsigma^2}}{\varsigma}$	$\frac{s}{s^2+2\varsigma\omega_n s+\omega_n^2}$
18	$1-\frac{1}{\sqrt{1-\varsigma^2}}e^{-\varsigma\omega_n t}\sin(\omega_n\sqrt{1-\varsigma^2}\,t+\phi)$ $\phi=\tan^{-1}\frac{\sqrt{1-\varsigma^2}}{\varsigma}$	$\frac{\omega_n^2}{s(s^2+2\varsigma\omega_n s+\omega_n^2)}$

- # Laplace transformation properties:

$f(t)$: function in time domain $\hat{f}(s)$: transformed function in Laplace domain

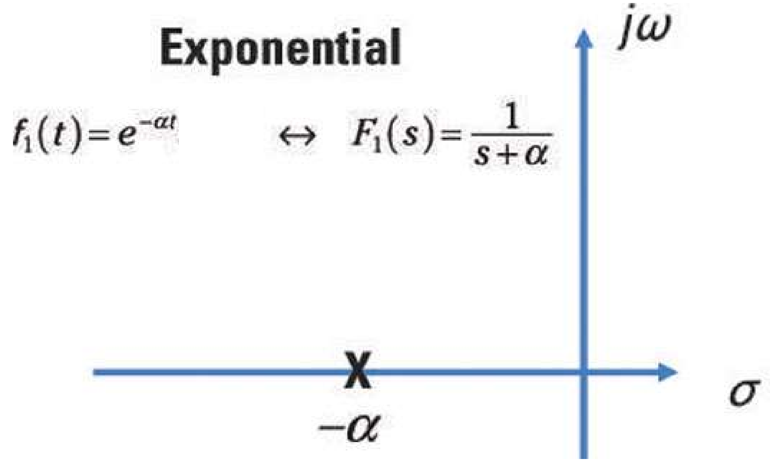
source: [19]

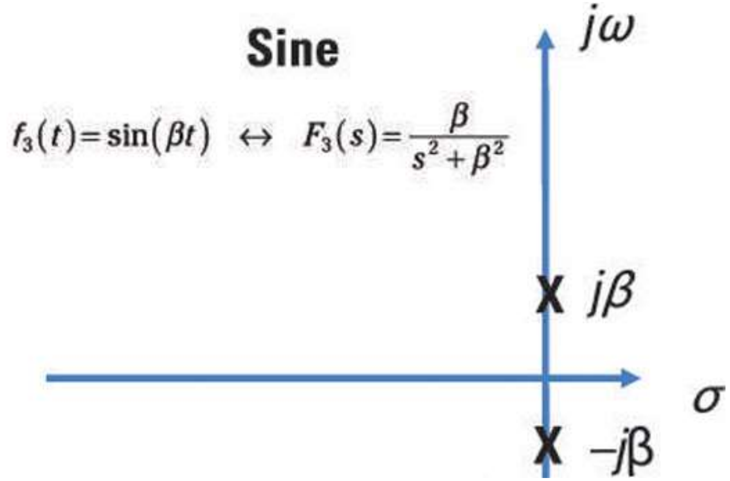
Time different-	$df(t)/dt$	$s\hat{f}(s) - f(0)$
iation	$d^k f(t)/dt^k$	$s^k \hat{f}(s) - [s^{k-1} f(0) + \cdots + f^{(k-1)}(0)]$
Frequency shift	$e^{-at} f(t)$	$\hat{f}(s+a)$
Time shift	$f(t-a)p(t-a), a > 0$	$e^{-as}\hat{f}(s)$
Scaling	$f(t/\alpha), \alpha > 0$	$\alpha \hat{f}(\alpha s)$
Convolution	$\int_0^t f(\tau) g(t-\tau) d\tau = f(t) * g(t)$	$\hat{f}(s)\hat{g}(s)$
Initial value	$\lim_{t \to 0^+} f(t) = f(0^+)$	$\lim_{s \to \infty} s\hat{f}(s)^{\dagger}$
Final value	$\lim_{t \to \infty} f(t)$	$\lim_{s \to 0} s\hat{f}(s)^{\ddagger}$

† If the limit exists. $p(t)$ represents the *unit step function*.

‡ If $s\hat{f}(s)$ has no singularities on the imaginary axis or in the right half s plane.

- # Simple examples:





- Main advantages of using Laplace domain

❑ Differentiation becomes multiplication with *s*

$$s \equiv \frac{d}{dt}$$

❑ Integration becomes division by *s*

$$\frac{1}{s} \equiv \int_{0^-}^{t} dt$$

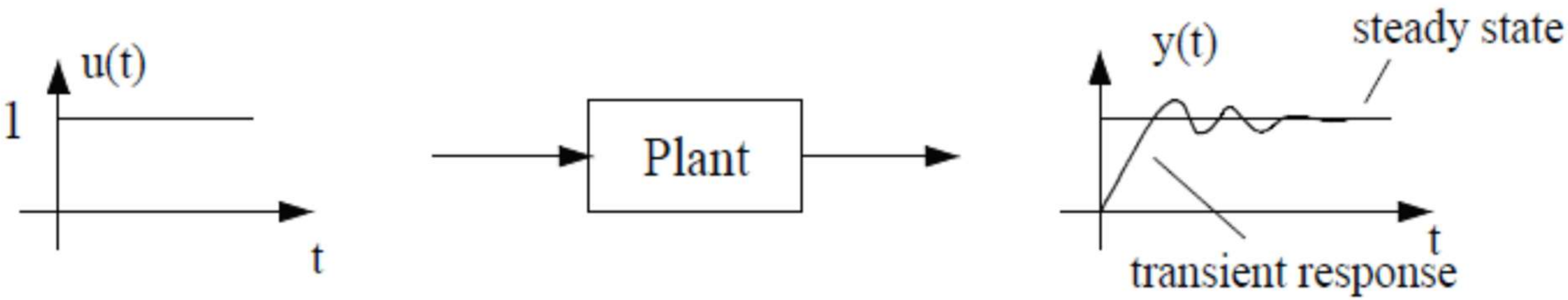
❑ Solving the ODEs with *t*-argument becomes solving the algebraic equations with *s*-argument

$$y^{(n)} + \ldots + a_1 \dot{y} + a_0 y = b_m u^{(m)} + \ldots + b_1 \dot{u} + b_0 u$$
$$\Downarrow$$
$$s^n Y + \ldots + a_1 s Y + a_0 Y = b_m s^m U + \ldots + b_1 s U + b_0 U$$

❑ Input-output system behavior described by an algebraic <u>transfer function</u>

$$G(s) = \frac{Y(s)}{U(s)} = \frac{m_m s^m + \ldots + b_2 s^2 + b_1 s + b_0}{s^n + \ldots + a_2 s^2 + a_1 s + a_0}$$

- # Transient analysis

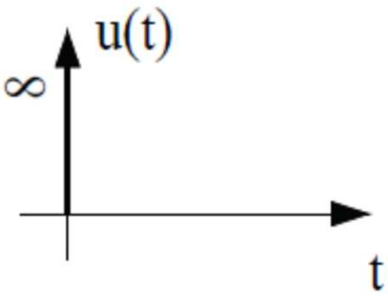


$$y(s) = G(s) \cdot u(s) \quad ; \qquad\qquad y(t) = L^{-1}\{y(s)\}$$

Typical (i.e. characteristic) input functions:

Impulse function

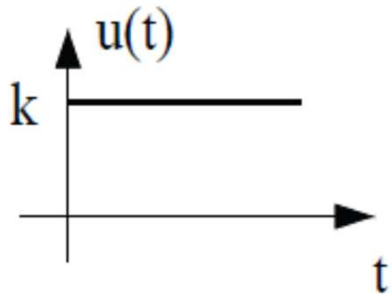
$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$



$$L\{\delta(t)\} = 1$$

Step function

$$u(t) = k; \quad t \geq 0$$

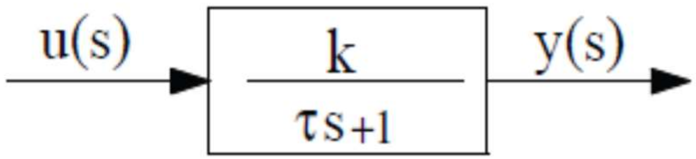


$$L\{u(t)\} = k / s$$

- **Example 9**: first-order system

$$\tau\frac{dy(t)}{dt}+y(t)=k\cdot u(t) \qquad \textit{Laplace transformation} \;\Rightarrow$$

$$\frac{y(s)}{u(s)}=\frac{k}{\tau\cdot s+1}$$

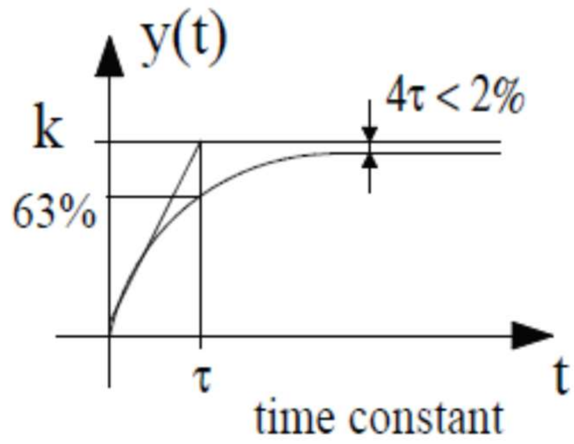


When applying unit input step

$$y(s)=\frac{k/\tau}{s(s+1/\tau)}=\frac{k}{s}+\frac{-k}{s+1/\tau}$$

Transforming back into time domain ⇒

$$y(t)=k\cdot(1-e^{-\frac{1}{\tau}\cdot t})$$



Evaluating the initial derivative

$$\left.\frac{dy(t)}{dt}\right|_{t=0}=\left.\frac{k}{\tau}e^{-\frac{t}{\tau}}\right|_{t=0}=\frac{k}{\tau}$$

Evaluating single points

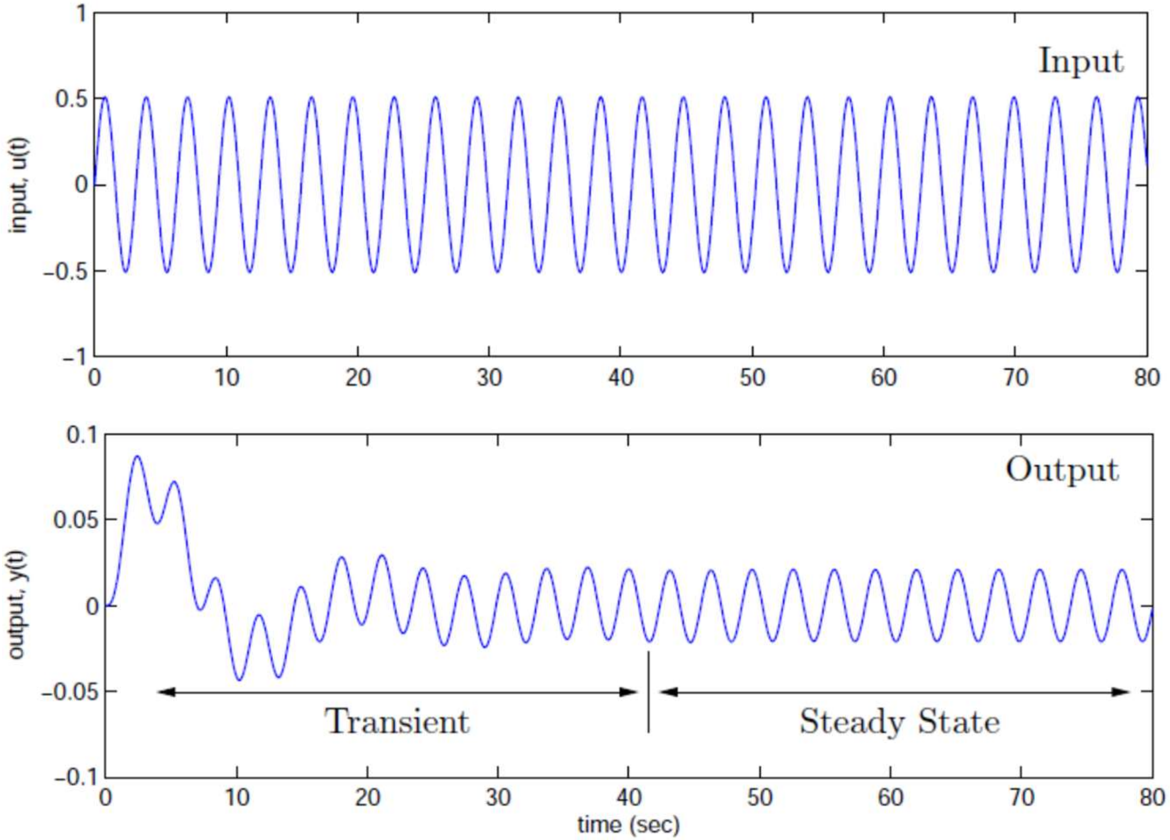
$t{:}\times\tau$	0	1	2	3	4
$e^{-t/\tau}$:	1	0.37	0.135	0.049	0.018

- **Example 9**: first-order system, cont.

Final value theorem: $\underbrace{\lim}_{t\to\infty} y(t) = s \cdot \left. \dfrac{k}{s \cdot (\tau \cdot s + 1)} \right|_{s=0} = k$

- What is steady-state if the input is not a *step*, but a *harmonic*?

Typical example for a harmonic input (excitation) and the resulted system response



- Fourier transform for system analysis in frequency domain

  - Transfer function of the system can be described in frequency domain

$$G(j\omega) = G(s)|_{s=j\omega} = R(\omega) + jX(\omega),$$

where

$$R(\omega) = \mathrm{Re}[G(j\omega)] \quad \text{and} \quad X(\omega) = \mathrm{Im}[G(j\omega)]$$

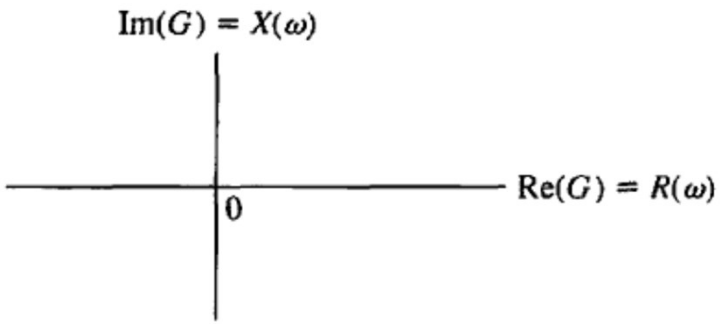
  - Alternatively (and commonly) with **magnitude** (M) and **phase** (φ) response

$$G(j\omega) = |G(j\omega)|e^{j\phi(\omega)} = |G(j\omega)|\underline{/\phi(\omega)},$$

where

$$\phi(\omega) = \tan^{-1}\frac{X(\omega)}{R(\omega)} \quad \text{and} \quad |G(j\omega)|^2 = [R(\omega)]^2 + [X(\omega)]^2$$

  - Also, in the polar coordinates (Nyquist plot)

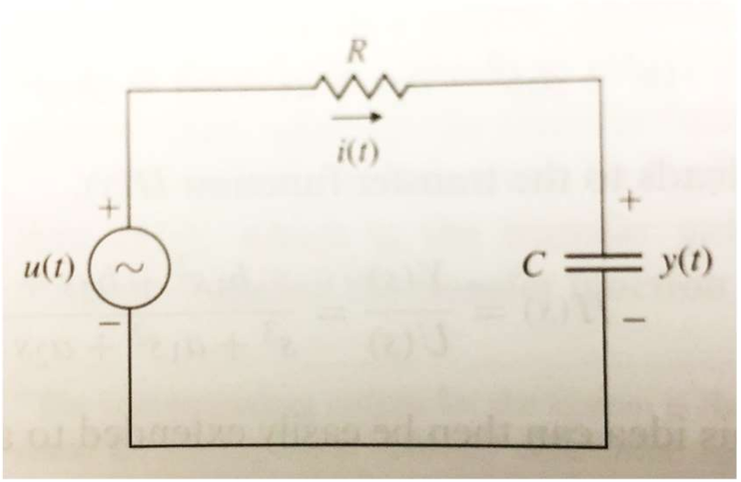


- What is behind Frequency Response Function (FRF)?

Example 10 (simple RC circuit)

$$Ri(t) + y(t) = u(t), \quad \text{with} \quad i(t) = C\frac{dy(t)}{dt}$$

$$\Rightarrow \quad RC\,\dot{y}(t) + y(t) = u(t)$$



source: [4]

Applying unit-impulse (i.e. Dirac) signal

$$RC\,\dot{y}(t) + y(t) = \delta(t)$$

Taking Laplace transform, and evaluating for zero initial condition, yields

$$RC\left(sY(s) - y(0^-)\right) + Y(s) = U(s) = 1$$

$$\Rightarrow \quad \frac{Y(s)}{U(s)} = \frac{1}{RCs+1} = \frac{k}{s+k}, \qquad \text{with} \quad k = \frac{1}{RC}$$

Evaluating output (as inverse Laplace transform) results in the <u>impulse response</u>

$$y(t) = g(t) = \frac{1}{RC}\,e^{-\frac{t}{RC}} \cdot 1(t) \qquad \Rightarrow \qquad G(s) = L\{g(t)\} = \frac{1}{RCs+1}$$

- What is behind Frequency Response Function (FRF)? cont.

When exciting LTI-system by a harmonic (sin or cos) input

$$A\cos(\omega t) = \frac{A}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) \qquad \text{(due to Euler's formula)}$$

then, an LTI system is replying with

$$y(t) = \frac{A}{2}\left[G(j\omega)e^{j\omega t} + G(-j\omega)e^{-j\omega t}\right] =$$

$$= \frac{A}{2}M\left[e^{j(\omega t+\varphi)} + e^{-j(\omega t+\varphi)}\right] = AM\cos(\omega t + \varphi)$$

with the magnitude and phase characteristics $M = \left|G(j\omega)\right|, \ \varphi = \angle G(j\omega)$

Back to Example 10 (from the previous slide):

$$G(s) = \frac{k}{s+k} \quad \Rightarrow \quad G(j\omega) = k\frac{1}{j\omega + k} \quad \Rightarrow \quad y(t) = AM\cos(\omega t + \varphi)$$

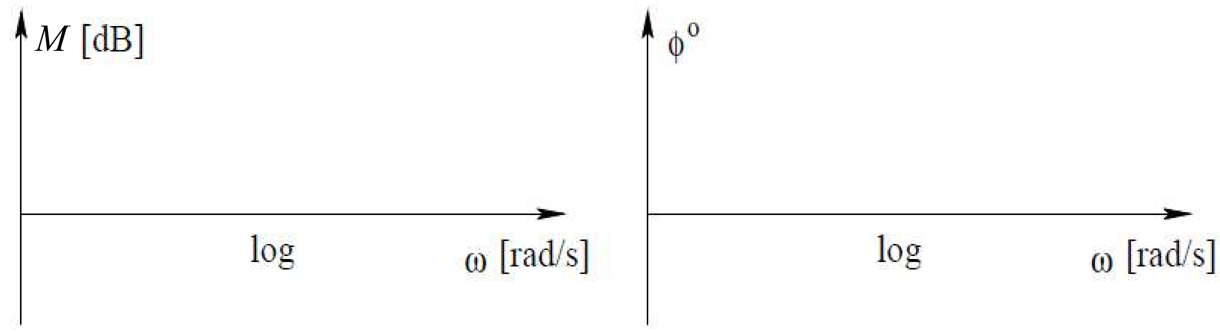
with $\quad M = k\left|\frac{1}{j\omega + k}\right| = \frac{k}{\sqrt{\omega^2 + k^2}}, \quad \varphi = -\tan^{-1}\left(\frac{\omega}{k}\right)$

- Bode plot (Bode diagram)

Steady-state characteristics of $G(j\omega)$

Bodeplot consists of two plots;

- The magnitude (gain) versus frequency
- Phase shift versus frequency



$[\mathrm{dB}] \equiv 20\log_{10}(|G|)$, where $|G|$ is the ratio (i.e. unitless)

Transfer functions can consist of the following elements:

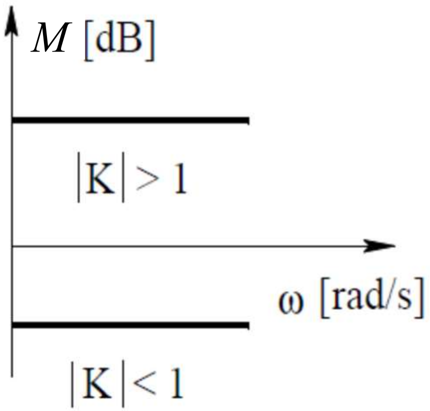
1. Constant gains
2. Poles and zeros at the origin
3. Real poles and zeros not at the origin
4. Complex poles and zeros
5. Ideal time delays

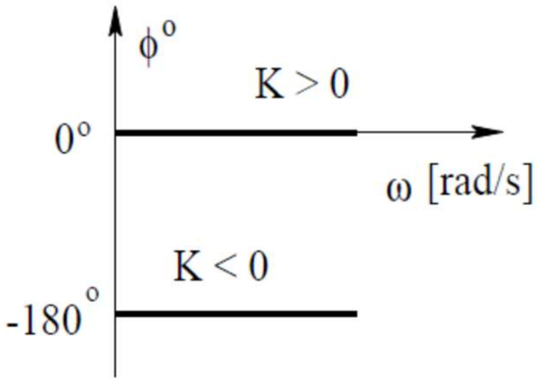
• Bode plot (Bode diagram), cont.

$$\boxed{G(s) = K \quad ; \quad dB = 20 \cdot \log|K| \quad ;}$$

$$K > 0 \quad ; \quad \phi = 0^o$$

$$K < 0 \quad ; \quad \phi = -180^o$$



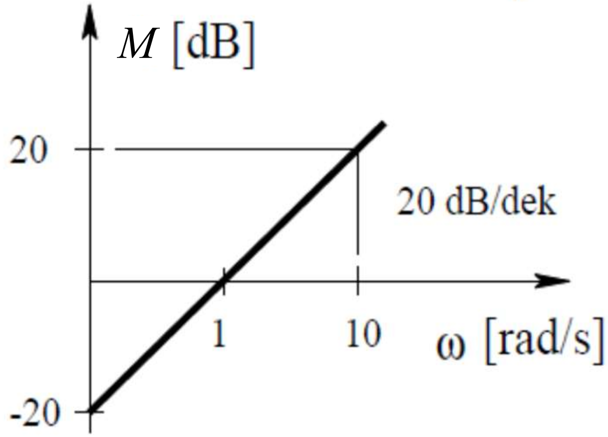


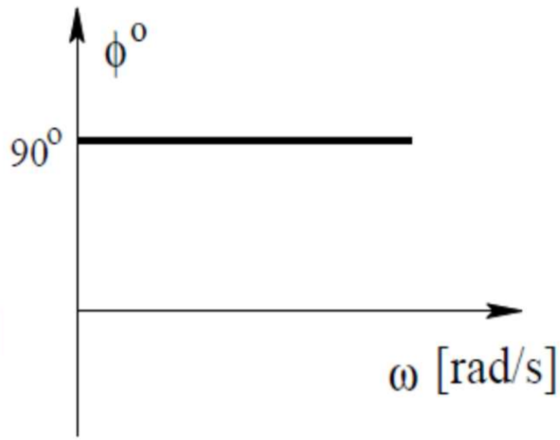
System gain

$$\boxed{\text{Zero:} \quad G(s) = s \quad ; \quad dB = 20 \cdot \log|j\omega| = 20 \cdot \log(\omega);}$$

$$\phi = 90^o$$

Differentiator





One can also consider only <u>the asymptotes</u> in Bode plot (for sake of simplicity)

- **Example 11**: constructing Bode plot

$$G(s)=\frac{k\cdot(1+\tau_3 s)}{(1+\tau_1 s)\cdot(1+\tau_2 s)}=\frac{k\cdot(1+s/\omega_3)}{(1+s/\omega_1)\cdot(1+s/\omega_2)}$$

ω_i´s are called *break frequencies*.

Magnitude: $$|G(j\omega)|=\frac{|k|\cdot|1+j\omega/\omega_3|}{|1+j\omega/\omega_1|\cdot|1+j\omega/\omega_2|}$$

Logarithms: $$log\left(\frac{a\cdot b}{c\cdot d}\right)=log(a\cdot b)-log(c\cdot d)$$
$$=log(a)+log(b)-log(c)-log(d)$$

Define decibel [dB]: $dB = 20\cdot log(a)$; $a \sim konstant$

Magnitude: $20\cdot\log|G(j\omega)|$

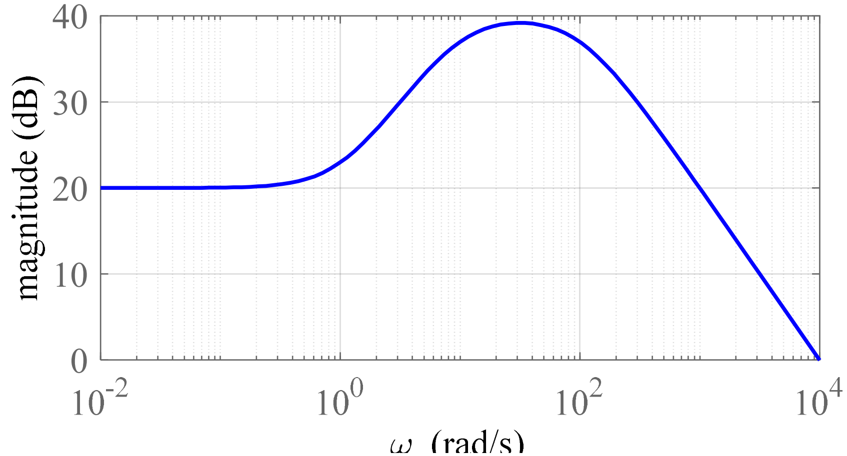
This means

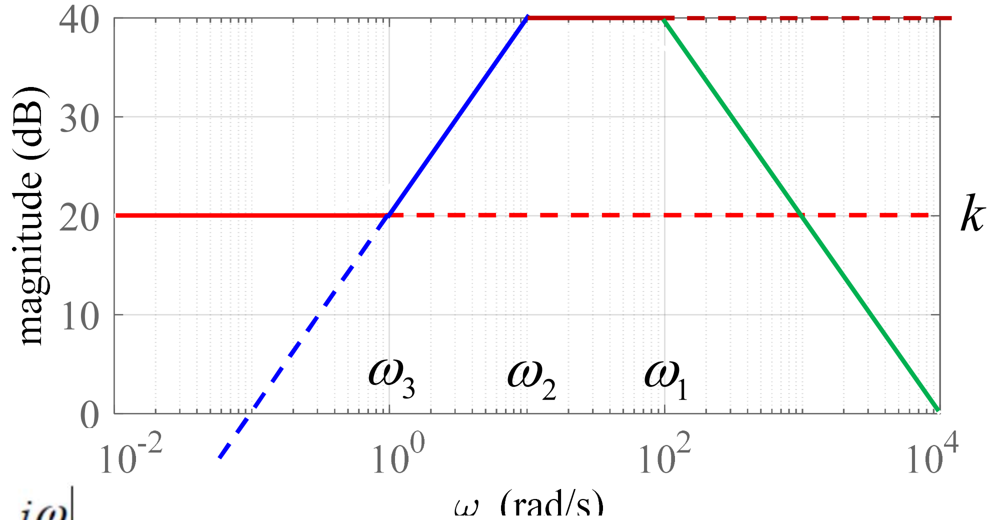
$$20\cdot log\,|G(j\omega)|=20\cdot log\frac{|k|\cdot|1+j\omega/\omega_3|}{|1+j\omega/\omega_1|\cdot|1+j\omega/\omega_2|}=$$

$$20\cdot log|k|+20\cdot log\left|1+\frac{j\omega}{\omega_3}\right|-20\cdot log\left|1+\frac{j\omega}{\omega_1}\right|-20\cdot log\left|1+\frac{j\omega}{\omega_2}\right|$$

with some example values:

$k=10,\ \omega_1=100,\ \ \omega_2=10,\ \omega_3=1$





• Practical (hand on) session 3

First-order ODE system

Consider a dynamic system described by

$$2\dot{y}(t) + y(t) = u(t), \quad y(0) = 0.$$

Assume the system is excited by

$u(t) = h(0),$ where $h(t)$ is the unit-step function.

1. First, solve the above homogenous ODE in time domain.
2. Then, solve the above non-homogenous ODE in Laplace domain.
3. For the back-transformed solution of 2., make a MATLAB implementation.

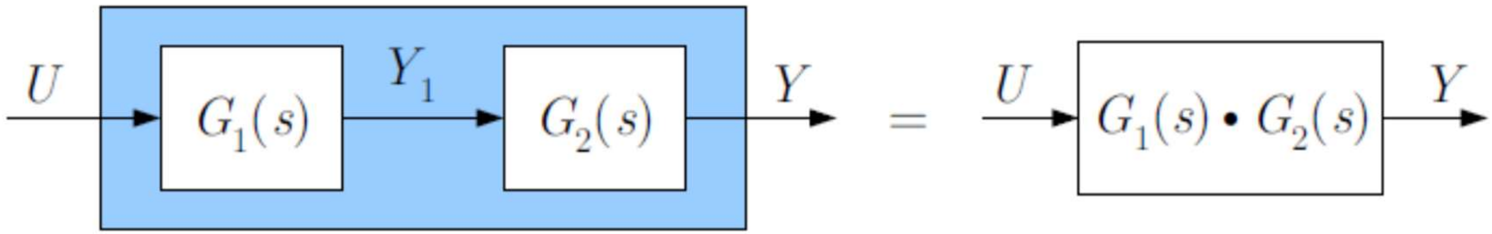
Feedback control systems for mechatronics and robotics

Michael Ruderman
michael.ruderman@uia.no

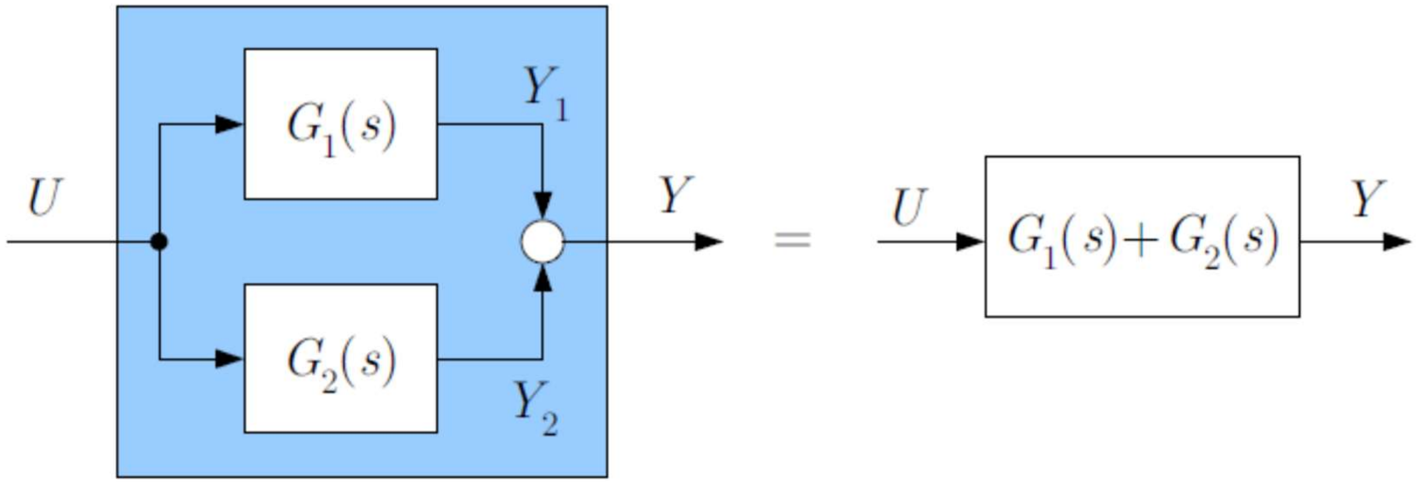
Transfer function analysis and state-space modeling

- Main rules of block diagrams algebra

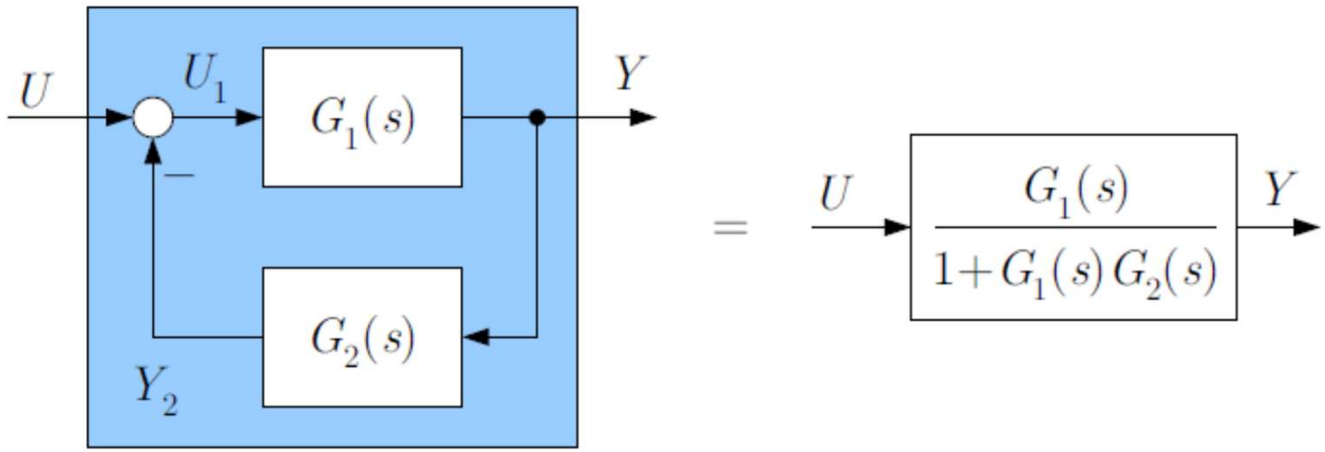
(i) in series



(ii) in parallel



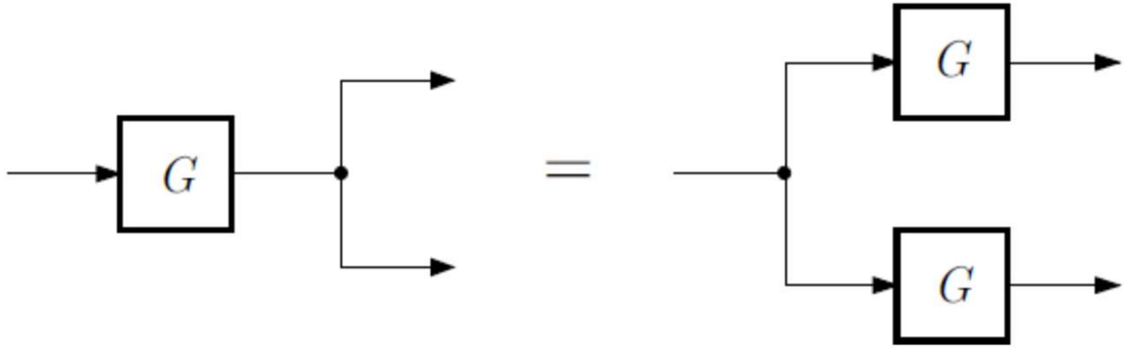
(iii) closing loop



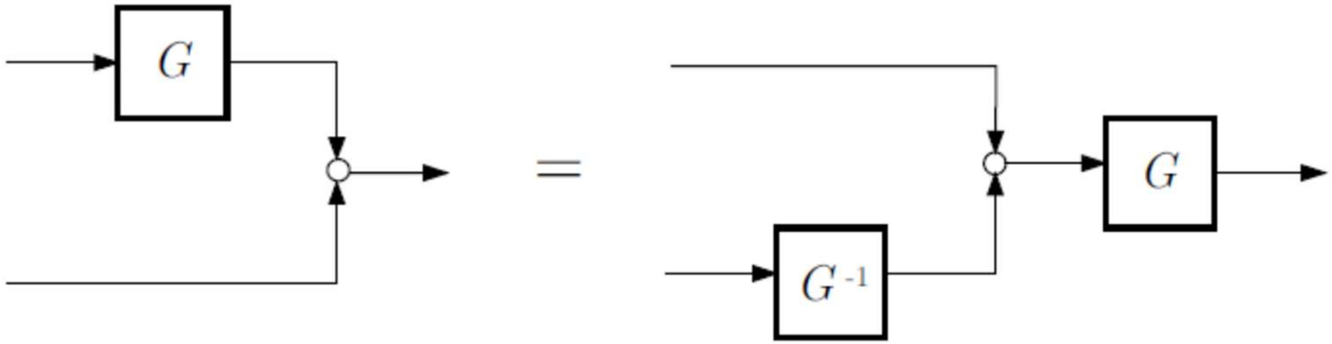
- Main rules of block diagrams algebra, cont.

(i) commutative

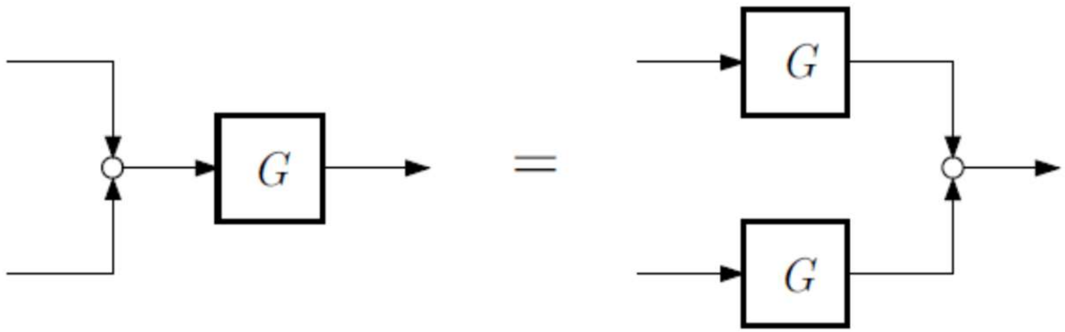
(ii) associative



(iii) invertible

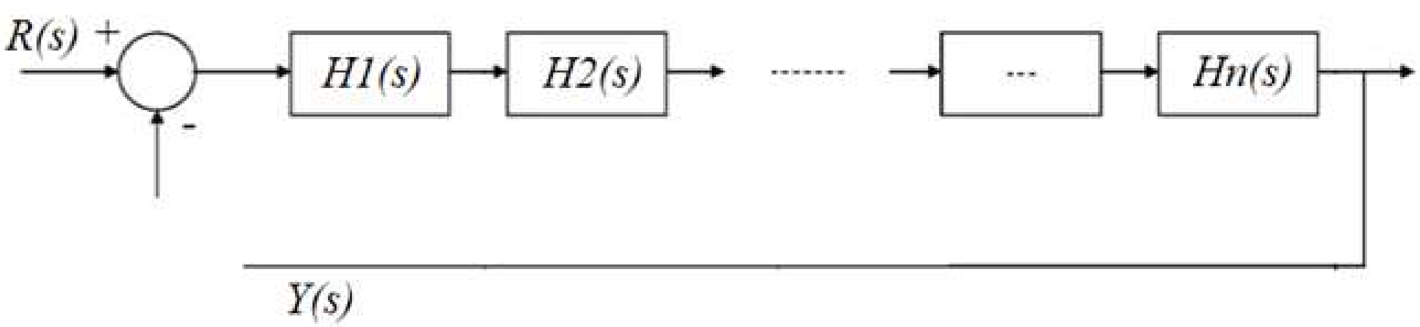


(iv) distributive



- Stability of feedback loops

<u>Gain</u> and <u>phase</u> <u>margins</u> of the <u>open-loop</u>



Open-loop transfer function
(all transfer blocks connected in series before closing the loop)

$$L(s) = \frac{Y(s)}{R(s)} = H1(s) \cdot H2(s) \cdot \ldots \cdot Hn(s)$$

Open-loop transfer function has <u>all</u> information about the <u>closed-loop stability</u>

$$\boxed{G(s) = \frac{L(s)}{1 + L(s)}}$$

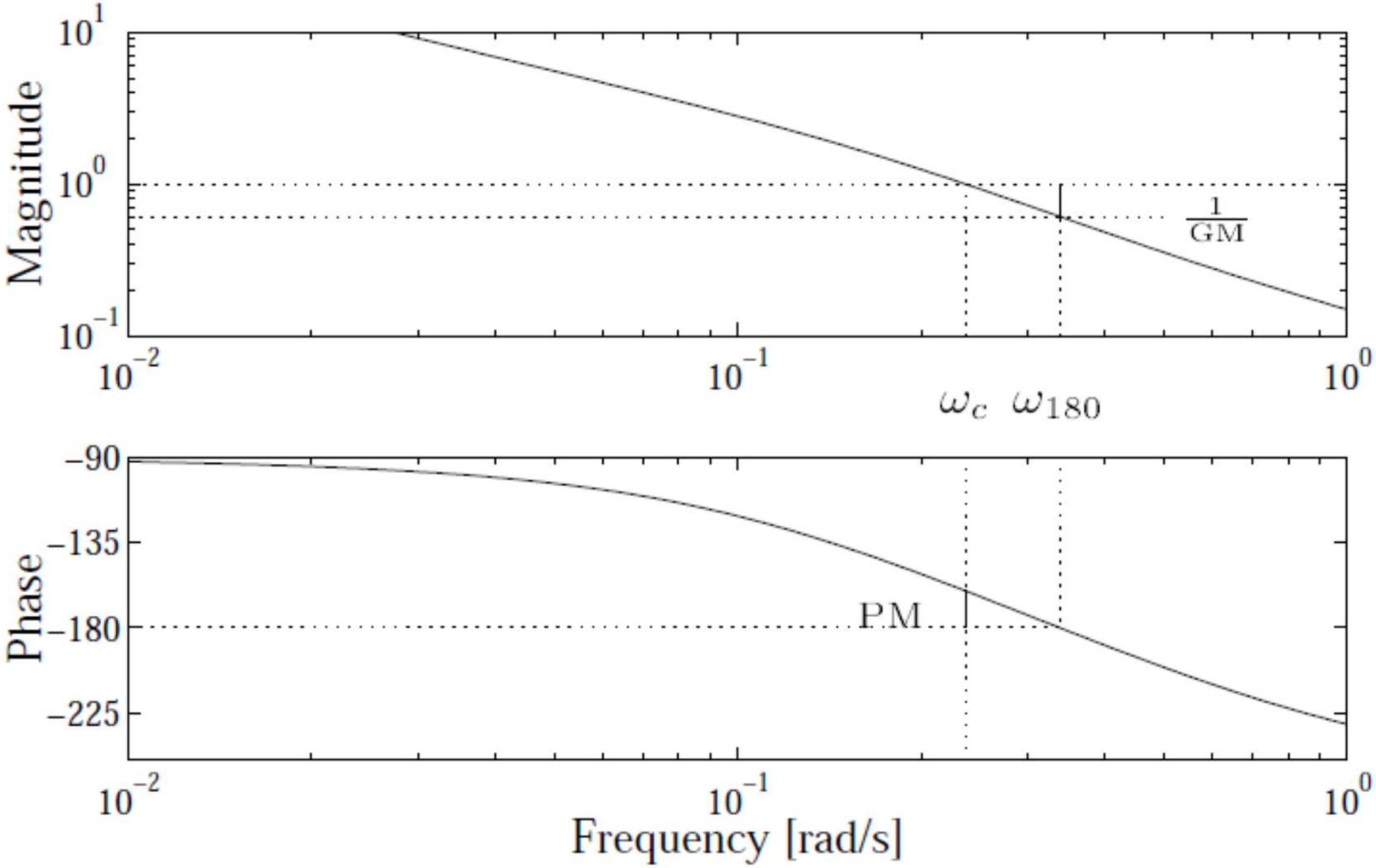
But (!) to consider only <u>characteristic polynomial</u> $D(s)$ of *L(s)* is not sufficient: $L(s) = \frac{N(s)}{D(s)} \Rightarrow G(s) = \frac{N(s)}{N(s) + D(s)}$

- Stability of feedback loops, cont.

Gain margin $\mathrm{GM} = 1/|L(j\omega_{180})|$ where the *phase crossover frequency* ω_{180}

Phase margin $\mathrm{PM} = \angle L(j\omega_c) + 180^\circ$ where $|L(j\omega_c)| = 1$

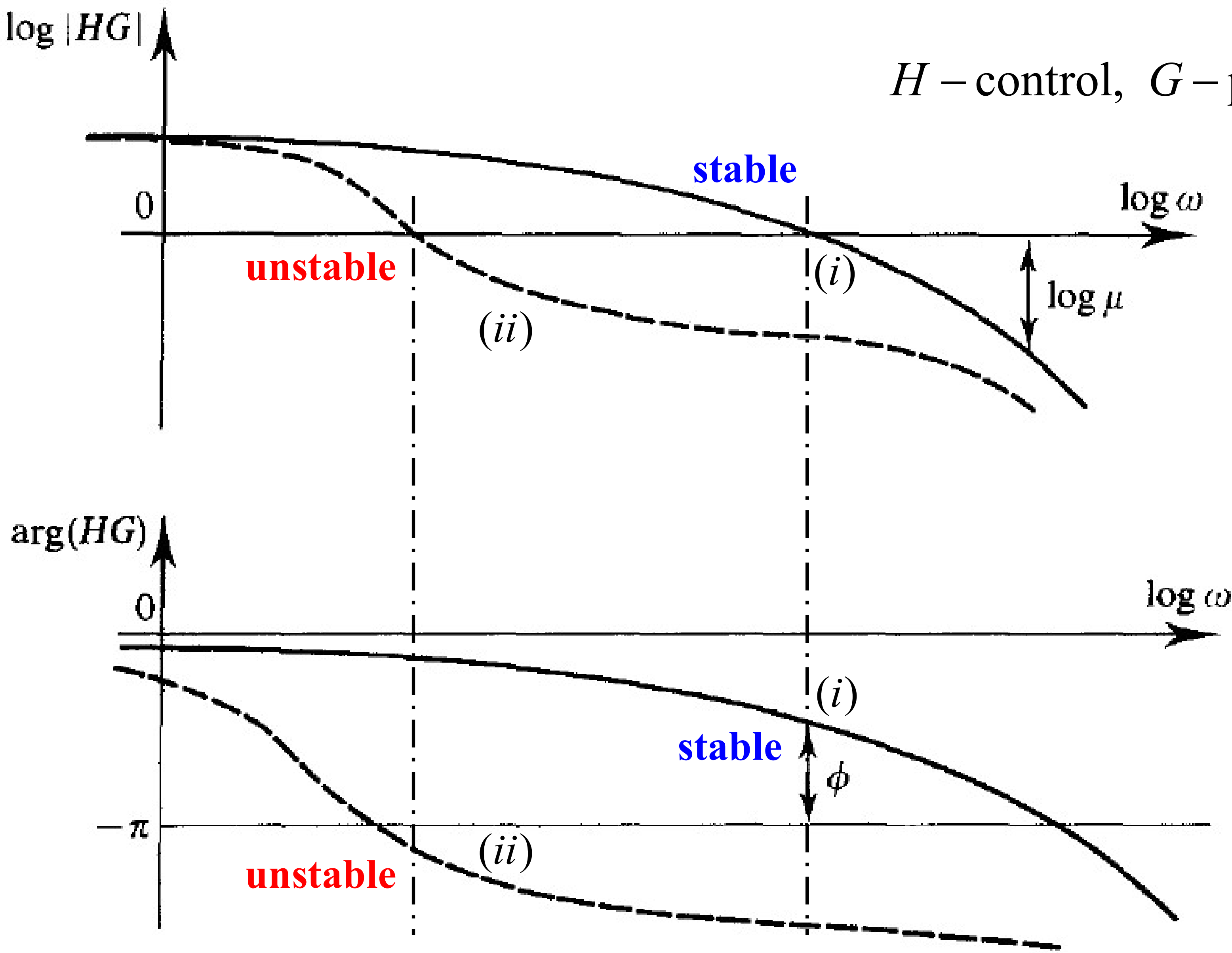
the *gain crossover frequency* ω_c is where $|L(j\omega)|$ first crosses 1 from above



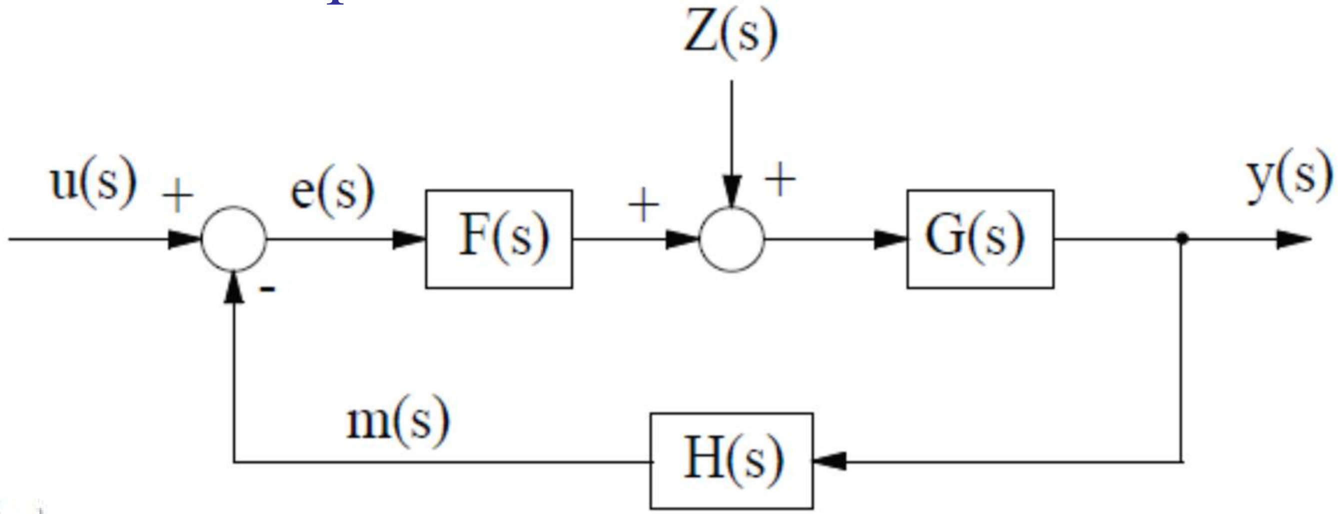
- Stability of feedback loops, cont.

Examples of open-loops for unstable and stable closed-loops

H – control, G – plant



- Control error of the closed-loop



Error: $e(s) = u(s) - m(s)$

$$e(s) = u(s) - H(s)G(s)\cdot\left[Z(s) + F(s)e(s)\right] \Rightarrow$$

$$e(s)\cdot\left[1 + H(s)G(s)F(s)\right] = u(s) - H(s)G(s)Z(s) \Rightarrow$$

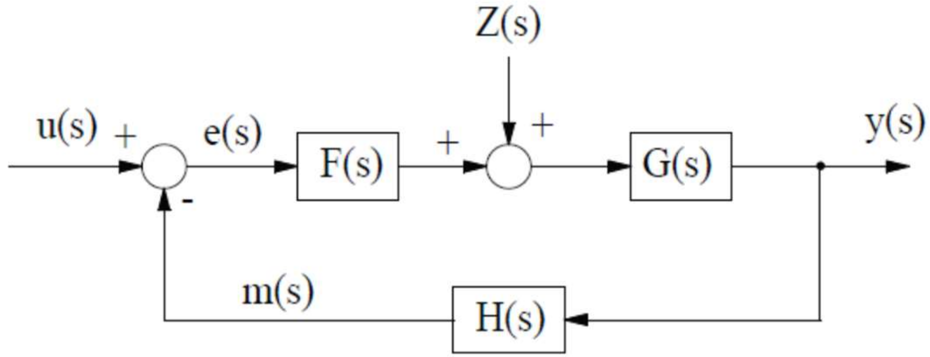
$$e(s) = \frac{1}{1 + H(s)F(s)G(s)}\cdot u(s) - \frac{H(s)G(s)}{1 + H(s)F(s)G(s)}\cdot Z(s)$$

Open-loop transfer function: $G_o(s) = F(s)G(s)H(s)$

$\Rightarrow \quad e(s) = \dfrac{1}{1+G_o(s)}u(s) - \dfrac{H(s)G(s)}{1+G_o(s)}Z(s) \quad \Rightarrow \quad G_o(s)$ must be shaped to minimize $e(s)$

• Control error of the closed-loop, cont.

Final value theorem



$$e_{u,ss} = \lim_{t\to\infty} e_u(t) = \lim_{s\to 0} s \cdot \frac{1}{1+L(s)} \cdot u(s)$$

$$e_{Z,ss} = \lim_{t\to\infty} e_Z(t) = \lim_{s\to 0} s \cdot \frac{-H(s)G(s)}{1+L(s)} \cdot Z(s)$$

$$L(s) = \frac{K\cdot(b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + 1)}{s^N \cdot (a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + 1)}$$

Typical input functions:

Step function: $u(t) = h \Rightarrow u(s) = \frac{h}{s}$

Ramp function: $u(t) = v \cdot t \Rightarrow u(s) = \frac{v}{s^2}$

Parabolic function: $u(t) = \frac{a}{2} \cdot t^2 \Rightarrow u(s) = \frac{a}{s^3}$

General: $u(t) = c\frac{t^k}{k!} \Rightarrow u(s) = \frac{c}{s^{k+1}}$

• Control error of the closed-loop, cont.

~ The number of free integrators, N, is called the *system type*.

$$\lim_{s \to 0} L(s) = \lim_{s \to 0} \frac{K}{s^N}$$

Steady-state errors for different reference input

$$e_{u,ss}(t) = \lim_{s \to 0} s \cdot \frac{1}{1 + \frac{K}{s^N}} \cdot \frac{c}{s^{k+1}} = \lim_{s \to 0} \frac{c}{s^k(1 + \frac{K}{s^N})}$$

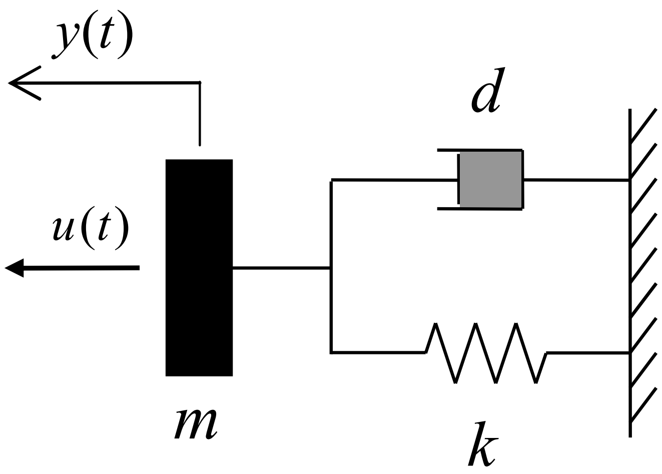
Steady-state error $e_{u,ss}$:

$$u(s) = \frac{c}{s^{k+1}} \Rightarrow$$

	$N=0$	$N=1$	$N=2$
Step $k=0,\ c=h$	$\frac{h}{1+K}$	0	0
Ramp $k=1,\ c=v$	∞	$\frac{v}{K}$	0
Parabolic $k=2,\ c=a$	∞	∞	$\frac{a}{K}$

- Motivation for state-space modeling
  - State-space model describe the entire LTI dynamic system (also MIMO) of the n-th order via a **vector-valued state** (n state-variables) and a set of n **first-order differential equations**
  - Each ODE (ordinary differential equation) of the n-th order can be transformed into n independent first-order ODEs
  - Compact and standard form of the **matrix equations** suitable for: modeling, analysis, state estimation, and control design

- **Example 12**: mass-spring-damper system



$$m\ddot{y}(t)+d\dot{y}(t)+ky(t)=u(t)$$

$$\ddot{y}(t)=\frac{1}{m}\big(u(t)-d\dot{y}(t)-ky(t)\big)$$

– Introduce (internal) dynamic state variables

$$x_1(t) = y(t), \quad x_2(t) = \dot{y}(t) \quad \Rightarrow$$

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -\frac{k}{m} x_1(t) - \frac{d}{m} x_2(t) + \frac{1}{m} u(t)$$

– Matrix notation

$$\Rightarrow \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{d}{m} \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ \dfrac{1}{m} \end{pmatrix}}_{\mathbf{B}} u, \quad y = \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{\mathbf{C}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

– Uniform system description (parameterization) through

$\mathbf{A}$ – system matrix, $\mathbf{B}$ – input coupling matrix, $\mathbf{C}$ – output coupling matrix.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \qquad \text{state vector: } \mathbf{x} = [x_1, x_2, \ldots, x_n]^T$$
$$y = \mathbf{C}^T \mathbf{x}$$

- Laplace- and time-domain solutions

First, consider 1st order scalar system (ODE) with an initial value

$$\dot{x}(t) = a\, x(t) + b\, u(t) \qquad x(0\_) = x_{0\_}$$

Applying the Laplace transformation

$$s\, X(s) - x(0\_) = a\, X(s) + b\, U(s)$$

$$X(s) = \frac{1}{s-a}\, x(0\_) + \frac{1}{s-a}\, b\, U(s)$$

With back-transformation into time-domain

$$\frac{1}{s-a} \quad ○\!\!-\!\!\bullet \quad e^{at}: \qquad x(t) = \underbrace{e^{at}\, x_{0\_}}_{\text{homogeneous solution (own dynamics)}} + \underbrace{\int_{0\_}^{t} e^{a(t-\tau)}\, b\, u(\tau)\, \mathrm{d}\tau}_{\text{particular solution (externally excited dynamics)}}$$

- Laplace- and time-domain solutions, cont.

Now, consider the vector of 1st order ODEs with an initial value

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \qquad \mathbf{x}(0\_) = \mathbf{x}_{0\_}$$

that is a state-space model

The solution in vector/matrix form

$$\mathbf{x}(t) = e^{\mathbf{A}t}\,\mathbf{x}_{0\_} + \int_{0\_}^{t} e^{\mathbf{A}(t-\tau)}\,\mathbf{B}\,\mathbf{u}(\tau)\,\mathrm{d}\tau$$

What is derivative of the matrix exponential function $e^{\mathbf{A}t}$?

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{\mathbf{A}t} = \mathbf{A}\,e^{\mathbf{A}t}$$

Matrix function as power series

$$e^{\mathbf{A}t} = \mathbf{E} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} + \ldots + \mathbf{A}^k\frac{t^k}{k!} + \ldots = \sum_{k=0}^{\infty}\mathbf{A}^k\frac{t^k}{k!}$$ it converges for $|t| < \infty$

- Laplace- and time-domain solutions, cont.

– Often used notation

$$\mathbf{x}(t) = \phi(t)\,\mathbf{x}_{0\_} + \int_{0\_}^{t} \phi(t-\tau)\,\mathbf{B}\,\mathbf{u}(\tau)\,\mathrm{d}\tau$$

with **state-transition matrix** (also denoted as **fundamental matrix**)

$$\phi(t) = e^{\mathbf{A}t}$$

i. has dominant role in describing the dynamic systems

ii. appears in both – homogenous (excitation-free) and particular (externally excited) – solutions in time domain

iii. determines the system state at each time *t*, for the given initial state and input values

– For the initial time $t_{0\_} \neq 0$ and time transformation $t' = t - t_{0\_}$ one obtains

$$\mathbf{x}(t-t_{0\_}) = e^{\mathbf{A}(t-t_{0\_})}\,\mathbf{x}(t_{0\_}) + \int_{t_{0\_}}^{t} e^{\mathbf{A}(t-\tau)}\,\mathbf{B}\,\mathbf{u}(\tau)\,\mathrm{d}\tau$$

$\Rightarrow$ basis for any numerical (discrete-time) simulation

- Laplace- and time-domain solutions, cont.

In analogy to the scalar case

$$\dot{x}(t) = a\, x(t) + b\, u(t) \quad \Rightarrow \quad s\, X(s) - x(0\_) = a\, X(s) + b\, U(s)$$

one can write the general matrix form in Laplace domain

$$\begin{aligned} s\,\mathbf{X}(s) - \mathbf{x}(0\_) &= \mathbf{A}\,\mathbf{X}(s) + \mathbf{B}\,\mathbf{U}(s) \\ \mathbf{Y}(s) &= \mathbf{C}\,\mathbf{X}(s) + \mathbf{D}\,\mathbf{U}(s) \end{aligned}$$

$\Rightarrow\ (s\,\mathbf{E} - \mathbf{A})\,\mathbf{X}(s) = \mathbf{x}(0\_) + \mathbf{B}\,\mathbf{U}(s)$ where $\mathbf{E}$ is the identity matrix

Provided the matrix $(s\,\mathbf{E} - \mathbf{A})$ is non-singular $\Rightarrow$

$$\mathbf{X}(s) = \underbrace{(s\,\mathbf{E} - \mathbf{A})^{-1}\mathbf{x}(0\_)}_{\text{homogeneous solution (own dynamics)}} + \underbrace{(s\,\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}\,\mathbf{U}(s)}_{\text{particular solution (externally excited dynamics)}}$$

State transition matrix in Laplace domain

$$\phi(t) = \mathrm{L}^{-1}\{\Phi(s)\}(t), \quad \Phi(s) = (s\,\mathbf{E} - \mathbf{A})^{-1}$$

• Laplace- and time-domain solutions, cont.

Output behavior in Laplace domain

$$\mathbf{Y}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{x}(0\_) + \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}\,\mathbf{U}(s) + \mathbf{D}\,\mathbf{U}(s)$$

For zero initial values $\mathbf{x}(0_{-}) = \mathbf{0}$ one obtains the **<u>transfer function matrix</u>**

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

$$\mathbf{G}(s) = \mathbf{C}\,\mathbf{\Phi}(s)\,\mathbf{B} + \mathbf{D}$$

SISO (particular case) ⇒ with input and output vectors **b**, **c**, and feedthrough *d*

$$G(s) = \frac{\mathbf{c}^{t}\operatorname{adj}(s\mathbf{E} - \mathbf{A})\,\mathbf{b} + d\,\left|s\mathbf{E} - \mathbf{A}\right|}{\left|s\mathbf{E} - \mathbf{A}\right|}$$

with adjugate matrix
$\operatorname{adj}(\mathbf{A}) = \det(\mathbf{A})\,\mathbf{A}^{-1}$,

Denominator is <u>characteristic polynomial</u> ⇒ matrix **A** determines the poles!

• Practical (hand on) session 4

Consider a 4th order flexible joint system (from page 53)

The system dynamics is given by

$$m\ddot{\theta} + d\dot{\theta} + K(\theta - q) = u$$

$$M\ddot{q} + D\dot{q} - K(\theta - q) = 0$$

For task 2., assume the following numerical parameter values

$$m = 1, \quad M = 0.5, \quad d = 10, \quad D = 1, \quad K = 1000.$$

1. Determine and write down the state-space model with $q(t)$ as output.
2. Make MATLAB implementation of the state-space model. Show the step response of the following dynamic states $\dot{\theta}(t)$ and $\dot{q}(t)$.

Feedback control systems for mechatronics and robotics

i. Introduction to feedback control systems
ii. Control-oriented modeling
iii. Dynamic system behavior in time and frequency domain
iv. Transfer function analysis and state-space modeling
v. Similarity forms, controllability, and observability of the systems
vi. Standard output feedback controllers
vii. State feedback controllers and prefilter extensions
viii. Stability analysis and robust control design
ix. Motion-, force- and impedance-control in mechatronics and robotics
x. Use of observers and estimators in feedback control systems

Michael Ruderman
michael.ruderman@uia.no

Similarity forms, controllability, and observability of the systems

- Similarity transformations

  – There are multiple ways to define the dynamic states of one and the same system $\Rightarrow$ redefine the state variables $\equiv$ change of the state-coordinates

$$\mathbf{x} = \mathbf{T}\,\mathbf{x}'$$

  – For ensuring no loss of information, require also

$$\mathbf{x}' = \mathbf{T}^{-1}\,\mathbf{x}$$

  – Applying the transformation to the state-space model

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{x}(t) + \mathbf{D}\,\mathbf{u}(t) \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathbf{T}\,\dot{\mathbf{x}}'(t) &= \mathbf{A}\,\mathbf{T}\,\mathbf{x}'(t) + \mathbf{B}\,\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{T}\,\mathbf{x}'(t) + \mathbf{D}\,\mathbf{u}(t) \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} \dot{\mathbf{x}}'(t) &= \mathbf{T}^{-1}\mathbf{A}\,\mathbf{T}\,\mathbf{x}'(t) + \mathbf{T}^{-1}\mathbf{B}\,\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{T}\,\mathbf{x}'(t) + \mathbf{D}\,\mathbf{u}(t)\;. \end{aligned}$$

  – Resulted transformation rules

$$\boxed{\mathbf{A}' = \mathbf{T}^{-1}\mathbf{A}\,\mathbf{T}\;,\quad \mathbf{B}' = \mathbf{T}^{-1}\mathbf{B}\;,\quad \mathbf{C}' = \mathbf{C}\,\mathbf{T} \quad \text{and} \quad \mathbf{D}' = \mathbf{D}}$$

  – Original and transformed (′) matrices describe <u>one and the same system</u>!

- Similarity transformations, cont.

– One can prove the invariance of characteristic equation under similarity transformations, while characteristic equation determines the system dynamics

$$\left| s\mathbf{E} - \mathbf{A}' \right| = \left| s\mathbf{E} - \mathbf{T}^{-1}\mathbf{A}\,\mathbf{T} \right| = \ldots = \left| (s\mathbf{E} - \mathbf{A}) \right|$$

also $\left| \mathbf{A}' \right| = \left| \mathbf{T}^{-1}\mathbf{A}\,\mathbf{T} \right| = \left| \mathbf{T}^{-1} \right| \left| \mathbf{A} \right| \left| \mathbf{T} \right| \Rightarrow \left| \mathbf{A}' \right| = \left| \mathbf{A} \right|$

$\Rightarrow$ determinant of the system matrix and the characteristic equation are <u>invariant to similarity transformations</u> $\Rightarrow$ free choice of state coordinates!

- **Example 13**: consider state-space model

$$\mathbf{A} = \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \mathbf{D} = \mathbf{0}$$

– Assume state transformations: $x_1 = x'_1 + x'_2, \;\; x_2 = 10x'_2 \;\Rightarrow \mathbf{T} = \begin{bmatrix} 1 & 1 \\ 0 & 10 \end{bmatrix}$

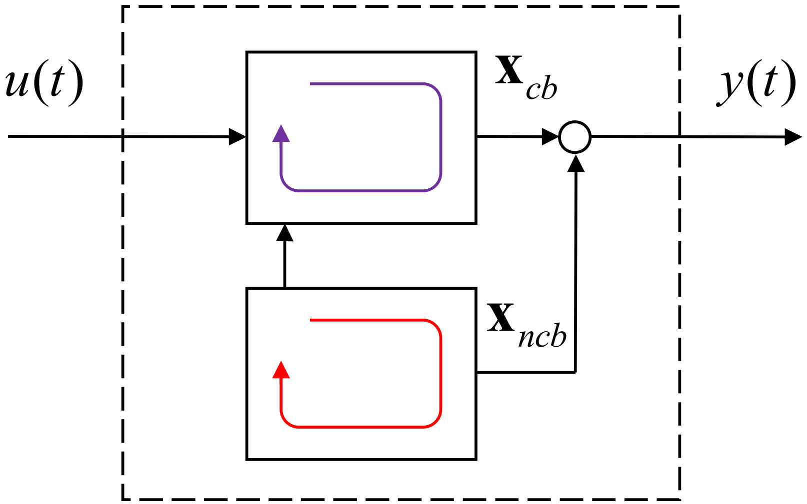
– Apply transformation rules: $\mathbf{A}' = \mathbf{T}^{-1}\mathbf{A}\,\mathbf{T}$, $\mathbf{B}' = \mathbf{T}^{-1}\mathbf{B}$, $\mathbf{C}' = \mathbf{C}\,\mathbf{T}$

– Compare in MATLAB both state-space models (original & transformed)

- Controllability and observability

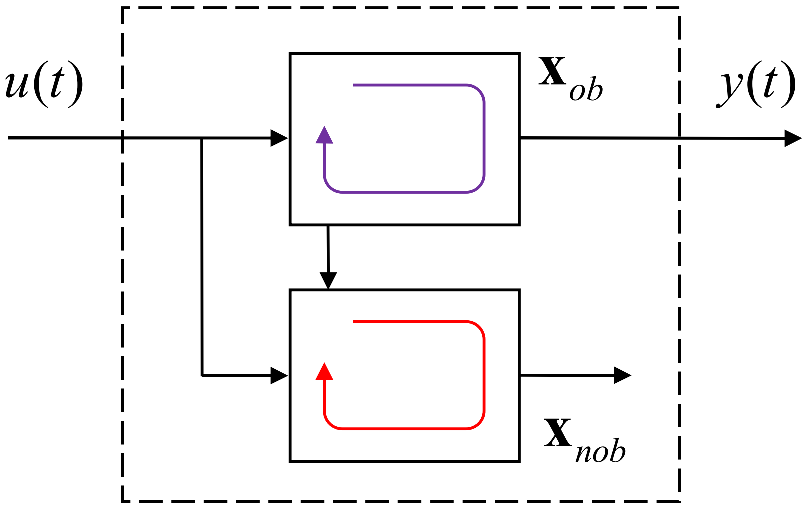
– Dynamic system (**A**,**B**) is fully **controllable** if the overall system state $\mathbf{x}(t)$ can be driven by an appropriate control $\mathbf{u}(t)$ to zero equilibrium, and that for any initial state $\mathbf{x}_0$.

– Reduced controllability: some non-controllable states $\mathbf{x}_{ncb}$



– Dynamic system (**A**,**C**) is fully **observable** if for any initial state $\mathbf{x}_0$ the overall state vector $\mathbf{x}(t)$ can be uniquely reconstructed (i.e. estimated) for the given input $\mathbf{u}(t)$ and output $\mathbf{y}(t)$.

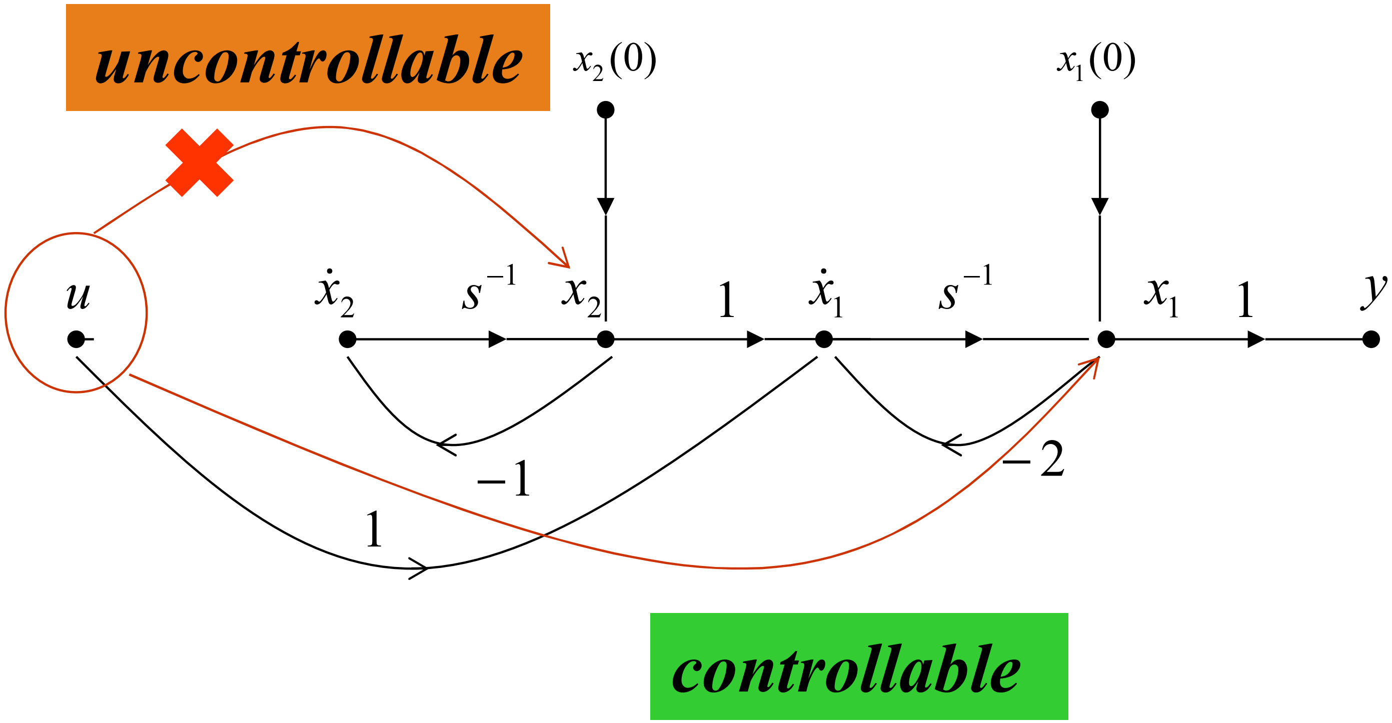
– Reduced observability: some non-observable states $\mathbf{x}_{nob}$



- Illustrative example of controllability

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-2 & 1\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\ 0\end{bmatrix}u \quad y=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Use of the signal-flow graphs for interpretation

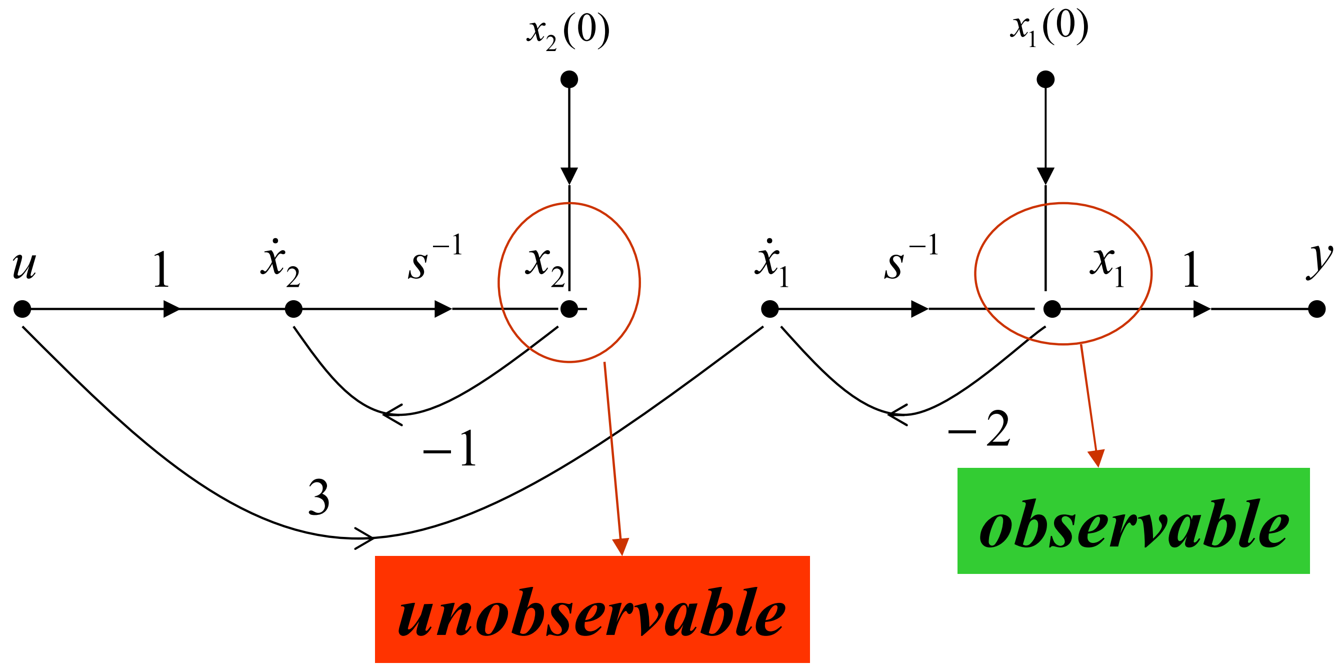


$$\dot{x}_1 + 2x_1 - x_2 = u$$
$$\dot{x}_2 + x_2 = 0$$

- Illustrative example of observability

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Use of the signal-flow graphs for interpretation



$$\dot{x}_1 + 2x_1 = 3u$$

$$\dot{x}_2 + x_2 = u$$

$$y = x_1$$

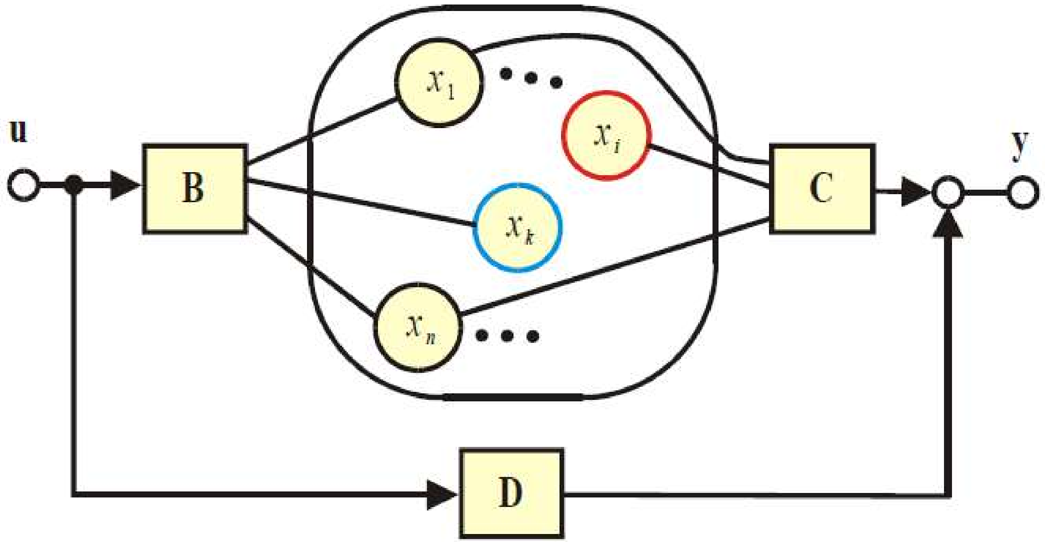
- Kalman criteria for controllability and observability

  – A dynamic system (**A,B**) is exactly then **fully** controllable **iff** the **controllability matrix** *Cr* fulfills

$$\operatorname{rank} Cr = \operatorname{rank}\left[\mathbf{B}, \mathbf{AB}, \ldots, \mathbf{A}^{n\text{-}1}\mathbf{B}\right] = n$$

  – A dynamic system (**A,C**) is exactly then **fully** observable **iff** the **observability** matrix *Ob* fulfills

$$\operatorname{rank} Ob = \operatorname{rank}\begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix} = n$$



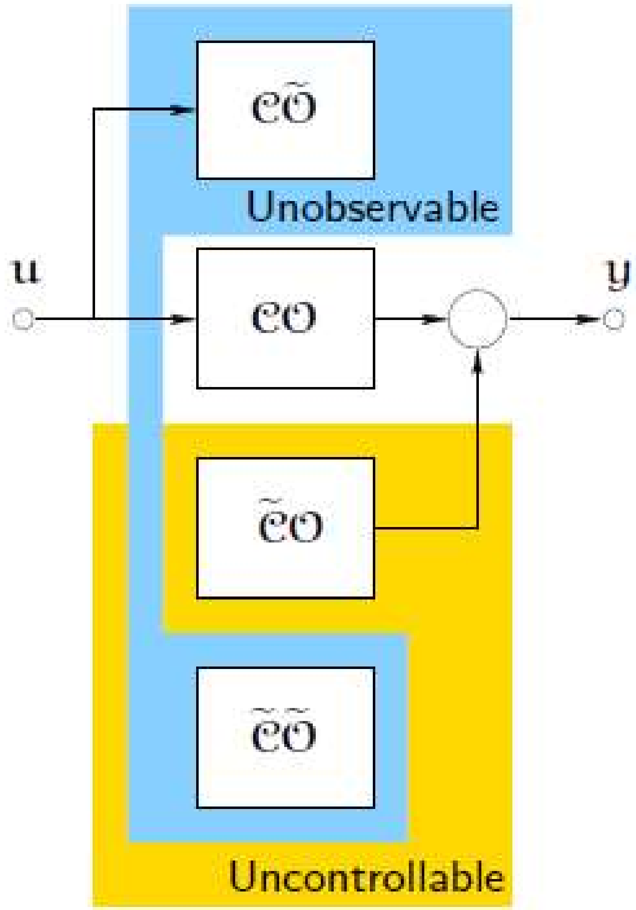
Remark:
controllability and observability of linear systems don't depend on feed-through matrix **D**

- # Kalman canonical decomposition

Suppose it is known that a SISO (single-input single-output) system has rank$(Cr)= n_c < n$, and rank$(Ob)= n_o < n$. In other words, the SISO system, in question, is neither controllable nor observable. We are interested in a transformation that rearranges the system modes (eigenvalues) into:

- modes that are both controllable and observable;
- modes that are neither controllable nor observable;
- modes that are controllable but not observable; and
- modes that are observable but not controllable.

Such a transformation is called *Kalman decomposition*.



Decomposed (desired) form without couplings

$$\begin{bmatrix} \dot{x}_{CO} \\ \dot{x}_{C\overline{O}} \\ \dot{x}_{\overline{C}O} \\ \dot{x}_{\overline{C}\overline{O}} \end{bmatrix} = \begin{bmatrix} A_{CO} & 0 & 0 & 0 \\ 0 & A_{C\overline{O}} & 0 & 0 \\ 0 & 0 & A_{\overline{C}O} & 0 \\ 0 & 0 & 0 & A_{\overline{C}\overline{O}} \end{bmatrix} \begin{bmatrix} x_{CO} \\ x_{C\overline{O}} \\ x_{\overline{C}O} \\ x_{\overline{C}\overline{O}} \end{bmatrix} + \begin{bmatrix} B_{CO} \\ B_{C\overline{O}} \\ 0 \\ 0 \end{bmatrix} u$$

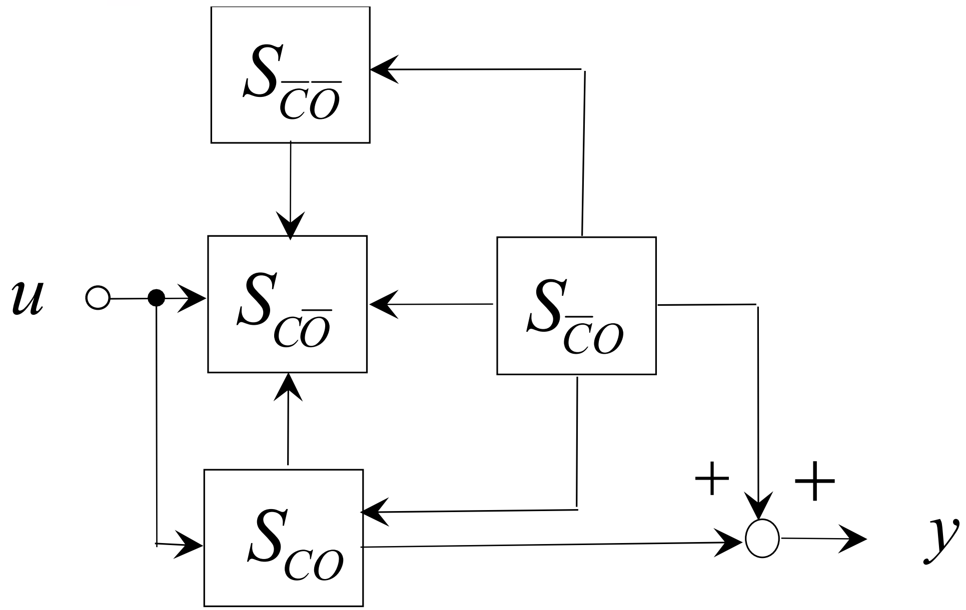
$$y = \begin{bmatrix} C_{CO} & 0 & C_{\overline{C}O} & 0 \end{bmatrix} x$$

- Example of a non-decomposed system

  – Initial situation with couplings

$$
\begin{bmatrix} \dot{\bar{x}}_{co} \\ \dot{\bar{x}}_{c\tilde{o}} \\ \dot{\bar{x}}_{\tilde{c}o} \\ \dot{\bar{x}}_{\tilde{c}\tilde{o}} \end{bmatrix}
=
\begin{bmatrix} \bar{A}_{co} & 0 & \bar{A}_{13} & 0 \\ \bar{A}_{21} & \bar{A}_{c\tilde{o}} & \bar{A}_{23} & \bar{A}_{24} \\ 0 & 0 & \bar{A}_{\tilde{c}o} & 0 \\ 0 & 0 & \bar{A}_{43} & \bar{A}_{\tilde{c}\tilde{o}} \end{bmatrix}
\underbrace{\begin{bmatrix} \bar{x}_{co} \\ \bar{x}_{c\tilde{o}} \\ \bar{x}_{\tilde{c}o} \\ \bar{x}_{\tilde{c}\tilde{o}} \end{bmatrix}}_{\bar{x}}
+
\begin{bmatrix} \bar{B}_{co} \\ \bar{B}_{c\tilde{o}} \\ 0 \\ 0 \end{bmatrix} u
$$

$$
y = \begin{bmatrix} \bar{C}_{co} & 0 & \bar{C}_{\tilde{c}o} & 0 \end{bmatrix} \bar{x} + Du
$$



  – Desired is decomposition into the canonical (diagonal) Kalman form (see previous page)

- Kalman canonical decomposition, cont.

– Diagonalization: $$\dot{x} = Ax + Bu$$ $$y = Cx + Du$$

where all eigen-values of A are distinct (!), i.e. $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \cdots \lambda_n$

– There exists a coordinate transformation $z = Tx$ such that

$$A_m = T^{-1}AT \quad \text{where} \quad A_m = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$$

– System in the new z-coordinates is

$$\dot{z} = A_m z + B_m u \qquad B_m = T^{-1}B = \begin{bmatrix} b_{m_1} \\ \vdots \\ b_{m_n} \end{bmatrix} \quad C_m = CT = \begin{bmatrix} c_{m_1} & \cdots & c_{m_n} \end{bmatrix}$$
$$y = C_m z$$

– Homogeneous solution of the above state equation is

$$z(t) = \boldsymbol{v}_1 e^{\lambda_1 t} z_1(0) + \cdots + \boldsymbol{v}_n e^{\lambda_n t} z_n(0)$$

- Kalman canonical decomposition, cont.

If $b_{m_i} \neq 0$ and $c_{m_i} \neq 0$, mode λ_i is controllable and observable

– Since all eigenvalues of A are distinct $\Rightarrow$ all eigenvectors are independent

$T = [\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n]$: transformation for diagonalization

– **Diagonalized system**:

as state-space form

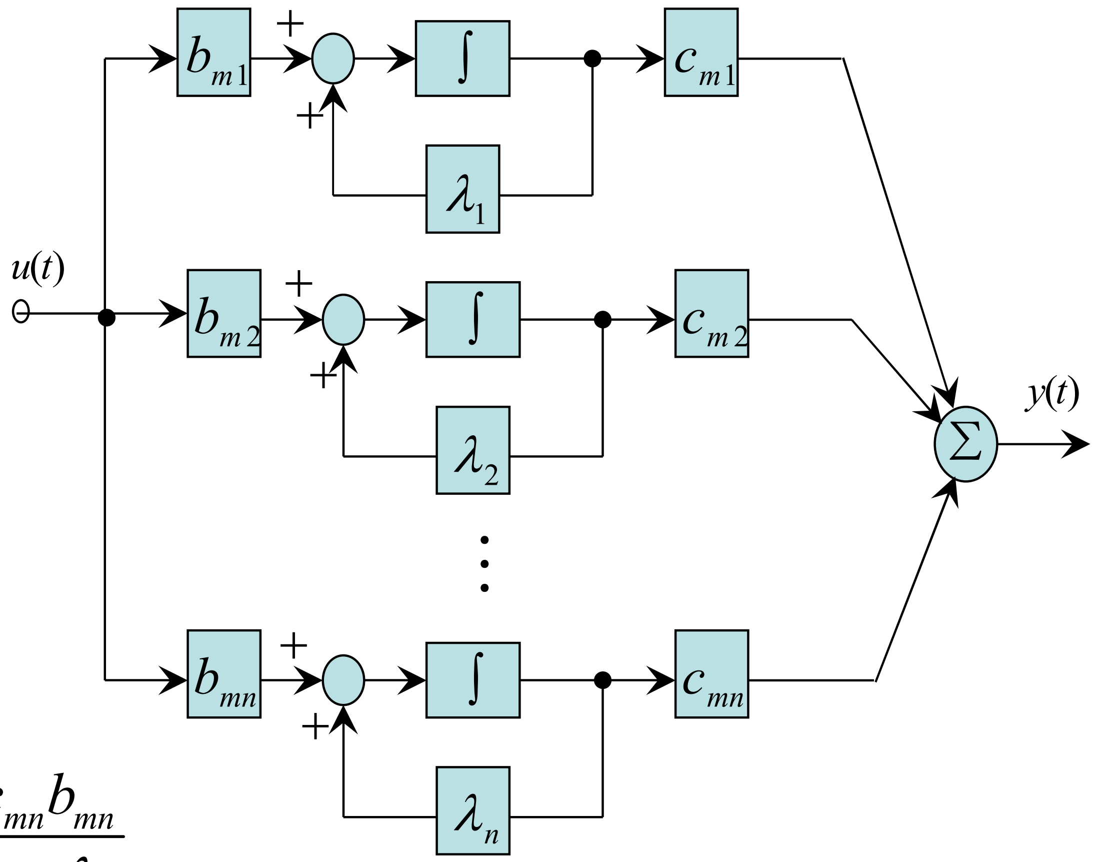
$$\dot{z}_1 = \lambda_1 z_1 + b_{m1} u$$

$$\dot{z}_2 = \lambda_2 z_2 + b_{m2} u$$

$$\vdots$$

$$\dot{z}_n = \lambda_n z_n + b_{mn} u$$

$$y = c_{m1} z_1 + c_{m2} z_2 + \cdots + c_{mn} z_n$$



as transfer function

$$\frac{y(s)}{u(s)} = \sum_{i=1}^{n} \frac{c_{mi} b_{mi}}{s - \lambda_i} = \frac{c_{m1} b_{m1}}{s - \lambda_1} + \cdots + \frac{c_{mn} b_{mn}}{s - \lambda_n}$$

• Controllable canonical form

– Given the general dynamic LTI model (i.e. ODE)

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t)$$
$$= \quad b_m\frac{d^m u(t)}{dt^m} + b_{m-1}\frac{d^{m-1}u(t)}{dt^{m-1}} + \cdots + b_1\frac{du(t)}{dt} + b_0 u(t) \qquad m < n$$

– Corresponding transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

– First, derive the controllable canonical form without input derivatives

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = u(t)$$

– Assign the state variables x_i at the outputs of each integrator $\Rightarrow$ n-integrators in total connected in series in a forward path

- Controllable canonical form, cont.

$$\begin{aligned} x_1 &= y, \\ x_2 &= \dot{y}, \\ &\vdots \\ x_n &= \overset{(n-1)}{y} \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -a_0 x_1 - a_1 x_2 - \cdots - a_{n-1} x_n + u, \\ y &= x_1, \end{aligned}$$

– In matrix form (here without system zeros) $\quad y = (1\ 0 \cdots 0) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$$\begin{pmatrix} \dot{x}_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & \cdots & -a_{n-1} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix} u$$

- Controllable canonical form, cont.

  – Then, for a general case with right-hand side derivatives, i.e. $\dot{u}, \ddot{u}, \ldots \neq 0$, one uses the following approach:

$$\begin{aligned} \overset{(n)}{\xi} + a_{n-1} \overset{(n-1)}{\xi} + \cdots + a_1 \dot{\xi} + a_0 \xi &= u \\ b_{n-1} \overset{(n-1)}{\xi} + b_{n-2} \overset{(n-2)}{\xi} + \cdots + b_1 \dot{\xi} + b_0 \xi &= y \end{aligned}$$

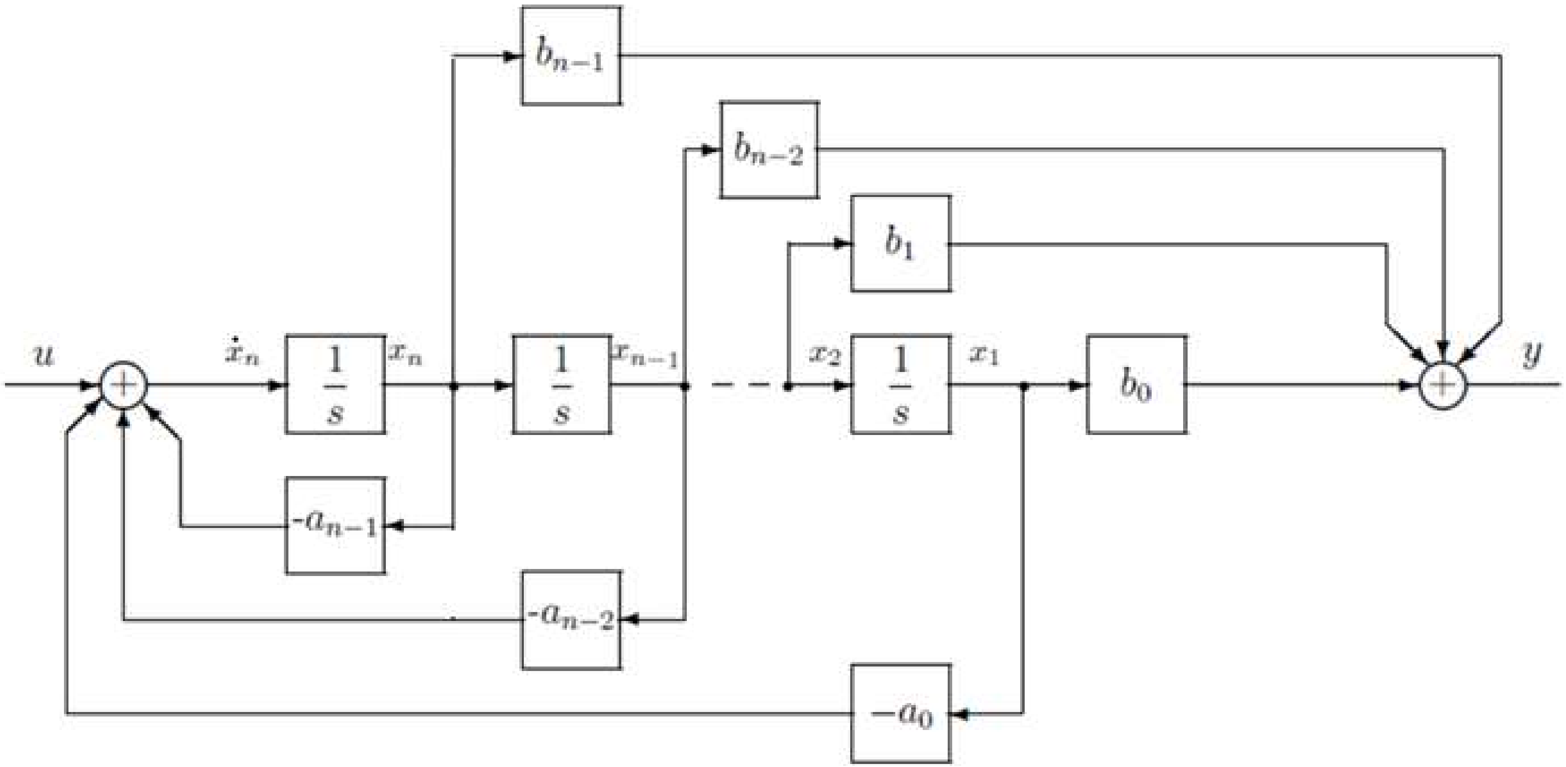
  – The matrix form with new states $\mathbf{x}_C \equiv \xi$:

$$\dot{\mathbf{x}}_C = \mathbf{A}_C \mathbf{x}_C + \mathbf{b}_C u, \quad y = \mathbf{c}_C^T \mathbf{x}_C.$$

$$y = \underbrace{(b_0 \; b_1 \cdots b_{n-1})}_{\mathbf{c}_C^T} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\begin{pmatrix} \dot{x}_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \dot{x}_n \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & \cdots & -a_{n-1} \end{pmatrix}}_{\mathbf{A}_C} \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ x_n \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix}}_{\mathbf{b}_C} u$$

- Controllable canonical form, cont.



– Compute transformation matrix:

i. controllability matrix: $Cr = \left[\mathbf{B}, \mathbf{AB}, \ldots, \mathbf{A}^{n\text{-}1}\mathbf{B} \right]$

ii. take last row of Cr^{-1}: $c_1 = [0\ 0 \ldots 1] Cr^{-1}$

iii. collect elements into the inverse transformation matrix:

$$\mathbf{T}^{-1} = \begin{bmatrix} c_1 \\ c_1\mathbf{A} \\ c_1\mathbf{A}^2 \\ \vdots \\ c_1\mathbf{A}^{n-1} \end{bmatrix}$$

- Observable canonical form
  - Similar approach as for the controllable canonical form. But (!) with the goal to have the last state (output) and input within the dynamics of each previous state:

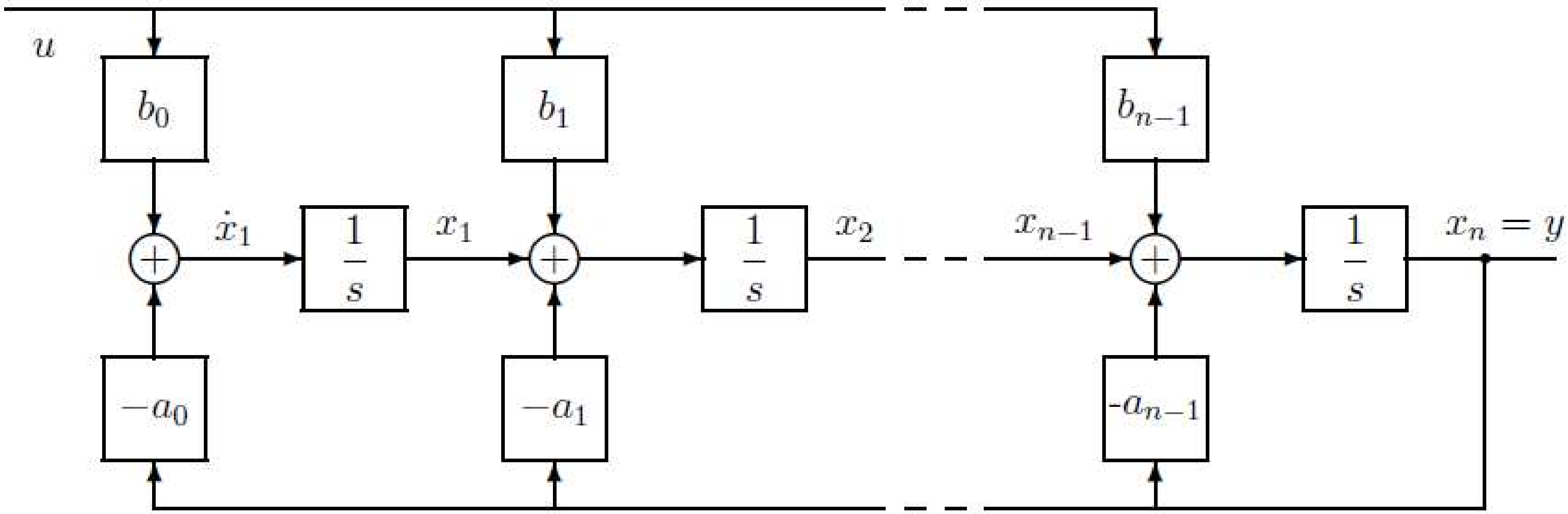
$$
\begin{aligned}
\dot{x}_1 &= -a_0 x_n + b_0 u \\
\dot{x}_2 &= x_1 - a_1 x_n + b_1 u \\
&\vdots \\
\dot{x}_{n-1} &= x_{n-2} - a_{n-2} x_n + b_{n-2} u \\
\dot{x}_n &= x_{n-1} - a_{n-1} x_n + b_{n-1} u
\end{aligned}
$$

$$
y = (0\ 0 \cdots 1) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}
$$

$$
\begin{pmatrix} \dot{x}_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & \cdots & 0 & -a_1 \\ 0 & 1 & \ddots & & \vdots & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & -a_{n-2} \\ 0 & 0 & \cdots & 0 & 1 & -a_{n-1} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ \vdots \\ \vdots \\ b_{n-1} \end{pmatrix} u
$$

- Observable canonical form

  – In the transformed matrix form: $\dot{\mathbf{x}}_O = \mathbf{A}_O \mathbf{x}_O + \mathbf{b}_O u, \quad y = \mathbf{c}_O^T \mathbf{x}_O.$



  – Compute transformation matrix:

    i. observability matrix: $Ob = \begin{bmatrix} \mathbf{C} & \mathbf{CA} & \dots & \mathbf{CA}^{n-1} \end{bmatrix}^T$

    ii. take last column of Ob^{-1}: $o_1 = Ob^{-1} \begin{bmatrix} 0 & 0 & \dots & 1 \end{bmatrix}^T$

    iii. <u>collect transformation matrix</u>: $\mathbf{T} = \begin{bmatrix} o_1 & \mathbf{A}o_1 & \mathbf{A}^2 o_1 & \dots & \mathbf{A}^{n-1} o_1 \end{bmatrix}$

- Duality of both canonical forms
  - Controllable and observable canonical forms are dual to each other, i.e. $\mathbf{A}_O = \mathbf{A}_C^T, \quad \mathbf{b}_O = \mathbf{c}_C^{\,T}, \quad \mathbf{c}_O = \mathbf{b}_C^{\,T}$
  - Both system matrices ($\mathbf{A}_O$ and $\mathbf{A}_C$) contain the coefficients of the characteristic polynomial and, thus, describe the same system dynamics
  - All coefficients b_i of the numerator polynomial are simultaneously in the coupling vectors $\mathbf{c}_C$ and $\mathbf{b}_O$ of the corresponding canonical forms

source: [19]

Consider the system described by

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \tag{5.18}$$

where $A \in R^{n\times n}, B \in R^{n\times m}, C \in R^{p\times n}$, and $D \in R^{p\times m}$. The *dual system* of (5.18) is defined as the system

$$\dot{x}_D = A_D x_D + B_D u_D, \quad y_D = C_D x_D + D_D u_D, \tag{5.19}$$

where $A_D = A^T, B_D = C^T, C_D = B^T$, and $D_D = D^T$.

Lemma 5.7. *System (5.18), denoted by $\{A, B, C, D\}$, is reachable (controllable) if and only if its dual $\{A_D, B_D, C_D, D_D\}$ in (5.19) is observable (constructible), and vice versa.*

• Practical (hand on) session 5

Consider DC motor without load M_L (from page 49)

For the task 4., assume also the following parameter values:

$$R = 1, \quad L = 0.0002, \quad \Psi = 0.04,$$
$$B = 0.0001, \quad J = 0.00005$$

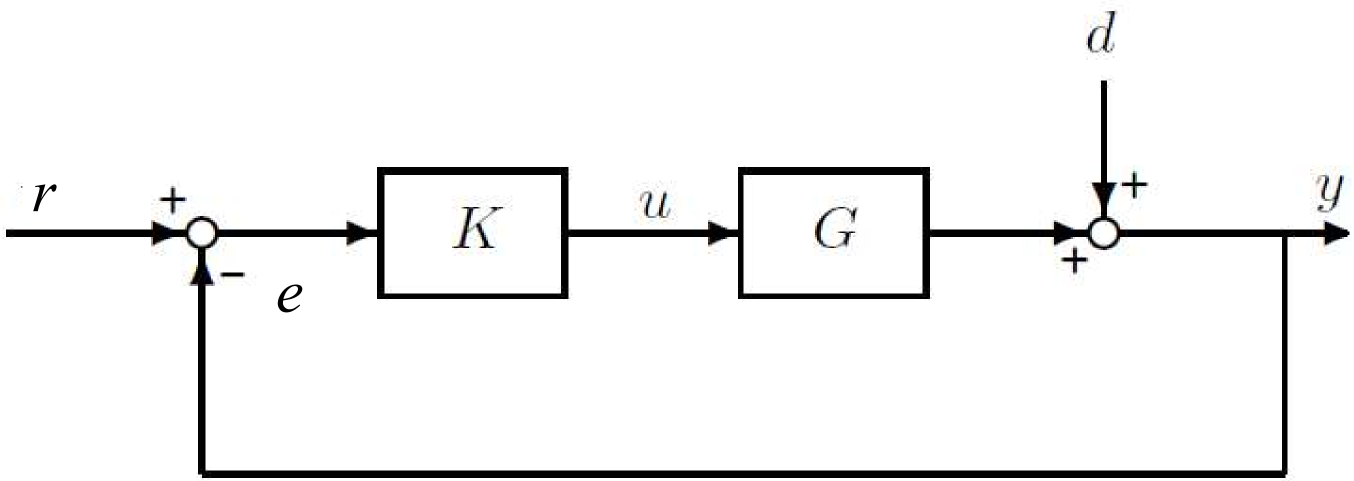
1. Write down the state-space model.
2. Derive the controllable canonical form of the state-space model.
3. Derive the observable canonical form of the state-space model.
4. Compute (using MATLAB) the canonical Kalman form. Check the system eigenvalues from the system matrix.

Feedback control systems for mechatronics and robotics

Michael Ruderman
michael.ruderman@uia.no

Standard output feedback controllers

- SISO closed-loop with reference r and disturbance d as inputs



- Basic notations for the control loop analysis

$L(s) = K(s)G(s)$ – (open) **loop transfer function** (control & plant without closing the feedback loop)

$S(s) = \dfrac{1}{1+L(s)}$ – **sensitivity function** (how sensitive is the control error to an external reference input)

$T(s) = \dfrac{L(s)}{1+L(s)}$ – **complementary sensitivity function** (same as the closed-loop transfer function)

- Requirements on SISO loop transfer function

i. Possibly large loop transfer function at lower frequencies

$$|L(j\omega)| \gg 1 \ \text{ for } \ \omega < \omega_{gc}$$

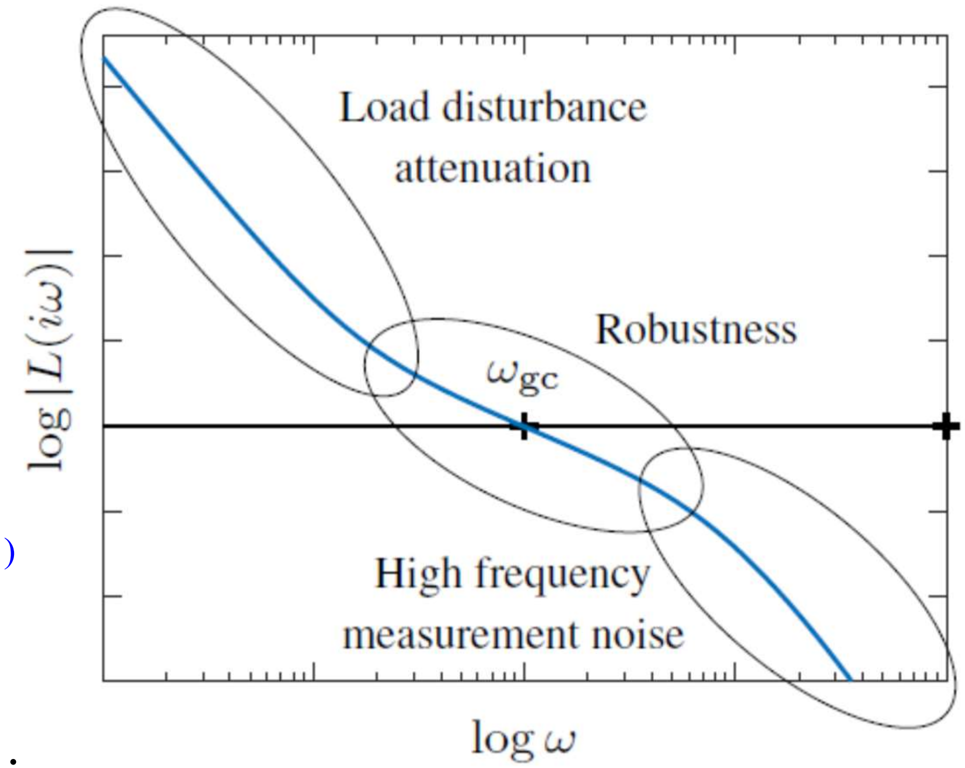
Possibly low loop transfer function at higher frequencies

$$|L(j\omega)| \ll 1 \ \text{ for } \ \omega > \omega_{(\gamma = -180 \text{ deg})}$$

Design of the loop transfer function is most critical for the frequency range between ω_{gc} and $\omega_{(\gamma = -180 \text{ deg})}$

Crossover frequency ω_{gc}:

$$|L(j\omega_{gc})| = 1 = 0 \text{ dB}$$



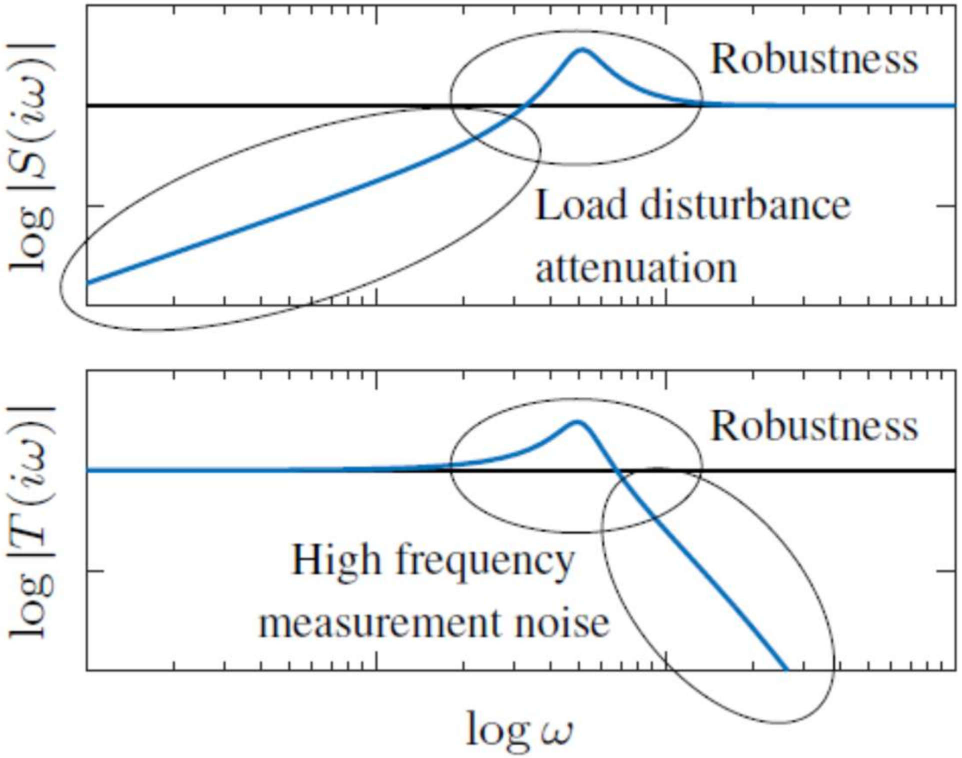
source: [20]

- Requirements on SISO loop transfer function, cont.

ii. Possibly <u>low</u> sensitivity function within possibly large bandwidth

$$\left|S(j\omega)\right| = \frac{1}{\left|1+L(j\omega)\right|} \approx 0$$

source: [20]



Since $e(s) = \underbrace{\frac{1}{1+L(s)}}_{S(s)} r(s),$

for steady-state $s \to 0 \;\Rightarrow$

$$e_{s.s.} \approx \frac{1}{1+1/s} = \left.\frac{s}{s+1}\right|_{s\to 0} \to 0$$

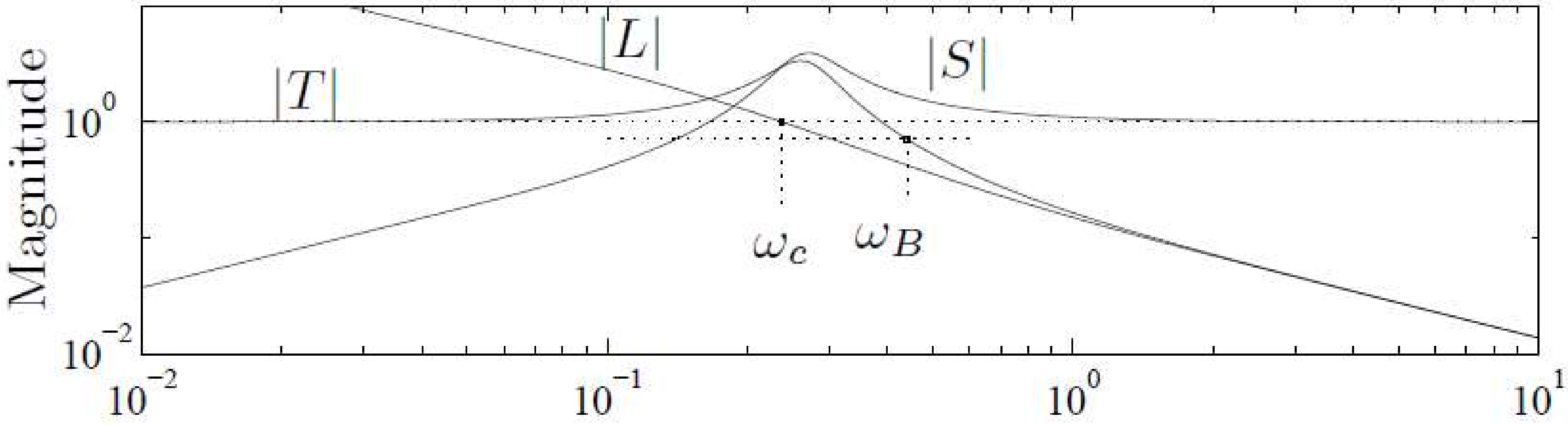
iii. <u>Unity</u> complementary sensitivity function within possibly large bandwidth

$$\left|T(j\omega)\right| = \frac{\left|L(j\omega)\right|}{\left|1+L(j\omega)\right|} \approx 1$$

- Unavoidable (!) limitations of feedback design

$$\boxed{S + T = 1}$$

source: [21]



- Always a <u>trade-off</u> when shaping the loop transfer function *L*

$$e(s) = \underbrace{\left(\frac{1}{1+L(s)}\right)}_{S(s)} r(s) - \frac{G(s)}{1+L(s)} d(s)$$

That means, from an "ideal" controller one wishes to have

$$e \approx 0 \cdot r - 0 \cdot d \quad \Rightarrow \quad S(s) \to 0$$

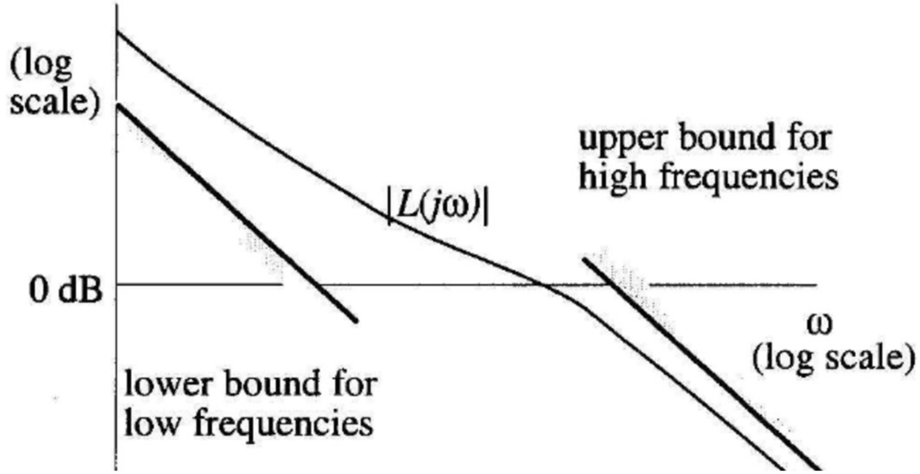
• Unavoidable (!) limitations of feedback design, cont.

– Low frequencies:

S small and T close to 1: $|L(j\omega)| >> 1$

– High frequencies:

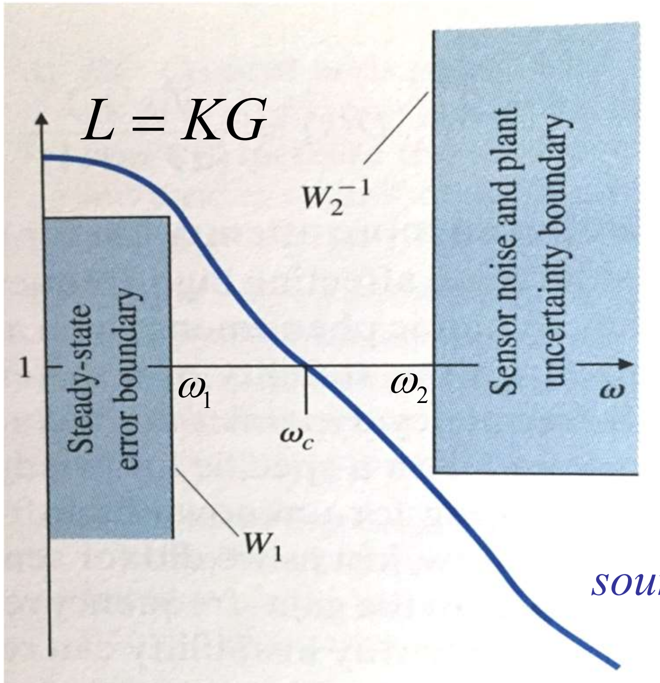
T small and S close to 1: $|L(j\omega)| << 1$



Reminder: **open-loop** transfer function L contains all characteristics of the closed-loop behavior. Thus:

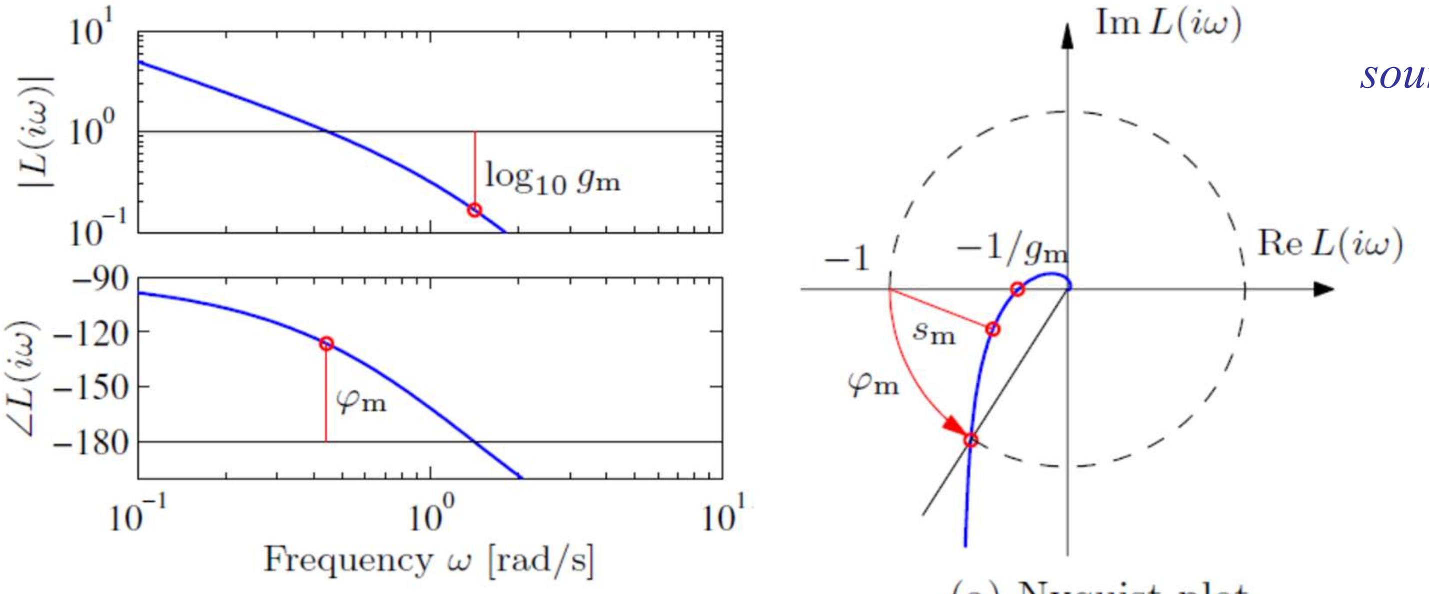
– make $|K(j\omega)G(j\omega)|$ large as possible within the specified **bandwidth** ω_1

– make $|K(j\omega)G(j\omega)|$ small as possible above the specified **bandwidth** ω_2



source: [4]

- Phase margin for shaping the loop transfer function L



(b) Bode plot

(a) Nyquist plot

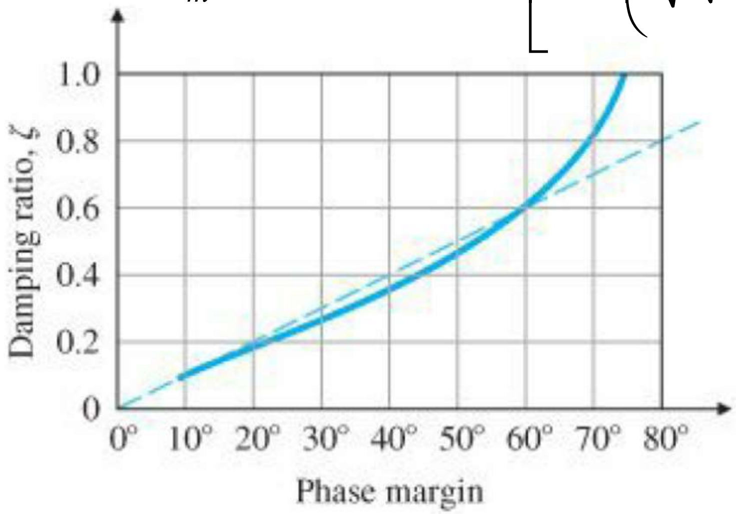
source: [20]

- Stability margins for 2nd order approximations

$$L(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)} \Rightarrow$$

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\varphi_m \equiv \text{PM} = \tan^{-1}\left[2\zeta\left(\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}\right)^{-1}\right]$$

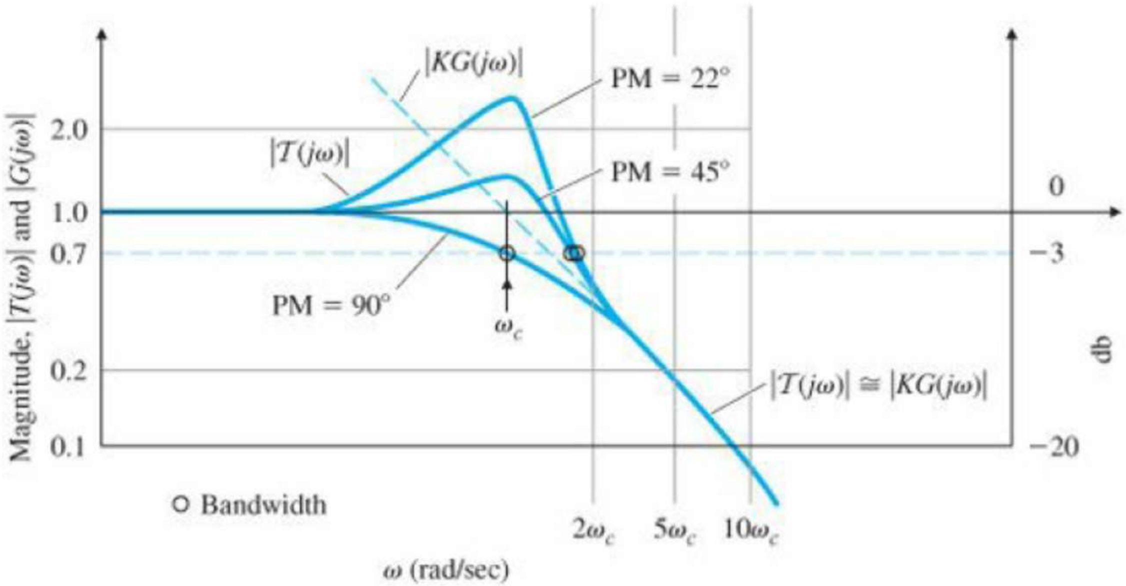


$$\zeta \approx \frac{\text{PM}}{100} \quad \text{for}$$

$$\text{PM} < 70 \text{ deg}$$

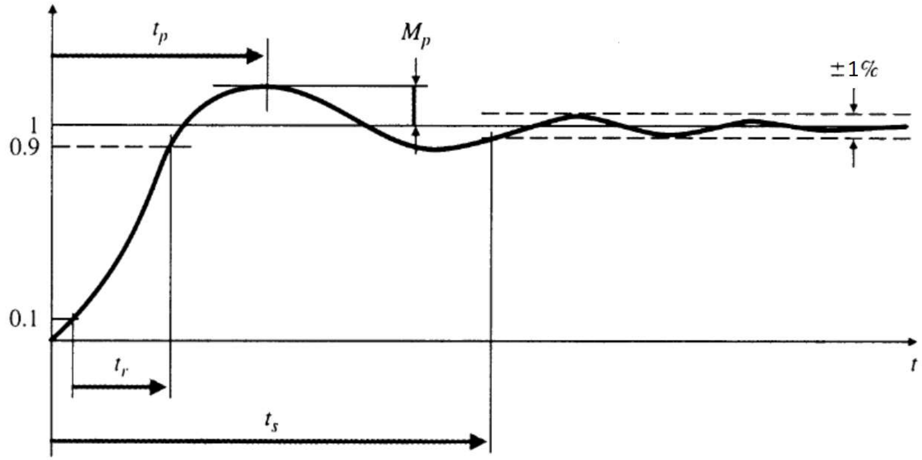
source: [4]

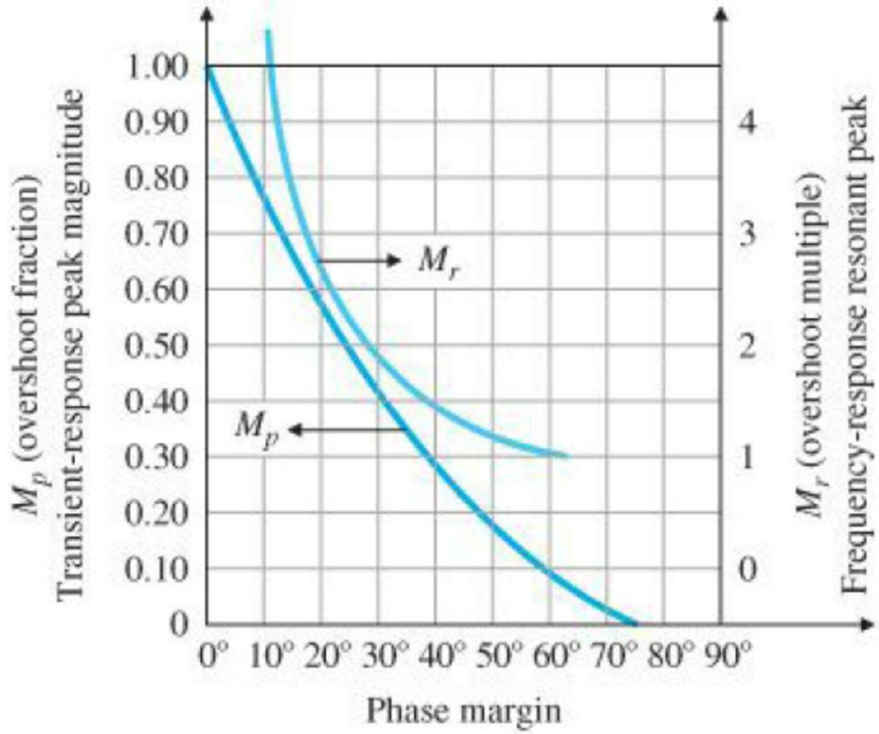
• Phase margin for shaping the loop transfer function L, cont.



source: [4]

– Overshoot M_p of transient-response in time domain, and resonant peak M_r in frequency domain versus PM





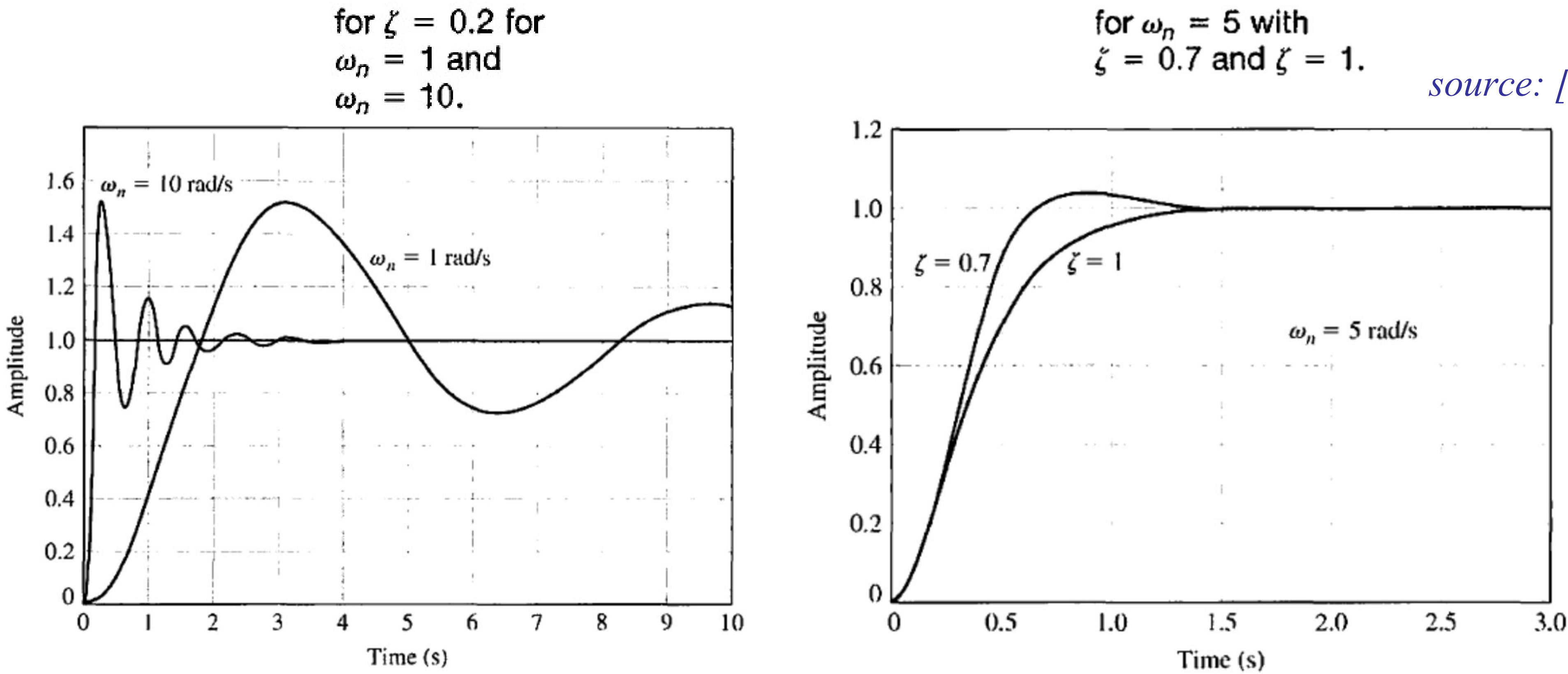
- Closed-loop control system as 2nd-order approximation

Transfer function in normalized form

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \Rightarrow H(s) = \frac{\omega_n^2}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}$$

Step response

source: [1]



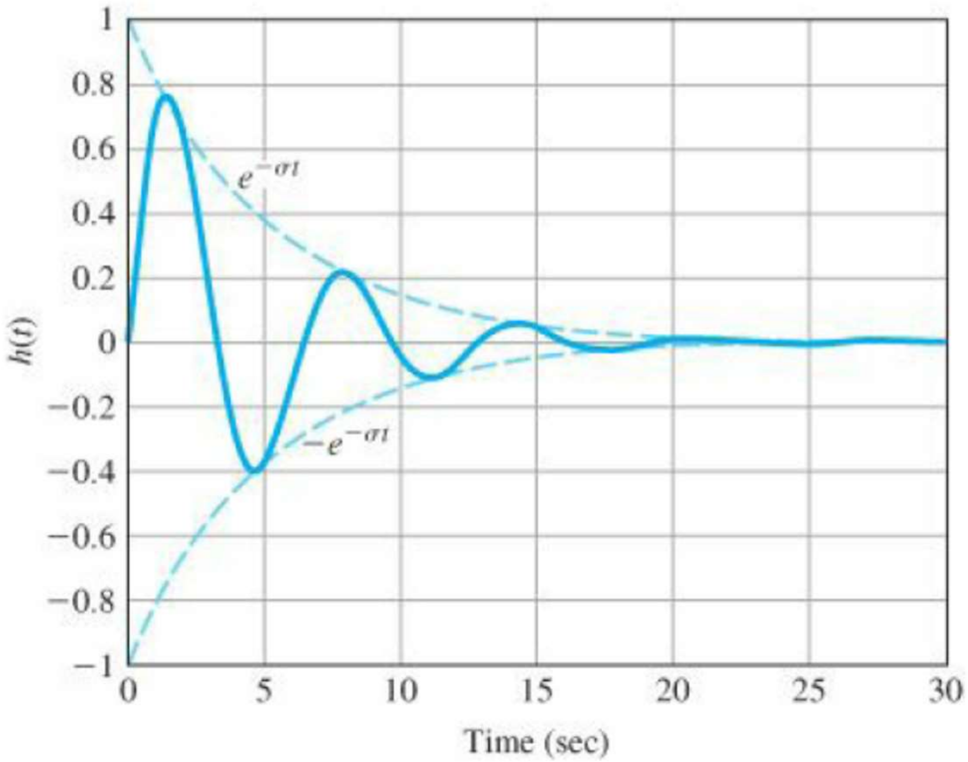
- Closed-loop control system as 2nd-order approximation, cont.

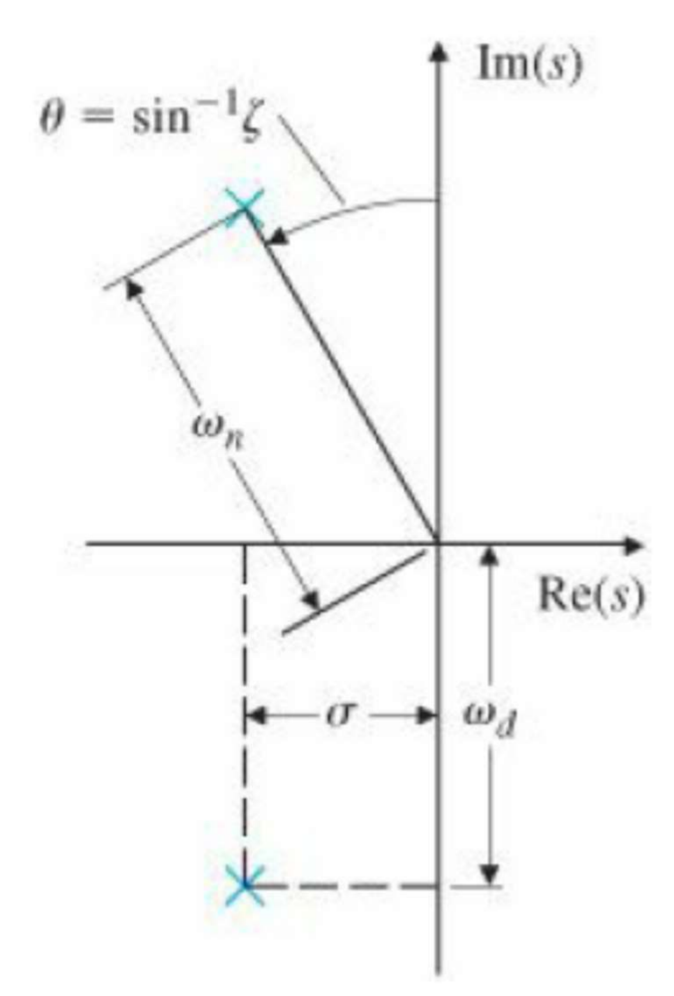
$$h(t) = L^{-1}\{H(s)\} = \frac{\omega_n}{\sqrt{1-\zeta^2}} \mathrm{e}^{-\sigma t} \sin(\omega_d t)$$

source: [4]

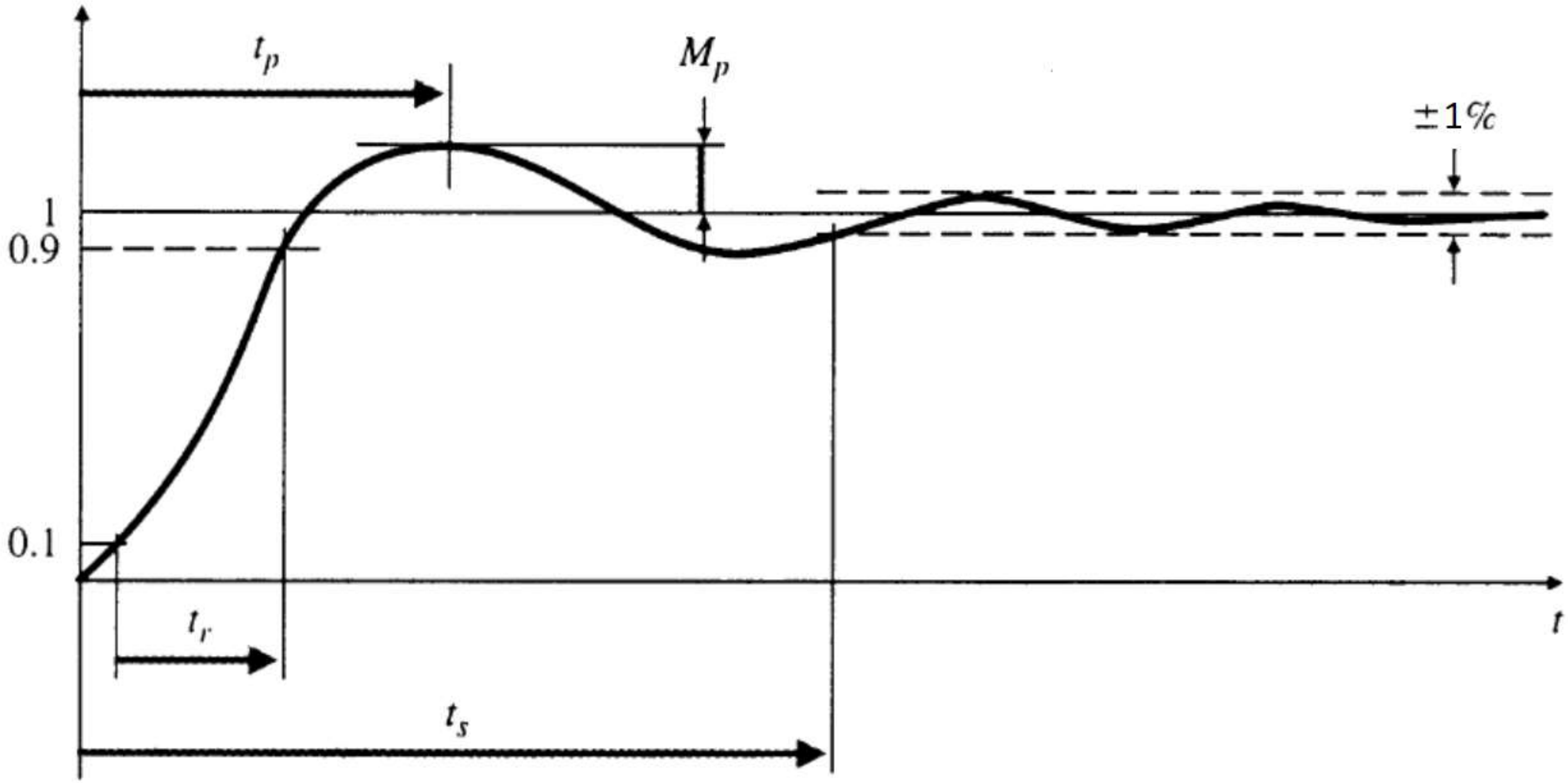
$\sigma = \zeta\omega_n$ – system damping

$\omega_d = \omega_n\sqrt{1-\zeta^2}$ – oscillation frequency





- Closed-loop control system as 2nd-order approximation, cont.



rise time (for ζ=0.5):

$$t_r \cong \frac{1.8}{\omega_n}$$

peak time:

$$t_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

settling time:

$$t_s \cong \frac{4.6}{\sigma} = \frac{4.6}{\zeta \omega_n}$$

- Transient overshoot versus damping ratio

Overshot appears when

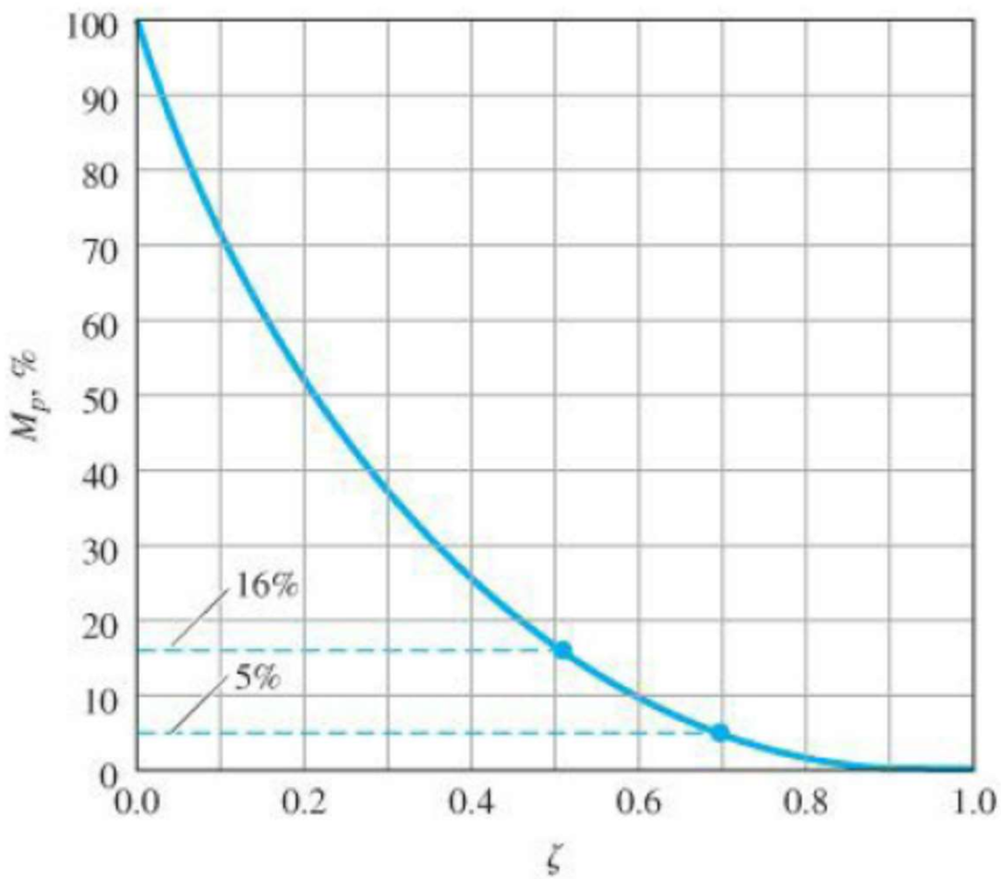
$$y(t) = L^{-1}\left\{H(s)\frac{1}{s}\right\}$$

reaches its maximum, i.e. when $\dot{y}(t) = 0$

$$\Rightarrow M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}},$$

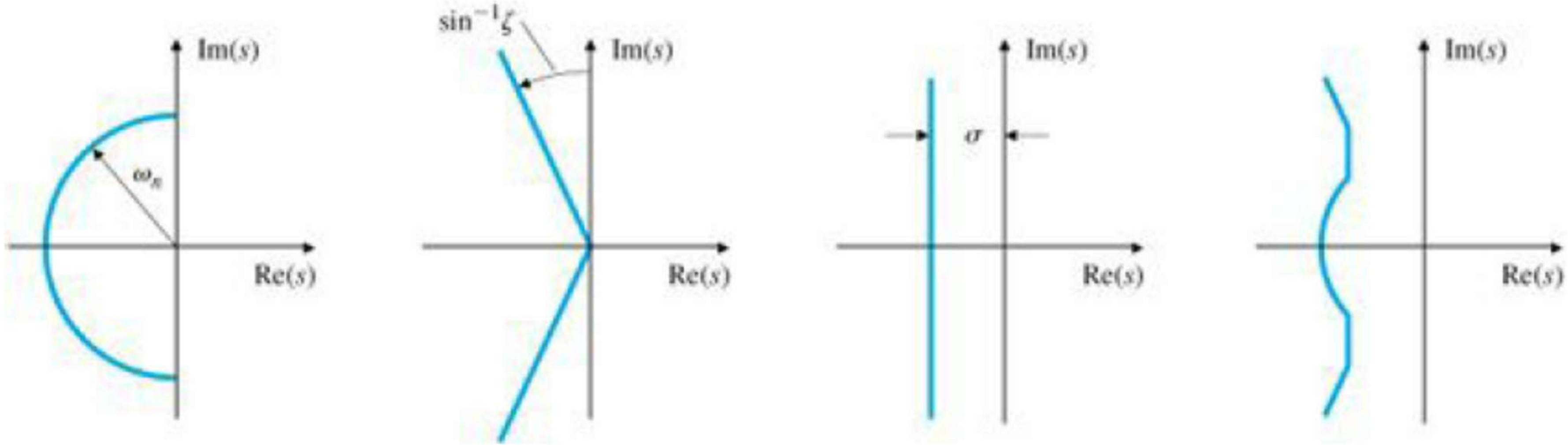
for $0 \le \zeta < 1$

$$M_p \cong \begin{cases} 5\% & \zeta = 0.7 \\ 16\% & \zeta = 0.5 \\ 35\% & \zeta = 0.3 \end{cases}$$



source: [4]

- Assignment of closed-loop poles based on time-domain response



Larger natural frequency $\Rightarrow$ smaller rise time

$$\omega_n \geq \frac{1.8}{t_r}$$

Larger angle $\Rightarrow$ smaller overshoot

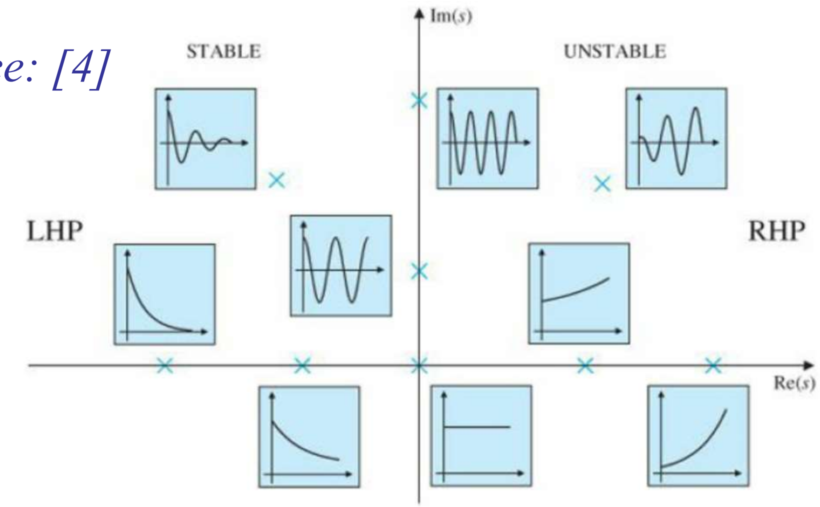
$$\zeta \geq \zeta(M_p)$$

Larger system damping $\Rightarrow$ smaller settling time

$$\sigma \geq \frac{4.6}{t_s}$$

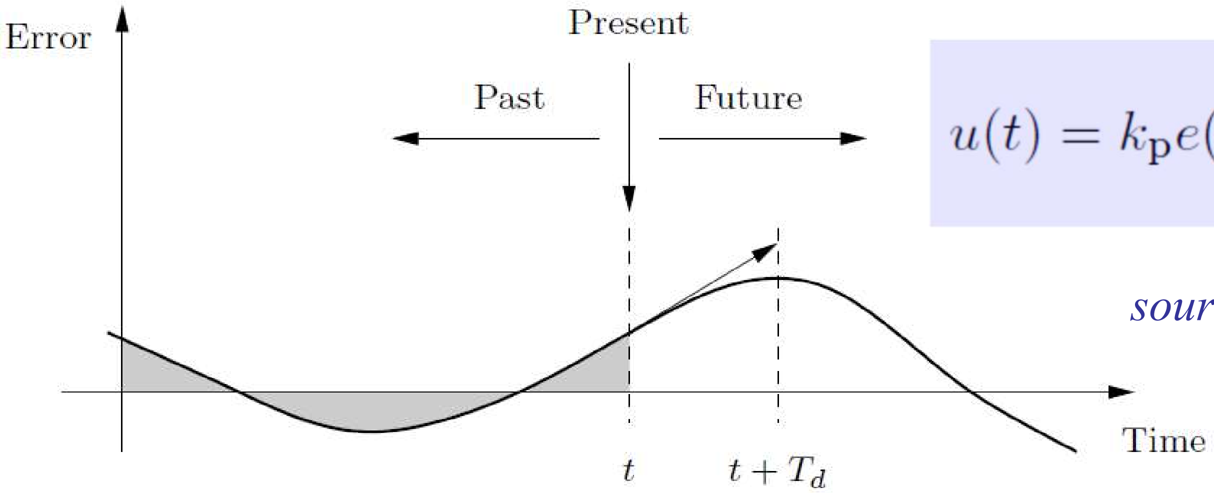
Superposition of all 3 curves determines the pole requirements

source: [4]



- Specification of <u>dominant pole pair</u> of the closed-loop control system in s-plane allows <u>shaping the transfer function</u> (i.e. design the controller)

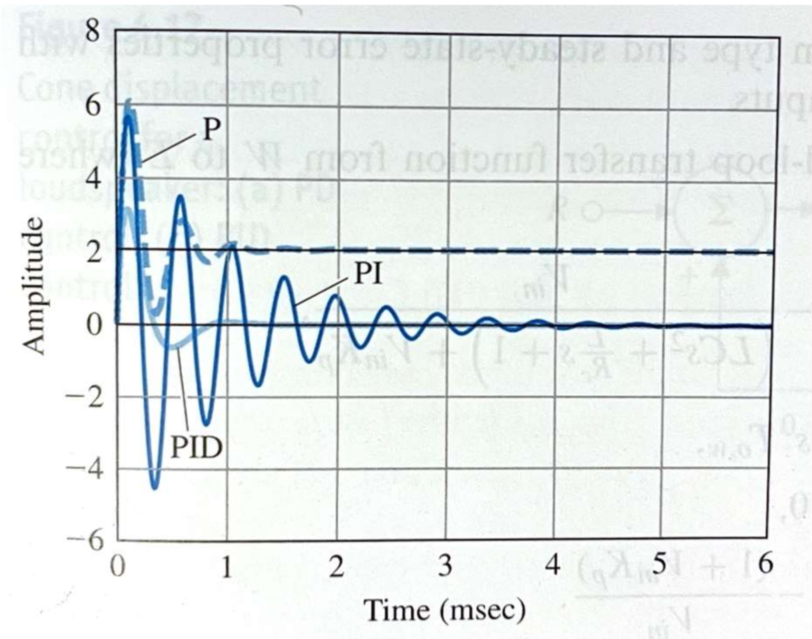
- Why PID (proportional-integral-derivative) control structure?



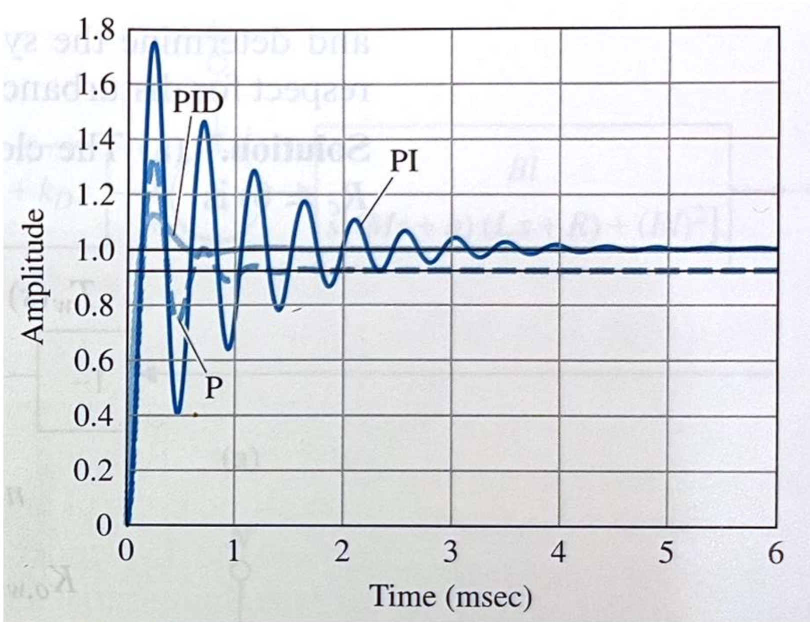
$$u(t) = k_{\mathrm{p}} e(t) + k_{\mathrm{i}} \int_0^t e(\tau)\, d\tau + k_{\mathrm{d}} \frac{de(t)}{dt}$$

source: [20]

source: [4]



response to the <u>step disturbance</u>



response to the <u>step reference</u>

- **Example 14**: impact of proportional control only

Let us control a 2nd-order dynamic system (plant)

$$G(s) = \frac{b_2}{s^2 + a_1 s + a_2}$$

by the proportional feedback only, i.e. $\dfrac{U(s)}{E(s)} = k_p$

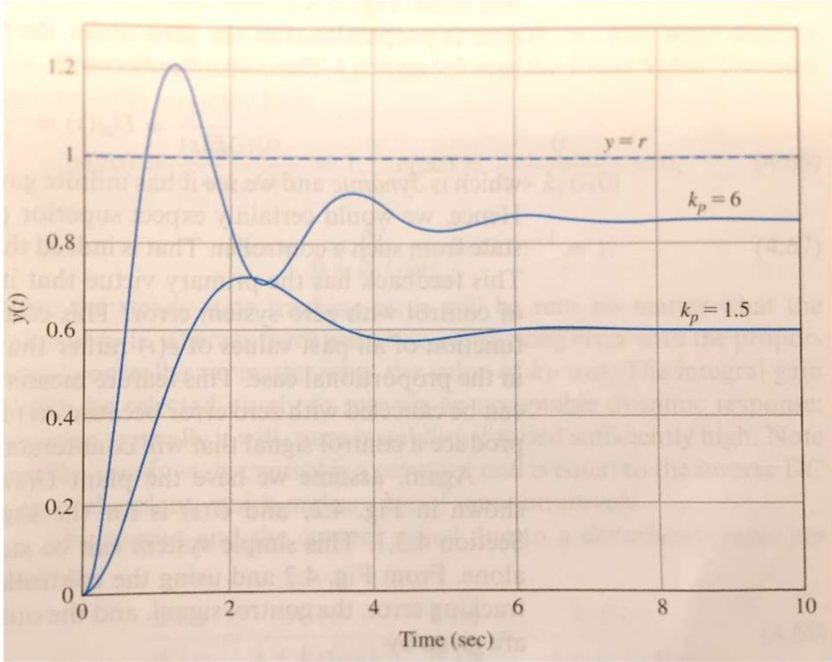
Characteristic equation (of the closed-loop):

source: [4]

$$1 + k_p G(s) = 0$$

$$\Rightarrow \quad s^2 + a_1 s + \underbrace{a_2 + k_p b_2}_{} = 0$$

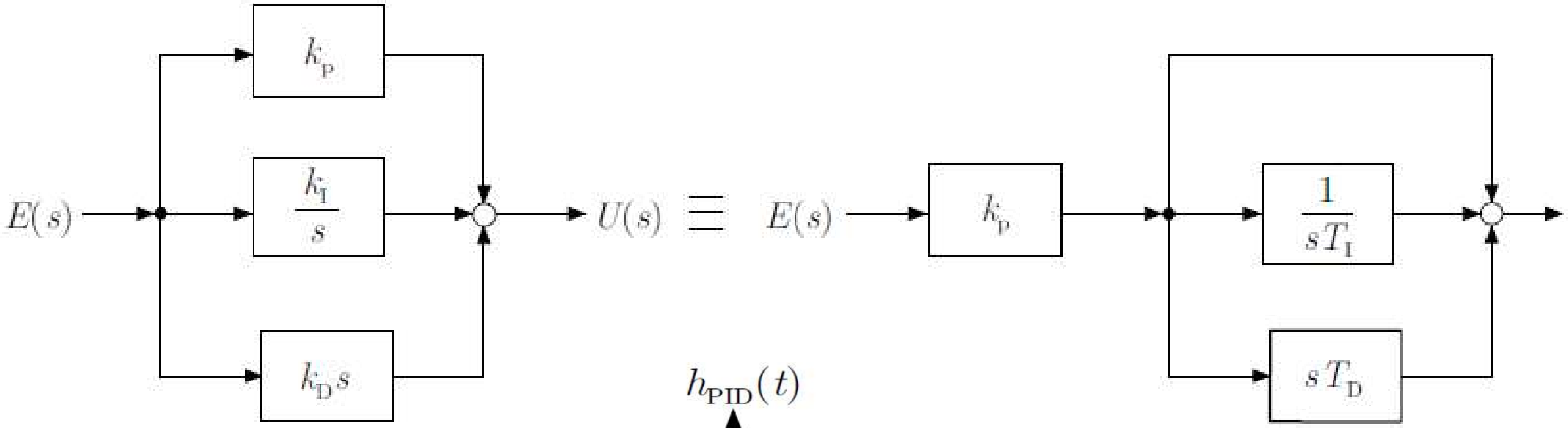
it determines natural frequency, but cannot change the system damping



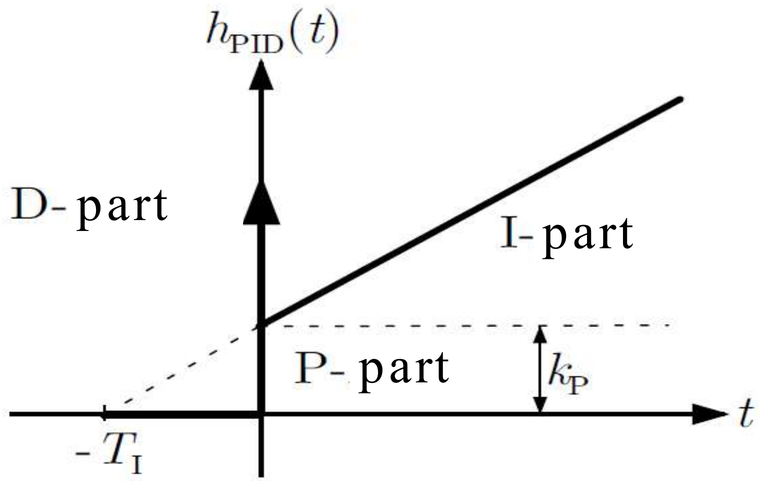
For steady-state (if $G(0)=1$) we obtain

$$\frac{Y(s)}{R(s)} = \left.\frac{k_p G(s)}{1 + k_p G(s)}\right|_{s \to 0} = \frac{k_p}{1 + k_p}$$

- PID controller: **P**roportional, **I**ntegral, **D**erivative



Step response of the PID control element



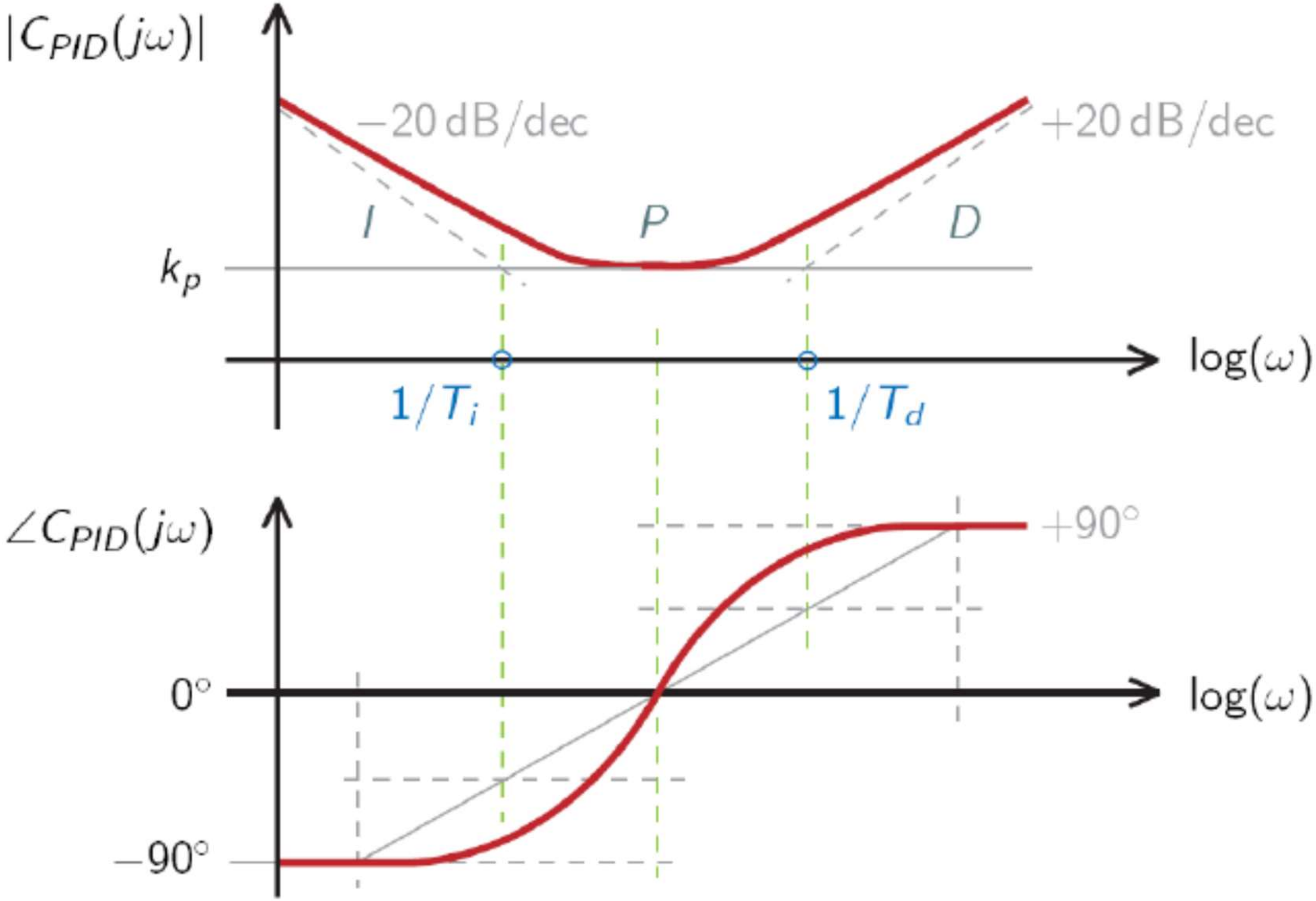
Overall control law, with two equivalent parametrizations $T_i = \frac{k_p}{k_i}, \; T_d = \frac{k_d}{k_p}$

$$u(t) = k_p e(t) + k_i \int_0^\infty e(t)dt + k_d \frac{de(t)}{dt} = k_p \left[e(t) + \frac{1}{T_i} \int_0^\infty e(t)dt + T_d \frac{de(t)}{dt} \right]$$

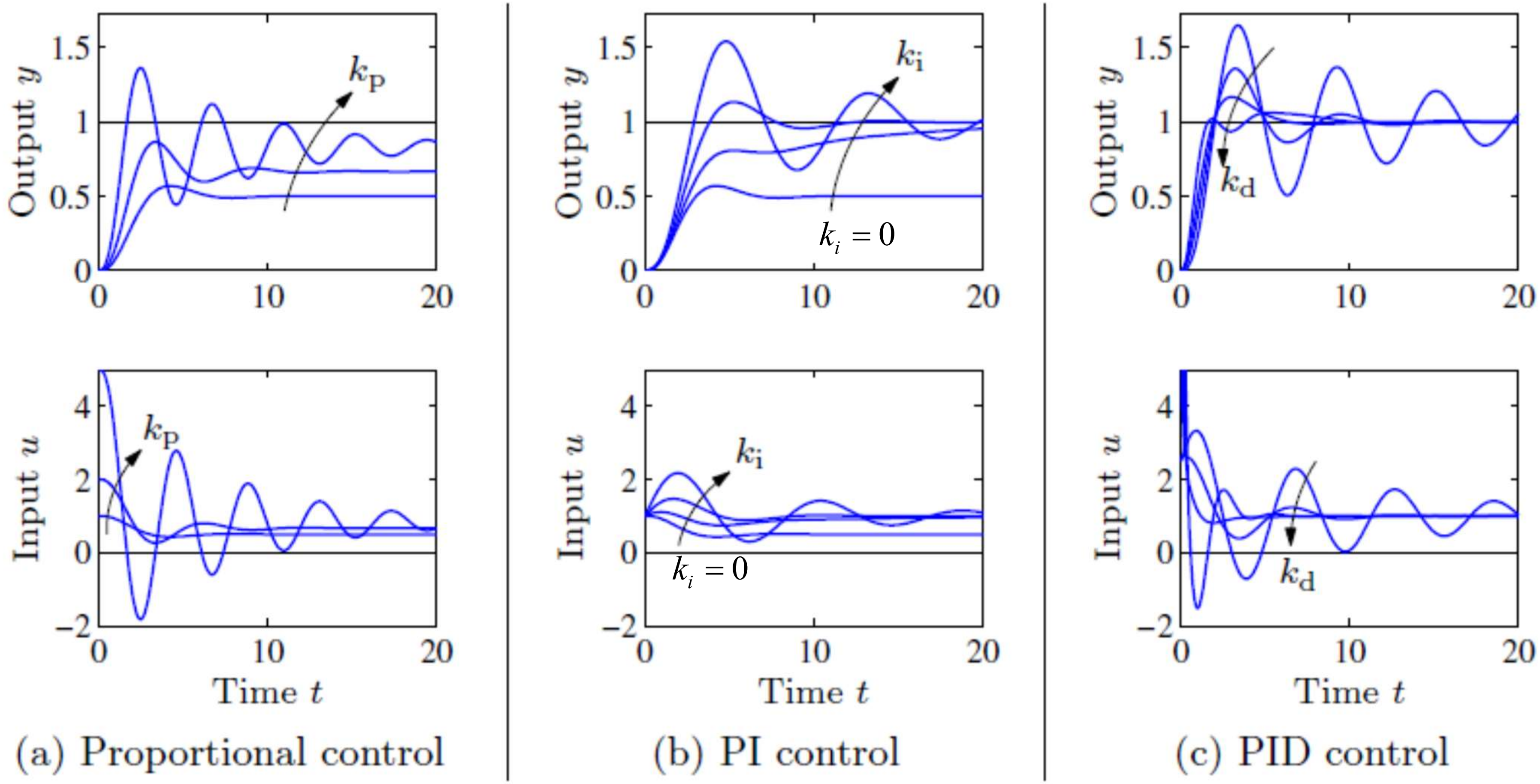
- PID controller: **P**roportional, **I**ntegral, **D**erivative, cont.

As transfer function
(with Bode diagram)

$$C(s) = k_p \cdot \left[\underbrace{1}_{P} + \underbrace{\frac{1}{s \cdot T_i}}_{I} + \underbrace{s \cdot T_d}_{D} \right]$$



- Principal impact provided by P, I, and D control terms

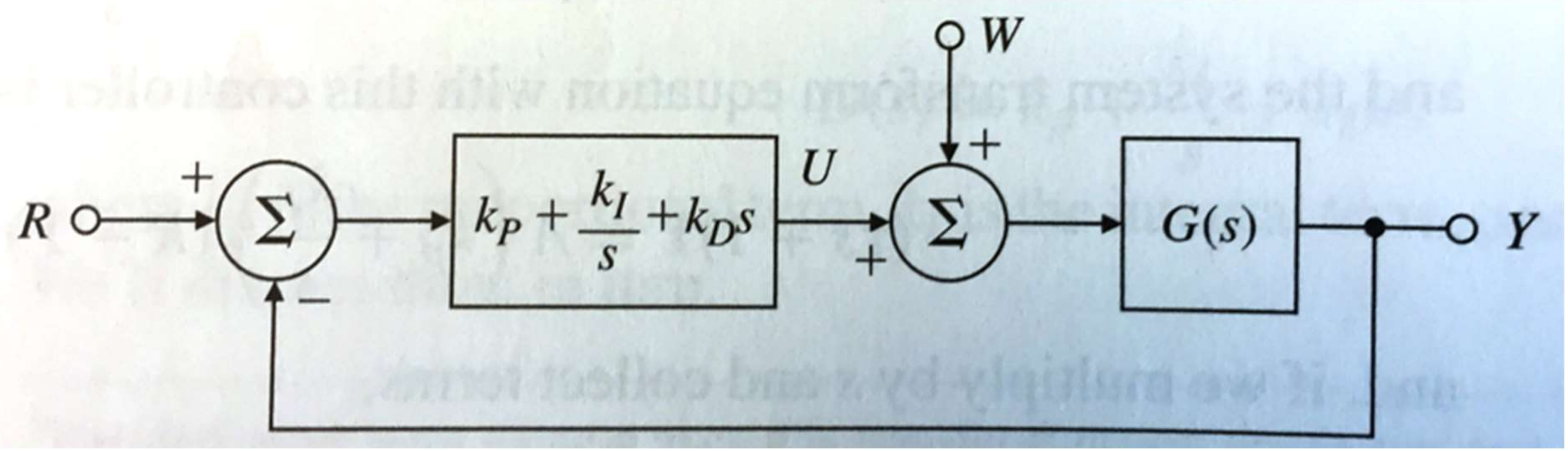


source: [20]

Responses to step changes in the reference value for a system with a proportional controller (a), PI controller (b), and PID controller (c). The process has the transfer function $P(s) = 1/(s+1)^3$, the proportional controller has parameters $k_p = 1$, 2, and 5, the PI controller has parameters $k_p = 1$, $k_i = 0$, 0.2, 0.5, and 1, and the PID controller has parameters $k_p = 2.5$, $k_i = 1.5$, and $k_d = 0$, 1, 2, and 4.

- PID control in frequency domain

Standard structure



For practical implementation
(for ensuring PID transfer function is proper, i.e. no free differentiator of D-part)

$$\boxed{C_{PID}(s) = k_p \left(1 + \frac{1}{T_i s} + \frac{T_d s}{\tau_d s + 1} \right) \quad \text{with} \quad \tau_d \ll T_d}$$

$\tau_d > 0$ time constant of low-pass filter

- PID control in frequency domain, cont.

As transfer function: once in a <u>parallel-</u> and once in a <u>serial-connection</u>

$$C_{PID}(s) = K_p + \frac{K_i}{s} + K_d s = K_p\left(1 + \frac{1}{T_i s} + T_d s\right) = K_{PID}\frac{(T_1 s + 1)(T_2 s + 1)}{s}$$

with parameters relationship

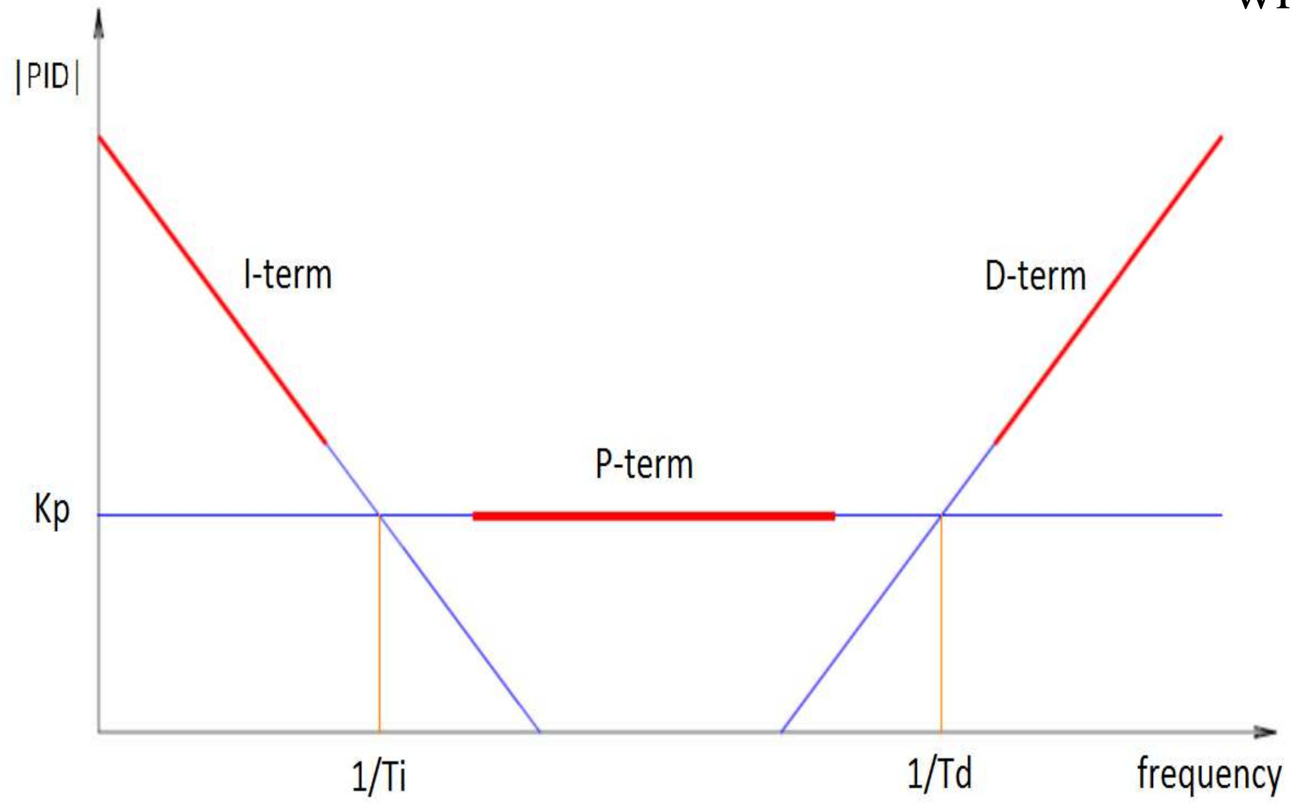
$$T_i = \frac{K_p}{K_i}$$

$$T_d = \frac{K_d}{K_p}$$

$$K_p = K_{PID} T_i$$

$$T_i = T_1 + T_2$$

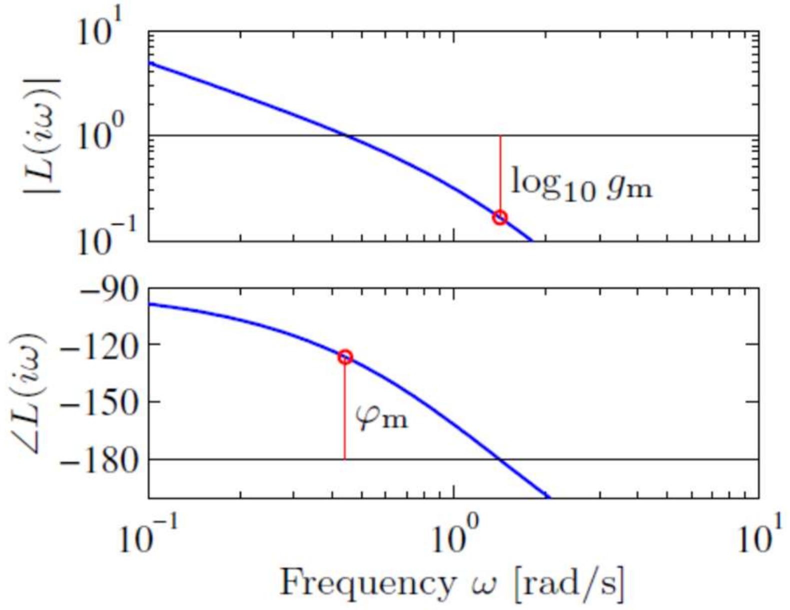
$$T_d = \frac{T_1 T_2}{T_1 + T_2}$$

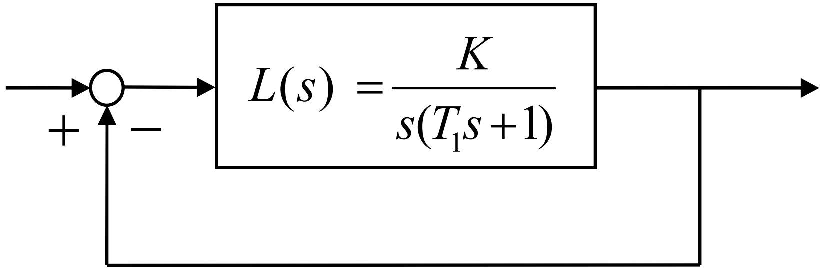


- Loop shaping by using 2nd-order approximation

Optimal open-loop transfer function in form of

$$\boxed{L(s) = \frac{K}{s(T_1 s + 1)}}$$





The still unknown parameters K, T_1 of the open-loop transfer function $L(s)$

$$L(s) \quad \Rightarrow \quad G_{cl}(s) = \frac{K}{T_1 s^2 + s + K} = \frac{1}{(T_1 / K) s^2 + (1/K) s + 1}$$

Parameter T_1 (as a time-constant) can be assigned first, by a control specification, since it affects directly the bandwidth of the closed-loop control system. Then, one finds a suitable K

- Loop shaping by using 2nd-order approximation, cont.

<u>Magnitude optimum approach</u>

$$G_{cl}(s) = \frac{1}{(T_1/K)s^2 + (1/K)s + 1} \quad \text{with} \quad \boxed{K = \frac{1}{2T_1}}$$

$$\Rightarrow \quad G_{cl}(s) = \frac{1}{2T_1^2 s^2 + 2T_1 s + 1} = \frac{1}{\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

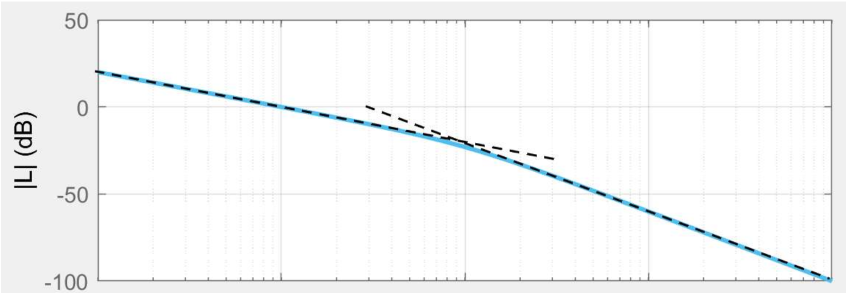
$$\text{with} \quad \omega_n = \frac{1}{\sqrt{2}T_1} \quad \text{and} \quad \zeta = T_1\omega_n = \frac{1}{\sqrt{2}} = 0.707$$

This yields 2nd-order system with damping $\zeta \approx 0.71 \;\Rightarrow\;$ overshoot ≈ 4.7%, that is mostly acceptable and, even, sometimes desired in applications (when the plant's damping is higher)

Open-loop in the range of –20 dB/dec

$$\Rightarrow \quad |L(j\omega)| = \frac{K}{\omega} = \frac{1}{2T_1\omega} = \frac{1}{2}\frac{\omega_1}{\omega}$$

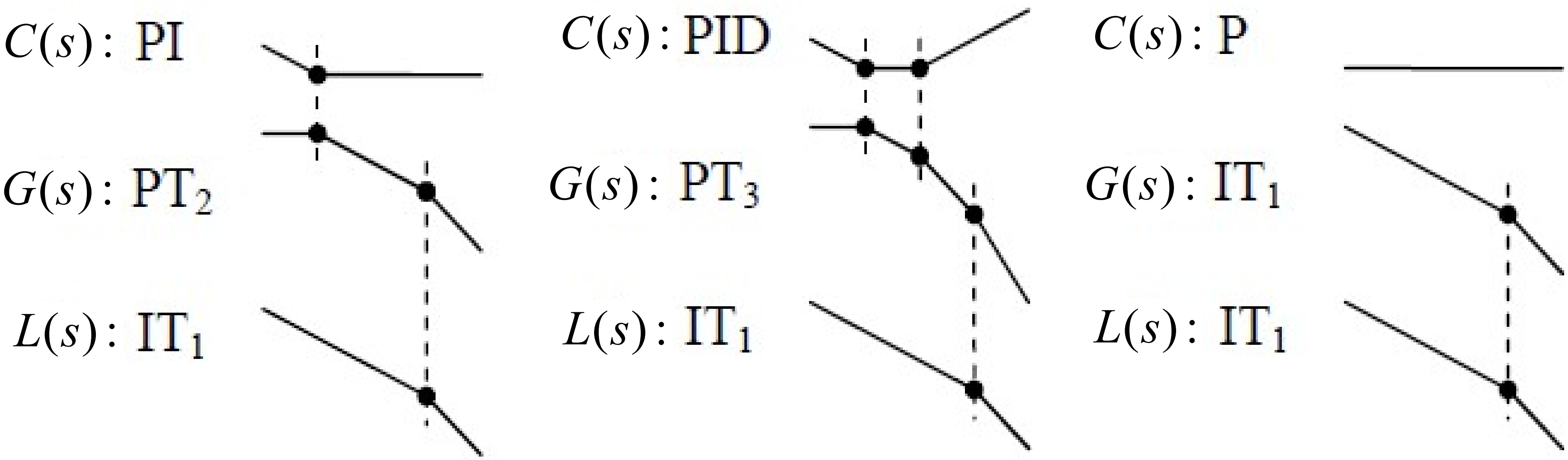
$$\Rightarrow \quad |L(\omega = \omega_{cr})| = \frac{1}{2} \approx -6 \text{ dB}$$



$\omega_1 = 1/T_1$ is the characteristic *corner*-frequency of intersection between –20 dB/dec and –40 dB/dec

- Optimal loop shaping with different types of controller

Some typical *corner*-frequencies for plant G, control C, and open-loop L



$$C_{PID}(s) = \frac{K_{PID}(sT_1+1)(sT_2+1)}{s}, \quad C_{PI} = \frac{K_{PI}(sT_1+1)}{sT_1}$$ transfer functions of PID, PI controls

$$G_{PT_3}(s) = \frac{K}{(s\tau_1+1)(s\tau_2+1)(s\tau_3+1)}$$ transfer function of PT_3 and PT_2 (τ_3=0) plants

$$L(s) = \frac{K}{s(T_1 s+1)}$$ transfer function IT_1 (integrator × 1st order system) of the open-loop $L(s)$

• Practical (hand on) session 6

Consider DC motor without load M_L (from page 49)

Assume the following parameter values: $R = 1, \quad L = 0.0002, \quad \Psi = 0.04,$

$B = 0.0001, \quad J = 0.00005$

1. Design PI (proportional-integral) control of angular velocity, using an optimal loop shaping. Here, determine first the integrator time constant, then the gain.
2. Implement the closed-loop system (system plant and controller), in either MATLAB or Simulink, and show the controlled velocity step response.

Feedback control systems for mechatronics and robotics

Michael Ruderman
michael.ruderman@uia.no

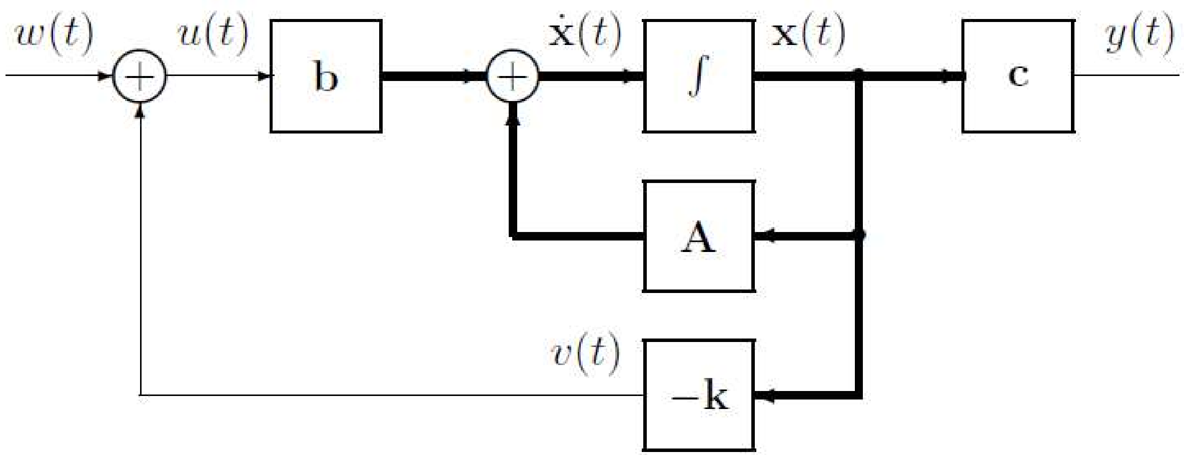
State feedback controllers and prefilter extensions

- Basic principle of state feedback control

– System of n-th order has <u>n roots</u> which determine its natural response

– Changing dynamics for each of <u>n states</u> means changing location of the roots

$\Rightarrow$ This can be done by appropriate assignment of the state feedback gains



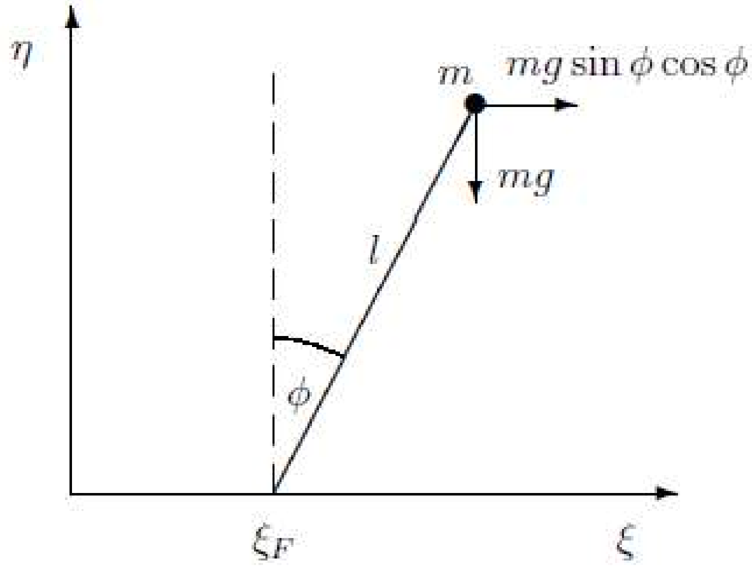
Here, consider first without reference value, i.e. $w=0$

$$u = -\mathbf{k}\mathbf{x} = -\begin{bmatrix} k_1 & k_2 & \ldots & k_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

– Resulted <u>state-</u> and <u>characteristic-equation</u>:

$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{b}\mathbf{k}\mathbf{x}$ and $\det\left[s\mathbf{E} - \left(\mathbf{A} - \mathbf{b}\mathbf{k} \right) \right] = 0$ $\quad \Rightarrow$ roots are ‘placeable’ through selection of $\mathbf{k}$

- **Motivating Example 15**: control of unbalanced rod

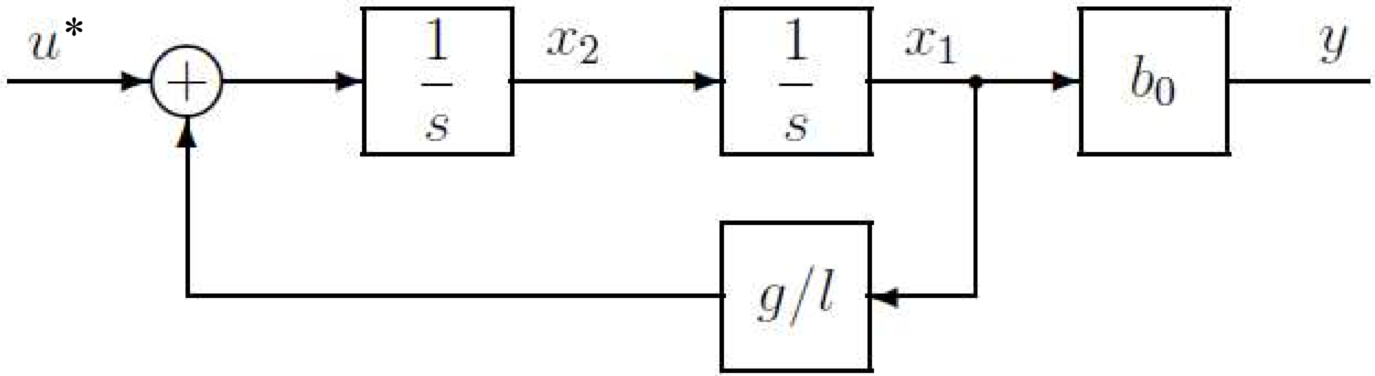


– Linearized system dynamics

$$\ddot{\phi}(t) - \frac{g}{l}\phi(t) = -\frac{1}{l}\ddot{\xi}_F(t)$$

$$\Rightarrow \quad \ddot{y}(t) - \frac{g}{l}y(t) = -u(t)^*$$

– As block diagram with $b_0 = -1$



– State-space model $\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ g/l & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)^*,$

$$y(t) = (b_0 \quad 0) \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},$$

- **Motivating Example 15**: control of unbalanced rod, cont.
  - First, use standard PD control for balancing the rod

$$u^{*} = K_P(w-y) + K_D\left(\dot{w}-\dot{y}\right)$$

  - Closed-loop transfer function with new input = reference value, i.e. (*u*=*w*)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{-(K_D s + K_P)}{s^2 - K_D s - (g/l + K_P)} = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0}$$

  - Polynomial coefficients

$$a_0 = -(g/l + K_P),\ a_1 = -K_D,\ b_0 = -K_P,\ b_1 = -K_D.$$

$\Rightarrow$ Two control gains determine two roots of the characteristic equation

  - Closed-loop state-space model, already in controllable canonical form (!)

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ g/l + K_P & K_D \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)$$

$$y(t) = \begin{pmatrix} -K_P & -K_D \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

- **Motivating Example 15**: control of unbalanced rod, cont.

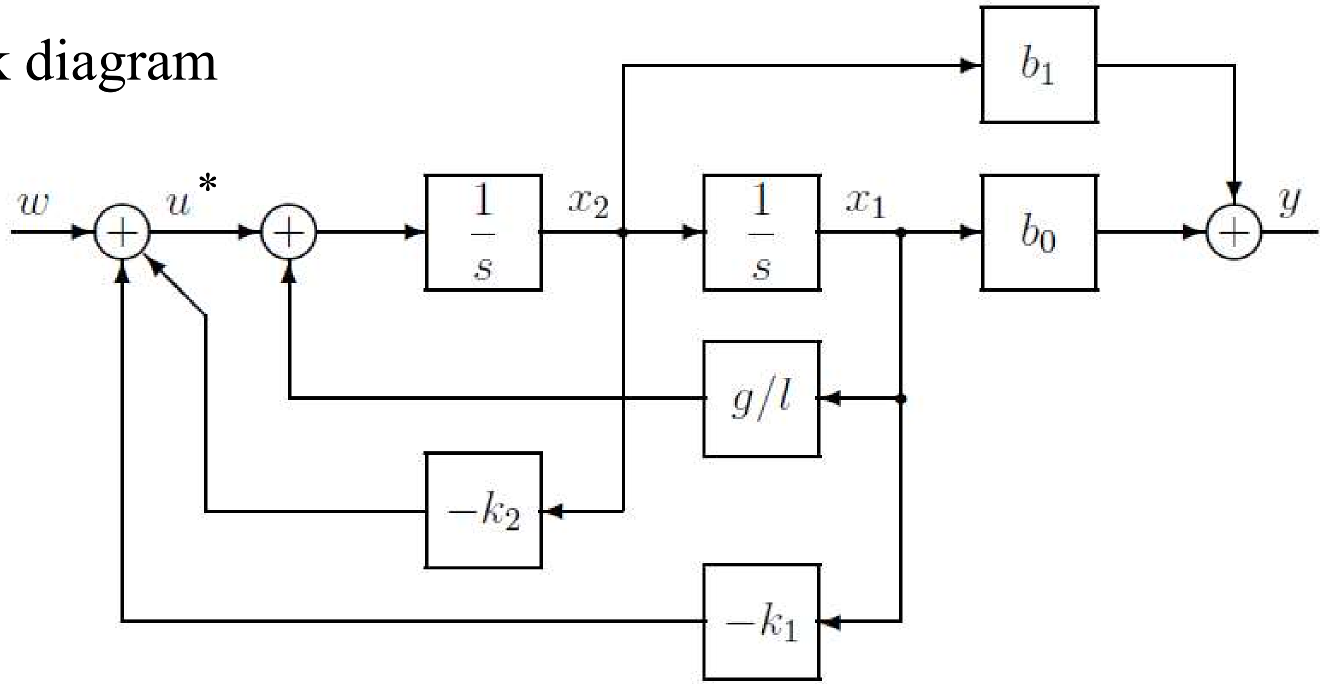
– Closed-loop state-space model with decomposed system matrix

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ g/l & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -k_1 & -k_2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w(t)$$

$$y(t) = (b_0 \quad b_1) \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

with $k_1 = -K_P$, $k_2 = -K_D$, $b_0 = -K_P$ and $b_1 = -K_D$.

– Corresponding block diagram



– To recognize:
the (initial) system control input
$u^* = w + \mathbf{k}^T \mathbf{x}$

- **Motivating Example 15**: control of unbalanced rod, cont.

– System matrix of the closed-loop (in controllable canonical form)

$$\mathbf{A}_F = \begin{pmatrix} 0 & 1 \\ g/l + K_P & K_D \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ g/l & 0 \end{pmatrix}}_{\mathbf{A}} + \underbrace{\begin{pmatrix} 0 & 0 \\ -k_1 & -k_2 \end{pmatrix}}_{\mathbf{A}_{bk}}$$

– Control-related term

$$\mathbf{A}_{bk} = -\mathbf{b}\mathbf{k}^T = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} -k_1 & -k_2 \end{pmatrix}$$

– Let us evaluate here a numerical example

For the given $\frac{g}{l} = 1$, design PD control with closed-loop poles $\mu_{1,2} = [-1, -2]$

Compute polynomial coefficients: with MATLAB `poly([-1,-2])`

$$\Rightarrow \; s^2 + \underbrace{3}_{a_1} s + \underbrace{2}_{a_0} = 0 \;\Rightarrow\; \mathbf{A}_F = \underbrace{\begin{bmatrix} 0 & 1 \\ g/l + K_P & K_D \end{bmatrix}}_{\textit{given}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}}_{\textit{in C.C. form}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{\textit{what we want}}$$

$$\Rightarrow \; K_P = -3, \;\; K_D = -3$$

- Control design via pole placement

  – Closed-loop control system (incl. feed-forward) in the state-space form

$$\dot{\mathbf{x}}(t) = \left(\mathbf{A} - \mathbf{bk}\right)\mathbf{x}(t) + \mathbf{bV}w(t)$$
$$y(t) = \mathbf{cx}(t)$$

  – Control with 2 degrees of freedom: state feedback & reference feed-forward

$$u(t) = u_{fb}(t) + u_{ff}(t) = -\mathbf{kx} + \mathbf{V}w$$

  – System dynamics

$$\left.\begin{aligned} &\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t), \qquad y(t) = \mathbf{c}^T\mathbf{x}(t), \\ &a(s) = |s\mathbf{I} - \mathbf{A}| = s^n + a_{n-1}s^{n-1} + \cdots + a_0 \end{aligned}\right\} \text{open-loop (plant) behavior}$$

$$\left.\begin{aligned} &\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{b}\mathbf{k}^T)\mathbf{x}(t) + \mathbf{b}w(t), \qquad y(t) = \mathbf{c}^T\mathbf{x}(t), \\ &a_k(s) = |s\mathbf{I} - \mathbf{A} + \mathbf{b}\mathbf{k}^T| \end{aligned}\right\} \text{closed-loop (control system) behavior}$$

  – For desired dynamics $\Rightarrow$ desired poles $\Rightarrow$ characteristic equation

$$\boxed{\alpha(s) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_0}$$

Goal: determine such feedback gain **k** which provides $a_k(s) = \alpha(s)$

- Control design via pole placement, cont.

– Method by **Bass-Gura algorithm** *source: [22]*

Assume that the desired closed-loop poles are μ_i with $i = 1, \ldots, n$

$$\alpha(s) = \prod_{i=1}^{n} (s - \mu_i) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_{n-1} s + \alpha_n$$

The open-loop characteristic equation of the original (given) plant model

$$a(s) = \det(s\boldsymbol{I} - \boldsymbol{A}) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n$$

$$\Rightarrow \quad [\alpha - a]^T = \mathbf{k}\, C_r T \qquad \text{for} \quad [\alpha - a]^T = \left[(\alpha_1 - a_1), \ldots, (\alpha_n - a_n)\right]$$

with the controllability C_r and Toeplitz matrices T (both invertible)

$$C_r = (\mathbf{b} \quad \mathbf{A}\mathbf{b} \quad \cdots \quad \mathbf{A}^{n-1}\mathbf{b}),$$

$$T = \begin{pmatrix} 1 & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \\ 0 & 1 & a_{n-1} & \cdots & a_3 & a_2 \\ 0 & 0 & 1 & \ddots & a_4 & a_3 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 & a_{n-1} \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

The resulted Bass-Gura formula:

$$\mathbf{k} = [\alpha - a]^T C_r^{-1} T^{-1}$$

- Control design via pole placement, cont.

  – Alternatively, transform state-space model into controllable canonical form

  – For n-th order systems:

$$\mathbf{A}_C - \mathbf{b}_C \mathbf{k}_C^T = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ \vdots & \vdots & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \\ \underbrace{-(a_0+k_1)}_{=-\alpha_0} & \underbrace{-(a_1+k_2)}_{=-\alpha_1} & \underbrace{-(a_2+k_3)}_{=-\alpha_2} & \cdots & \cdots & \underbrace{-(a_{n-1}+k_n)}_{=-\alpha_{n-1}} \end{pmatrix}$$

  – Example: to show the relation to the Bass and Gura approach

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad C_r = (\mathbf{b} \quad \mathbf{A}\mathbf{b}) = \begin{pmatrix} 0 & 1 \\ 1 & -a_1 \end{pmatrix}$$

$$C_r T = \begin{pmatrix} 0 & 1 \\ 1 & -a_1 \end{pmatrix} \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (C_r T)^{-1}$$

$\Rightarrow$ feedback: $(k_1 \quad k_2) = (\alpha_1 - a_1 \quad \alpha_0 - a_0) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (\alpha_0 - a_0 \quad \alpha_1 - a_1)$

- LQR (linear quadratic regulator) design by Riccati method

– For the state-space model

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}\mathbf{x}(t)$$

with state feedback control law

$$u(t) = -\mathbf{k}\mathbf{x}(t)$$

consider **design of control gains** as an **optimization problem**

$$\min_{\mathbf{k}} J$$

– **Assumption**: the system is stable and steady-state-accurate

$$J = \int_0^\infty (y - y_\infty)^2 dt$$ *objective function to be minimized*

- LQR (linear quadratic regulator) design by Riccati method, cont.

  – For steady-state values (denoted by subindex ∞), i.e. for $\dot{\mathbf{x}}(t) = 0$

$$\mathbf{x}_\infty = -\mathbf{A}^{-1}\mathbf{B}\,u_\infty \qquad y_\infty = \mathbf{C}^T\mathbf{x}_\infty$$

  we introduce the new variables

$$\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}_\infty \qquad \tilde{u} = u - u_\infty \qquad \tilde{y} = y - y_\infty$$

  so that the objective function becomes

$$J = \int_0^\infty \tilde{y}^2 dt = \int_0^\infty \tilde{\mathbf{x}}^T \mathbf{C}\mathbf{C}^T \tilde{\mathbf{x}} dt$$

  $\Rightarrow$ Thus, the original problem is transformed to the **regulation** towards the equilibrium state $\mathbf{x}_\infty$, starting from any initial state $\mathbf{x}$

  – Next, introduce the **weight matrix** of the states

  $\mathbf{Q}$ instead of $\mathbf{C}\mathbf{C}^T$, to be placed in the objective function J

- LQR (linear quadratic regulator) design by Riccati method, cont.

– The corresponding resulted objective function

$$J = \int_0^\infty \mathbf{x}^T \mathbf{Q} \mathbf{x} \, dt = \int_0^\infty \mathbf{x}_0{}^T e^{\mathbf{A}^T t} \mathbf{Q} \, \mathbf{x}_0 e^{\mathbf{A}t} \, dt = \mathbf{x}_0{}^T \mathbf{P} \, \mathbf{x}_0$$

with $\mathbf{P} = \int_0^\infty e^{\mathbf{A}^T t} \mathbf{Q} \, e^{\mathbf{A}t} \, dt$

– After partial integration of **P** one obtains

$$\mathbf{P} = -\mathbf{Q}\mathbf{A}^{-1} - \mathbf{A}^T \underbrace{\int_0^\infty e^{\mathbf{A}^T t} \mathbf{Q} \, e^{\mathbf{A}t} \, dt}_{\mathbf{P}} \mathbf{A}^{-1}$$

and, hence, one obtains the well-known **Lyapunov equation**

$$\mathbf{P} = -\mathbf{Q}\mathbf{A}^{-1} - \mathbf{A}^T \mathbf{P} \mathbf{A}^{-1} \quad \Rightarrow \quad \mathbf{A}^T \mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q}$$

- LQR (linear quadratic regulator) design by Riccati method, cont.

– For the closed-loop system with system matrix

$$\bar{\mathbf{A}} = \mathbf{A} - \mathbf{B}\mathbf{k}$$

consider the objective function

$$J = \int_0^\infty \mathbf{x}^T(t)\bar{\mathbf{Q}}\mathbf{x}(t)\,dt \quad \text{with} \quad \bar{\mathbf{Q}} = \mathbf{Q} + \mathbf{k}^T\mathbf{R}\mathbf{k}$$

and **R** as **weight matrix** for control value(s)

Q *weights ("penalizes") the states,* **R** *weights ("penalizes") the control effort, i.e. energy consumption*

– *Remark:*
larger **R** weight matrix values ⇒ higher "penalty" for the control action ⇒ less energy consumption and less workload of the control/actuator elements

– For the global minima of objective function, it is required:

$$\frac{\partial J}{\partial \mathbf{k}_{ij}} \overset{!}{=} 0 \quad \text{for} \quad i = 1\ldots m,\ j = 1\ldots n$$

with m inputs and n states

- LQR (linear quadratic regulator) design by Riccati method, cont.

We also require the **solution** to be **independent of the initial states**

for $J = \mathbf{x}_0^T \mathbf{P} \mathbf{x}_0$

$$\Rightarrow \frac{\partial J}{\partial k_{ij}} = \mathbf{x}_0^T \frac{\partial J}{\partial k_{ij}} \mathbf{x}_0 \overset{!}{=} 0$$

– Take the Lyapunov equation of the closed-loop system

$$\bar{\mathbf{A}}^T \mathbf{P} + \mathbf{P} \bar{\mathbf{A}} = -\bar{\mathbf{Q}} \qquad \text{(i)}$$

$$\text{with} \quad \left.\begin{aligned} \bar{\mathbf{A}} &= \mathbf{A} - \mathbf{B}\mathbf{k} \\ \bar{\mathbf{Q}} &= \mathbf{Q} + \mathbf{k}^T \mathbf{R} \mathbf{k} \end{aligned}\right\} \qquad \text{(ii)}$$

Since both terms in eq. (ii) are dependent of $\mathbf{k}$ $\Rightarrow$

$$\frac{\partial \bar{\mathbf{A}}^T}{\partial \mathbf{k}_{ij}} \mathbf{P} + \mathbf{P} \frac{\partial \bar{\mathbf{A}}}{\partial \mathbf{k}_{ij}} = -\frac{\partial \bar{\mathbf{Q}}}{\partial \mathbf{k}_{ij}}$$

- LQR (linear quadratic regulator) design by Riccati method, cont.

$\vdots$

some intermediate computation steps (omitted here)

$\vdots$

$$\mathbf{Rk} - \mathbf{B}^{T}\mathbf{P} = \mathbf{0}$$

– Solving the above equation with respect to **k** results in the **optimal control**

$$\mathbf{k} = \mathbf{R}^{-1}\mathbf{B}^{T}\mathbf{P} \qquad \text{(iii)}$$

– Putting equations (ii) and (iii) into (i) results in

$$\mathbf{A}^{T}\mathbf{P} + \mathbf{PA} - \mathbf{PBR}^{-1}\mathbf{B}^{T}\mathbf{P} + \mathbf{Q} = \mathbf{0} \qquad \text{(iv)}$$

Algebraic matrix Riccati equation

which is then used to calculate the **P-solution** for the given system (**A**,**B**) and the specified (by control design) weight matrices **Q** and **R**

*Solving Riccati equation means finding **P** solution of (iv) and then computing **k** by (iii)*

- LQR design by Riccati method: the related steps

– Define the weight matrices **Q** and **R** (mostly with only diagonal elements)

  - Increasing all q_{ii} elements $\Rightarrow$ faster total system dynamics, but also higher control values will be required
  - Increasing a particular q_{ii} element $\Rightarrow$ the dynamic behavior of the corresponding state becomes faster (than other)
  - Increasing of r_{ii} values $\Rightarrow$ suppression of the required control magnitude (e.g. required in case of actuator limits/saturation)

– Find the positive definite solution **P** of the Riccati equation

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q} = \mathbf{0}$$

– Compute the state feedback gains

$$\mathbf{k} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}$$

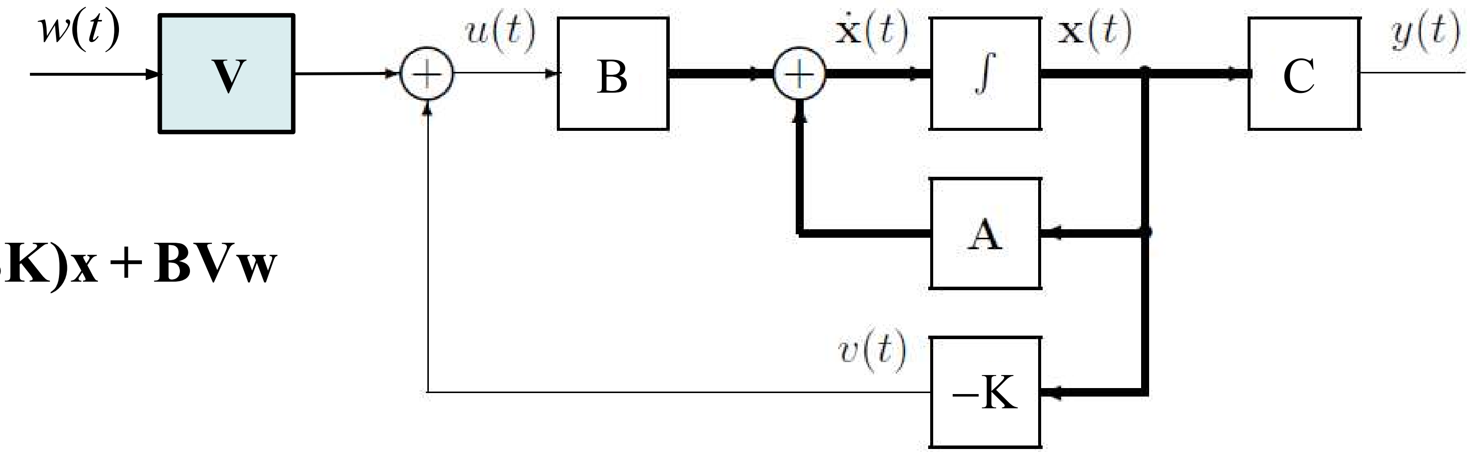
– Riccati state feedback controller minimizes

$$J = \frac{1}{2}\int_0^\infty \left[\underline{x}^T(\tau)\,\underline{Q}\,\underline{x}(\tau) + \underline{u}^T(\tau)\,\underline{R}\,\underline{u}(\tau)\right]d\tau$$

- State feedback control: design of prefilter

– State-feedback alone does not provide steady-state accuracy (!)

$\Rightarrow$ Include reference prefilter **V**, which is 2-nd degree of freedom of control



$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x} + \mathbf{BVw}$$
$$\mathbf{y} = \mathbf{Cx}$$

– For steady-state, it is required $\dot{\mathbf{x}} = \mathbf{0}, \;\; \mathbf{y} \overset{!}{=} \mathbf{w}$

$$\Rightarrow \quad \mathbf{0} = (\mathbf{A} - \mathbf{BK})\mathbf{x} + \mathbf{BVw}$$
$$\mathbf{y} = \mathbf{w} = \mathbf{Cx}$$

– Solving the above eq. for **x** and substituting into the output eq. results in

$$\Rightarrow \quad \mathbf{x} = (\mathbf{BK} - \mathbf{A})^{-1}\mathbf{BVw}, \quad \Rightarrow \quad \mathbf{y} = \mathbf{w} = \mathbf{C}(\mathbf{BK} - \mathbf{A})^{-1}\mathbf{BVw}$$

- State feedback control: design of prefilter, cont.
  - For the required $\mathbf{y} \overset{!}{=} \mathbf{w} \quad \Rightarrow \quad \mathbf{C}(\mathbf{Bk}-\mathbf{A})^{-1}\mathbf{BV} = \mathbf{E}$ where $\mathbf{E}$ is identity matrix
  - Then, the resulted prefilter

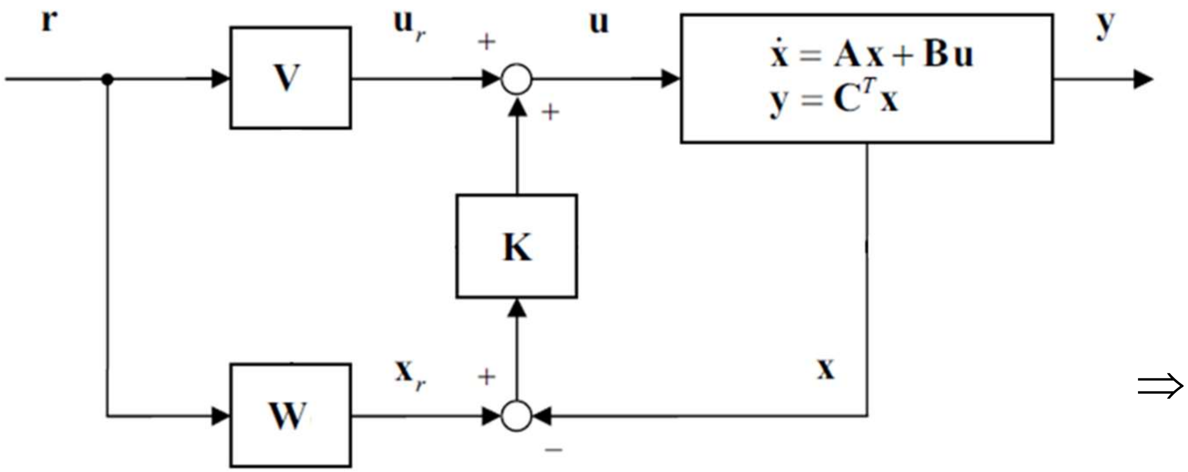
$$\mathbf{V} = \left(\mathbf{C}(\mathbf{Bk}-\mathbf{A})^{-1}\mathbf{B}\right)^{-1}$$

- Extended prefiltering with reference comparison

source: [23]

$$\dot{\mathbf{x}}_\infty = \mathbf{A}\,\mathbf{x}_\infty + \mathbf{B}\,\mathbf{u}_\infty = 0 \quad \Rightarrow \quad \mathbf{y}_\infty = \mathbf{C}^T\mathbf{x}_\infty = \mathbf{r}_\infty$$

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C}^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x}_\infty \\ \mathbf{u}_\infty \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathbf{r}_\infty \quad \Rightarrow \quad \begin{pmatrix} \mathbf{x}_\infty \\ \mathbf{u}_\infty \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C}^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathbf{r}_\infty$$



Require for steady-state:

$$\mathbf{e}_\infty = \mathbf{x}_r - \mathbf{x}_\infty = 0$$

$$\mathbf{u}_r(\mathbf{r}) = \mathbf{u}_\infty$$

$$\Rightarrow \begin{pmatrix} \mathbf{W} \\ \mathbf{V} \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C}^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

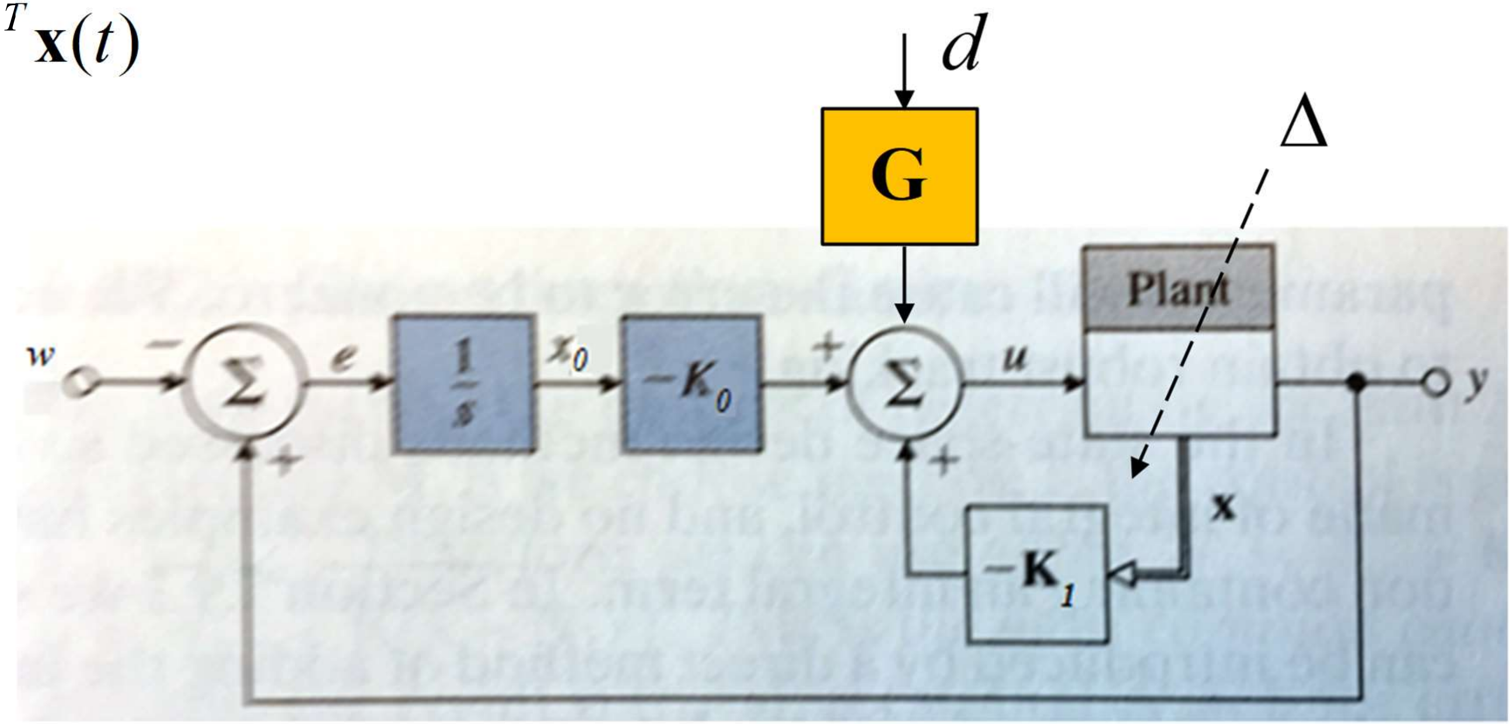
- Integral control combined with state-feedback

  – **Objective:** extend the designed state-feedback control so that to have the reference-output comparison (i.e. to have the output control error e)

    $\Rightarrow$ (i) to counteract the disturbances (d)

    (ii) to account for model uncertainties (Δ)

$$\dot{\mathbf{x}}(t) = \left(\mathbf{A} + \mathbf{\Delta}\right)\mathbf{x}(t) + \mathbf{B}u(t) + \mathbf{G}d(t)$$
$$y(t) = \mathbf{c}^T \mathbf{x}(t)$$



source: [4]

• Integral control combined with state-feedback, cont.

– Introduce an additional state variable of <u>negative</u> tracking control error (e)

$$x_0(t) = \int_0^t e\,dt \qquad \textit{new (additional) state}$$

$$\dot{x}_0(t) = \mathbf{c}^T \mathbf{x}(t) - w(t) = e(t) \qquad \textit{state dynamics (derivative of l.h.s. and r.h.s.)}$$

– For zero reference (w=0) $\Rightarrow$ control problem of a nonzero initial state

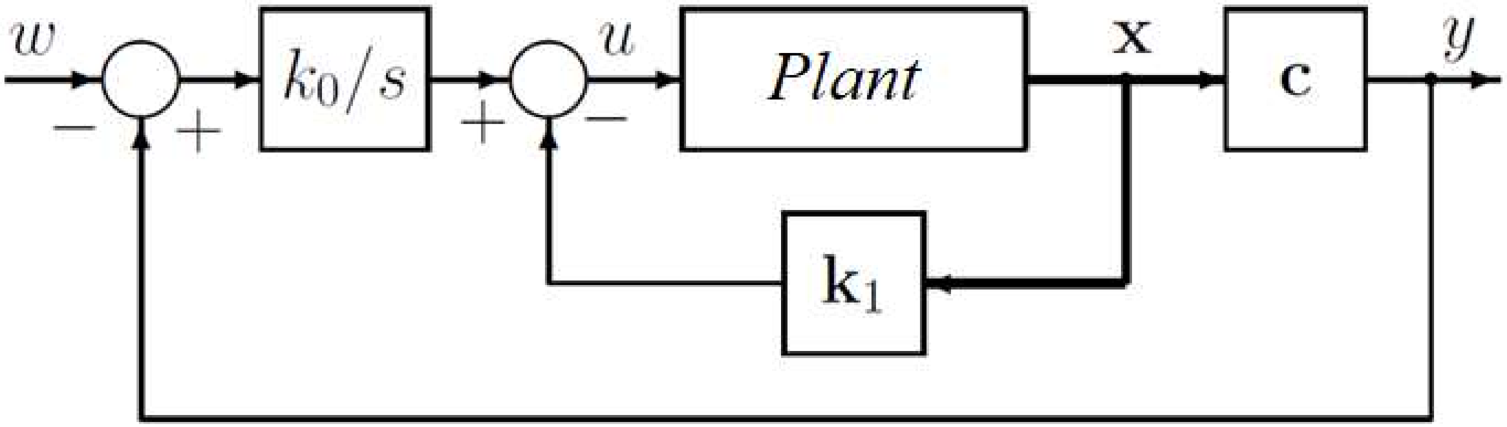
$$x_0(t) = \int_{t_0}^{t} y(t)dt = \int_{t_0}^{t} \mathbf{c}^T \mathbf{x}(t)dt$$

$$\begin{pmatrix} \dot{x}_0(t) \\ \dot{\mathbf{x}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{c}^T \\ \mathbf{0} & \mathbf{A} \end{pmatrix} \begin{pmatrix} x_0(t) \\ \mathbf{x}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ \mathbf{b} \end{pmatrix} u(t)$$

– State feedback control law $u(t) = -\begin{pmatrix} k_0 & \mathbf{k}_1^T \end{pmatrix} \begin{pmatrix} x_0(t) \\ \mathbf{x}(t) \end{pmatrix}$

- Integral control combined with state-feedback, cont.

  – Block diagram of the integral-state-feedback control



  – Overall closed-loop state-space model with reference

$$\begin{bmatrix} \dot{x}_0 \\ \dot{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} 0 & k_0 \mathbf{c}^T \\ 1 & \mathbf{A} - \mathbf{B}\mathbf{k}_1 \end{bmatrix} \begin{bmatrix} x_0 \\ \mathbf{x} \end{bmatrix} - \begin{bmatrix} k_0 \\ \mathbf{0} \end{bmatrix} w$$

  – Design of extended state-feedback control $[k_0 \; \boldsymbol{k}_1^T]$, i.e. for the integral error state and original system states, as before, e.g. by the pole placement

• Practical (hand on) session 7

Consider DC motor without load M_L (from page 49)

Assume the following parameter values: $R = 1, \quad L = 0.0002, \quad \Psi = 0.04,$

$$B = 0.0001, \quad J = 0.00005$$

1. Design the state feedback controller (by the pole placement) so that the poles of the closed-loop system are located at $\lambda_{1,2} = [-1000, -100]$.
2. For the designed state feedback controller from 1., calculate the prefilter in so as to guarantee the steady-state accuracy.
3. Implement (in MATLAB) the closed-loop control system, once from 1. without prefilter, and once from 2. with prefilter. Compare step responses.

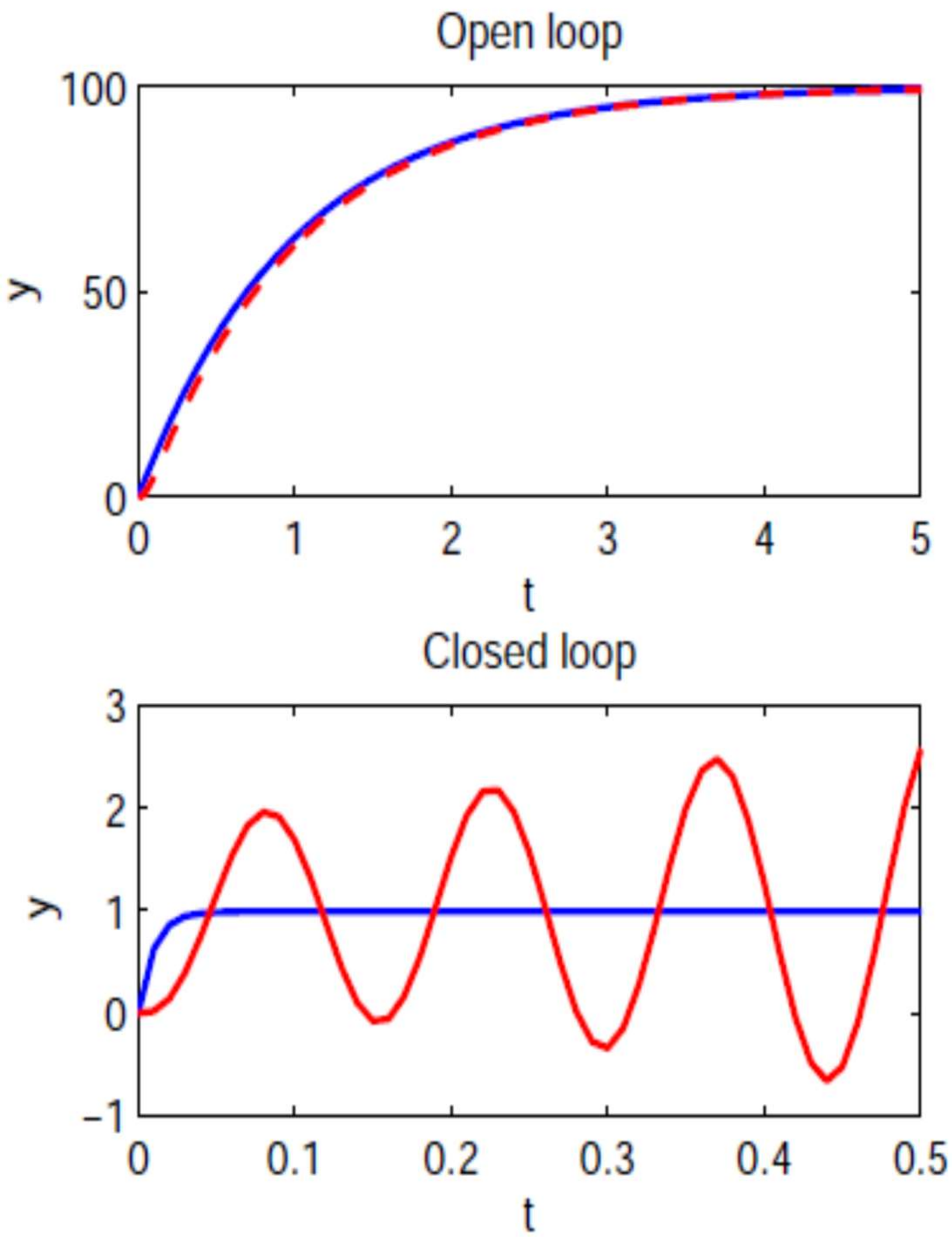
Feedback control systems for mechatronics and robotics

i. Introduction to feedback control systems
ii. Control-oriented modeling
iii. Dynamic system behavior in time and frequency domain
iv. Transfer function analysis and state-space modeling
v. Similarity forms, controllability, and observability of the systems
vi. Standard output feedback controllers
vii. State feedback controllers and prefilter extensions
viii. Stability analysis and robust control design
ix. Motion-, force- and impedance-control in mechatronics and robotics
x. Use of observers and estimators in feedback control systems

Michael Ruderman
michael.ruderman@uia.no

Stability analysis and robust control design

- Motivating examples of stable and unstable systems



1st example of two systems

Open-loop transfer functions (e.g. plants)

$$P_1(s) = \frac{100}{s+1}$$

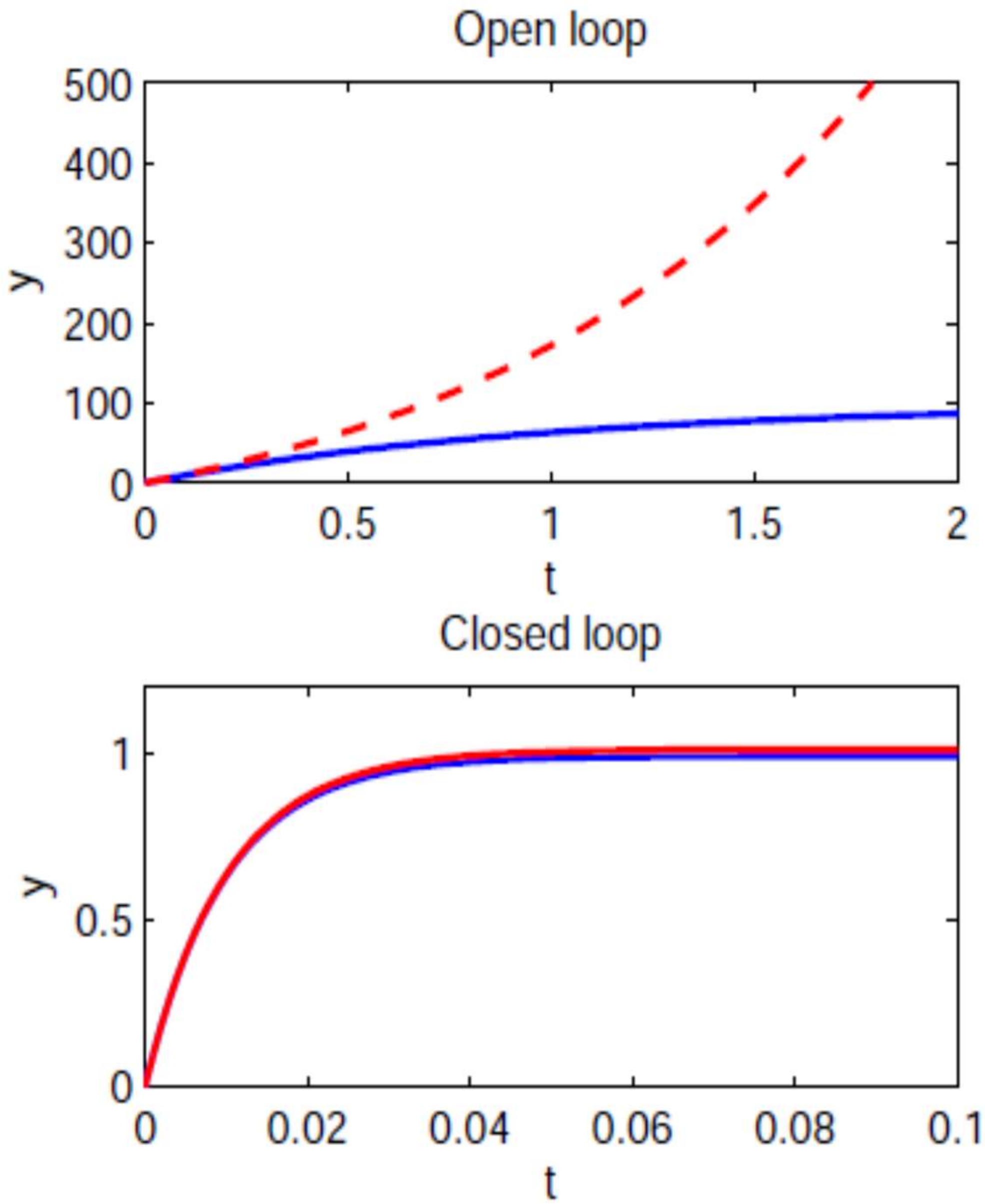
$$P_2(s) = \frac{100}{(s+1)(sT+1)^2}$$

Closed-loop transfer functions for the above plants $P_{1,2}$ (parameter T=0.025)

$$T_1 = \frac{100}{s+101}$$

$$\boxed{T_2 = \frac{100}{0.000625s^3 + 0.05063s^2 + 1.05s + 101}}$$

- Motivating examples of stable and unstable systems, cont.



2nd example of two systems

Open-loop transfer functions (e.g. plants)

$$P_1(s) = \frac{100}{s+1}$$

$$\boxed{P_2(s) = \frac{100}{s-1}}$$

Closed-loop transfer functions for the above plants $P_{1,2}$

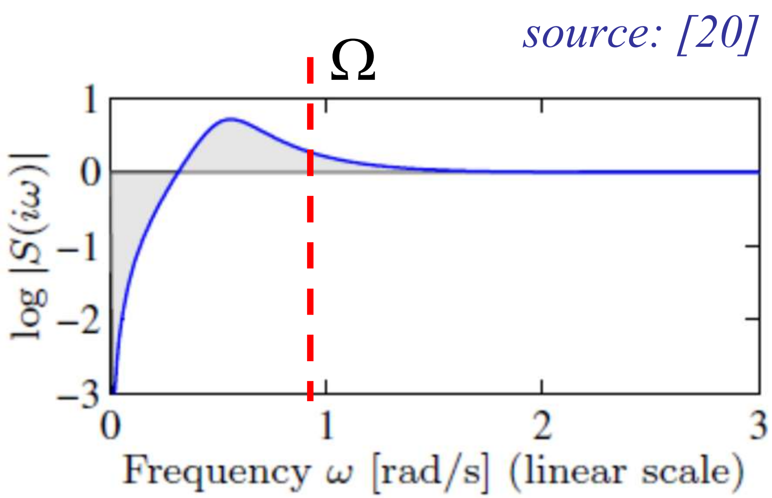
$$T_1(s) = \frac{100}{s+101}$$

$$T_2(s) = \frac{100}{s+99}$$

- ## Bode integral

  – Sensitivity function

$$S(j\omega)=\frac{E(j\omega)}{R(j\omega)}=\frac{1}{1+C(j\omega)P(j\omega)}$$



  – For stable plants: $\boxed{\int_0^{\infty}\log|S(j\omega)|\,d\omega=0}$

  – For unstable plants: $\int_0^{\infty}\log|S(j\omega)|\,d\omega=\pi\sum_i \mathrm{Re}(p_i)$, $\quad p_i$: the unstable poles

- ## Available bandwidth ($\Omega<\infty$)

  – is inherently limited due to:

  i. *uncertain or non-modeled dynamics of the plant*

  ii. *digital control implementation and power limits*

  iii. *nonlinearities and others*

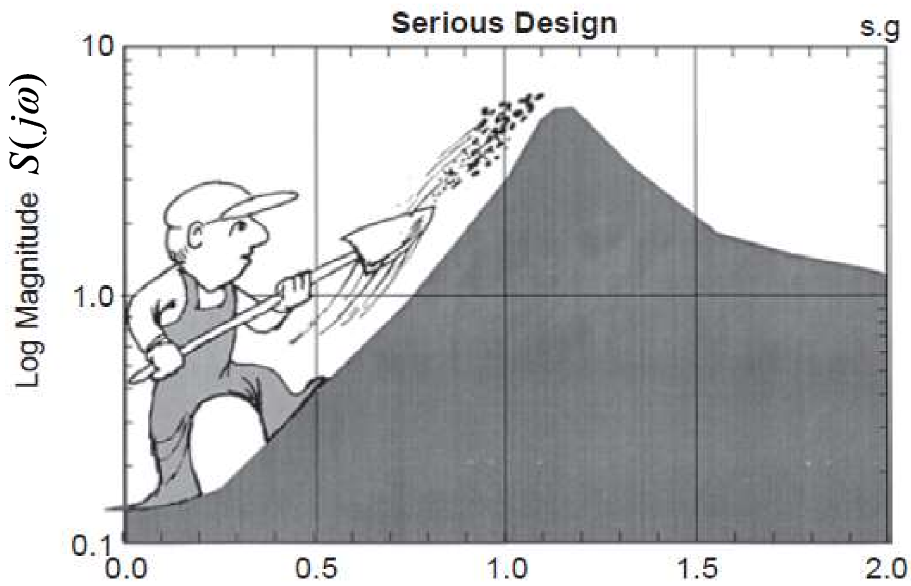
$$\Rightarrow \int_0^{\Omega}\log|S(j\omega)|\,d\omega=\frac{\text{const}}{\Omega}>0$$

- Interpretation of Bode integral

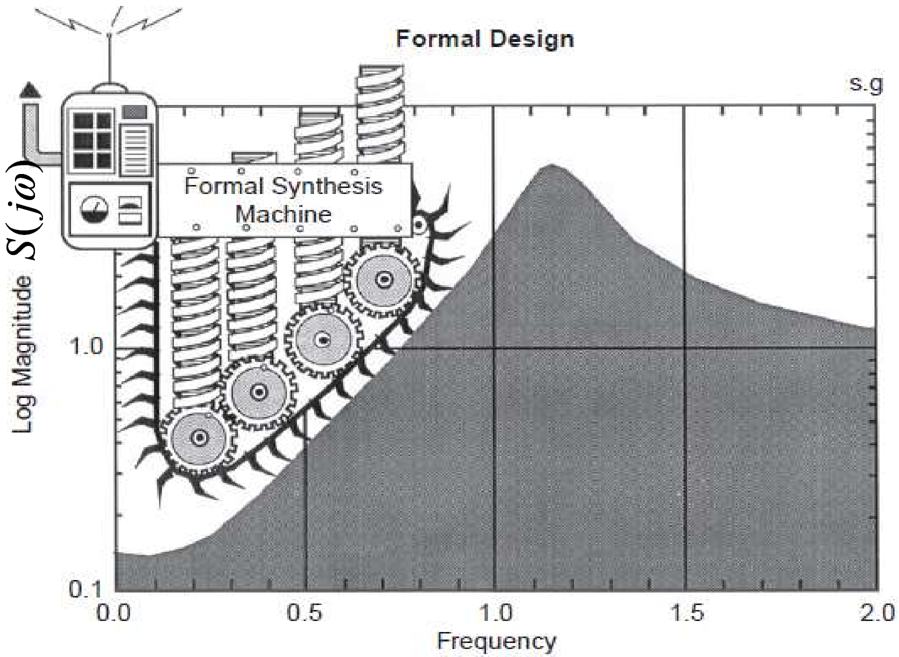
source: [6]

Sensitivity reduction at lower frequency leads unavoidably (!) to the sensitivity increase at the higher frequencies

– <u>“Serious” (but manual) design</u>



– <u>“Formal” (automatic) design</u>



More sophisticated (formal) design tools can provide a more “fine” shape/contour of $S(j\omega)$. But the Bode integral will held!

- BIBO (bounded-input-bounded-output) stability

– For LTI systems, with the impulse response $g(t)$ and output

$$y(t)=\int_0^\infty g(t-\tau)u(\tau)d\tau,$$

the BIBO stability requires that for any bounded input signal/function the output is also a bounded signal/function:

$$|u(t)| \le \max u < \infty \quad \Rightarrow \quad |y(t)| \le \max y < \infty$$

- LTI system with impulse response $g(t)$ is BIBO-stable iff

$$\boxed{\int_0^\infty |g(\tau)|d\tau < \infty} \quad \Rightarrow \quad \max y = \max u \cdot \int_0^\infty |g(\tau)|d\tau$$

Recall that integration of impulse response yields the step response function

$$h(t)=\int_0^\infty g(\tau)d\tau.$$ This implies for BIBO systems $\lim_{t\to\infty} h(t) = const$

- Example of not BIBO-stable system: water tank

ODE of the system plant $A\rho\dot{y}(t)=u(t)$

$$\Rightarrow \quad G(s)=\frac{y(s)}{u(s)}=\frac{1}{A\rho s} \quad \Rightarrow \quad g(t)=\frac{1}{A\rho}1(t)$$

$$\Rightarrow \quad \int_0^\infty |g(\tau)|\,d\tau=\frac{1}{A\rho}\int_0^\infty 1\,dt \;\rightarrow\infty \quad \text{not bounded}$$

- Example of BIBO-stable system: feedback-controlled water tank

ODE of the closed-loop system $A\rho\dot{y}(t)=K_p\left(y_{ref}(t)-y(t)\right)$

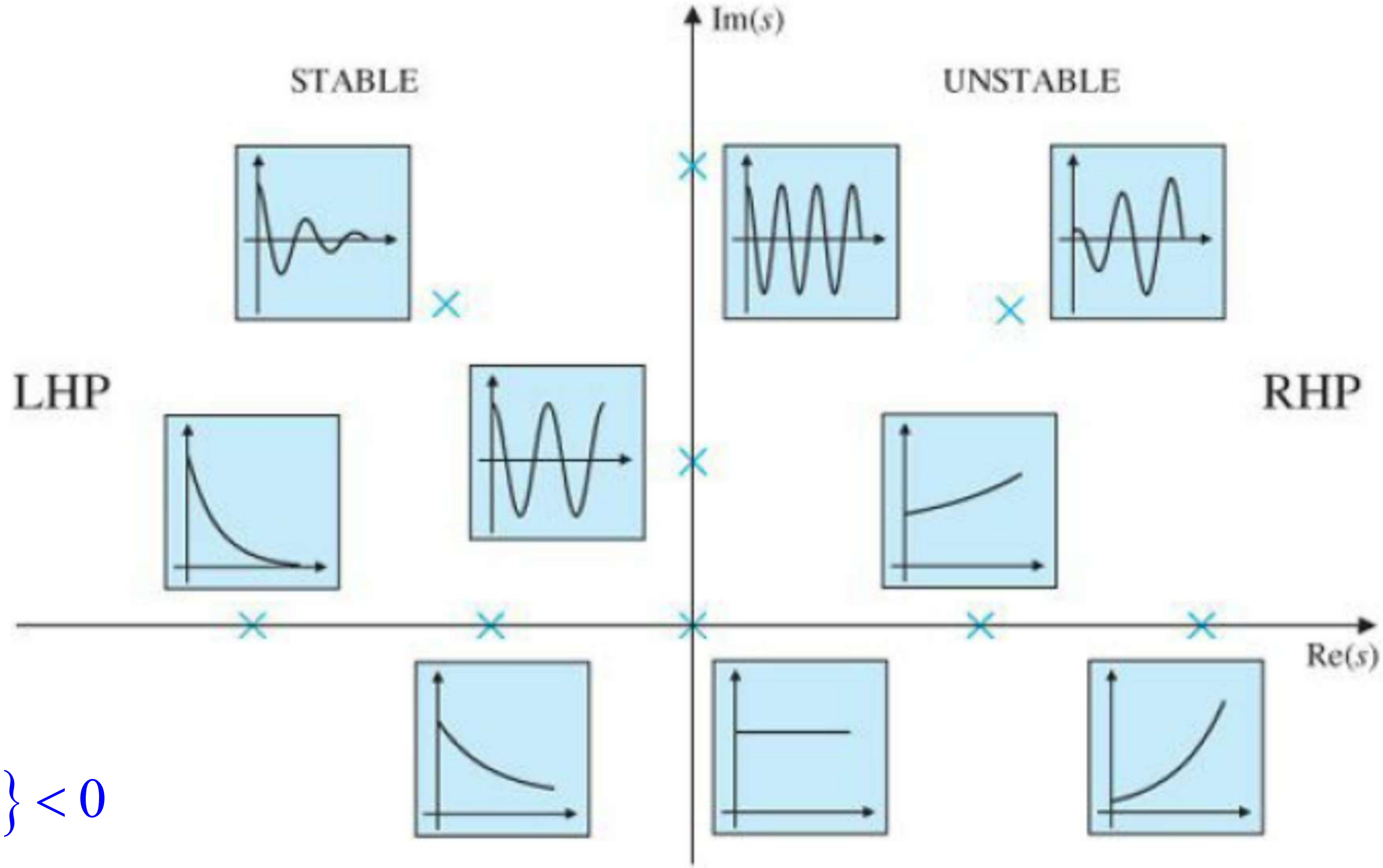
$$\Rightarrow \quad G(s)=\frac{y(s)}{y_{ref}(s)}=\frac{K_p}{A\rho s+K_p}=\frac{K_p(A\rho)^{-1}}{s+K_p(A\rho)^{-1}} \quad \Rightarrow \quad g(t)=\frac{K_p}{A\rho}e^{-\frac{K_p}{A\rho}t}$$

$$\Rightarrow \quad \int_0^\infty |g(\tau)|\,d\tau=1-e^{-\frac{K_p}{A\rho}t} \;\rightarrow 1 \quad \text{bounded}$$

- Necessary and sufficient condition for stability of LTI systems

An LTI system is said to be stable if all the roots of the transfer function denominator polynomial have negative real parts (i.e., they are all in the left hand s-plane) and is unstable otherwise.

source: [4]



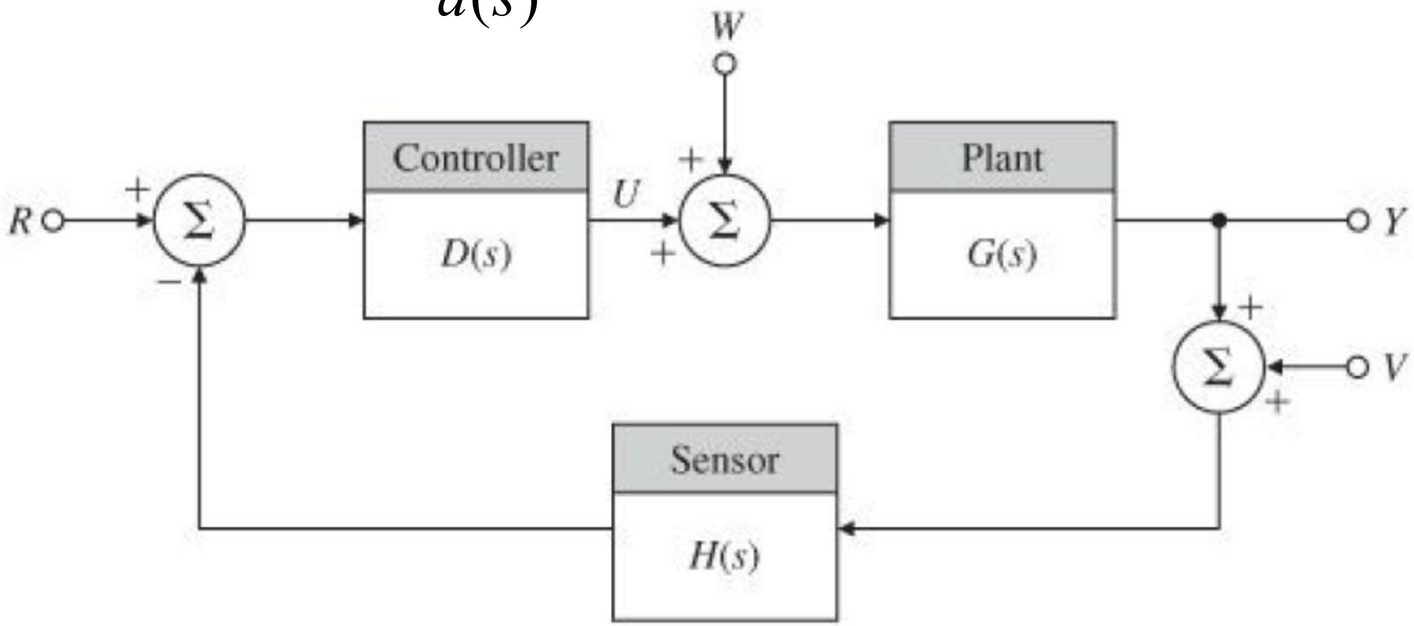
$$\mathrm{Re}\{\lambda_i\} < 0$$

- Root locus analysis for stability evaluation

Appliable to the open-loop transfer functions

$$L(s) = D(s)G(s)H(s) = \frac{b(s)}{a(s)}$$

source: [4]



To check characteristic equation of closed-loop system in dependency of K

$$1 + KL(s) = 0 \;\Rightarrow\; 1 + K\frac{b(s)}{a(s)} = 0 \;\Rightarrow\; a(s) + Kb(s) = 0 \;\Rightarrow\; L(s) = -\frac{1}{K}$$

• Root locus analysis for stability evaluation, cont.

Illustrative Example 16: loop transfer function G with one free parameter c

$$1 + G(s) = 1 + \frac{1}{s(s+c)}$$

given denominator of the closed-loop system

source: [4]

Same characteristic equation, now in the polynomial form

$$s^2 + cs + 1 = 0$$

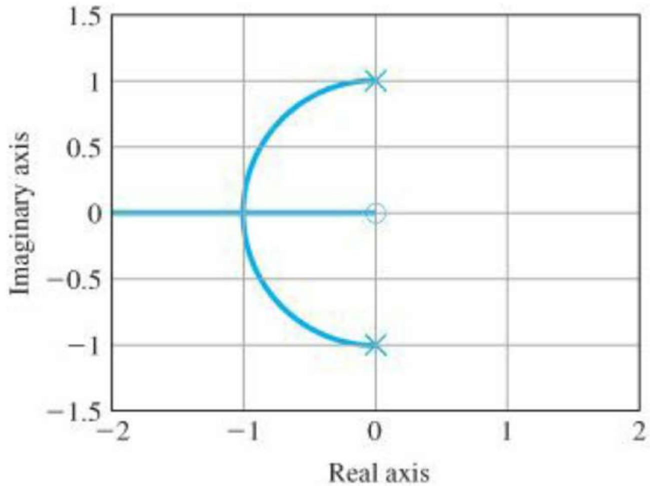
Rewriting the characteristic equation for root locus analysis

$$L = \frac{s}{s^2+1}, \qquad b(s) = s,$$
$$K = c, \qquad a(s) = s^2 + 1,$$

$$\Rightarrow \quad \boxed{1 + c\frac{s}{s^2+1} = 0}$$

The solutions, in terms of the roots depending on c

$$r_1, r_2 = -\frac{c}{2} \pm \frac{\sqrt{c^2-4}}{2}$$



- Routh stability criterion

Used for evaluating the roots of characteristic polynomial, that determines the location of poles, without solving explicitly the characteristic equation

$$a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s + a_n$$

- Necessary condition for stability by Routh criterion *source: [4]*

A necessary (but not sufficient) condition for stability is that all the coefficients of the characteristic polynomial be positive.

- Necessary and sufficient condition for stability by Routh

A system is stable if and only if all the elements in the first column of the Routh array are positive.

For constructing Routh array:

arrange the coefficients of characteristic polynomial $a(s)$ into two rows:
– first row beginning with 1 and followed by even-numbered coefficients
– second row beginning with a_1 and followed by odd-numbered coefficients

$$\begin{array}{lcllll} s^n & : & 1 & a_2 & a_4 & \cdots \\ s^{n-1} & : & a_1 & a_3 & a_5 & \ldots \end{array}$$

- # Routh stability criterion, cont.

Then, all subsequent rows of the Routh array are:

$$
\begin{array}{llll|c|ccc}
\text{Row} & n & & s^n: & 1 & a_2 & a_4 & \cdots \\
\text{Row} & n-1 & & s^{n-1}: & a_1 & a_3 & a_5 & \cdots \\
\text{Row} & n-2 & & s^{n-2}: & b_1 & b_2 & b_3 & \cdots \\
\text{Row} & n-3 & & s^{n-3}: & c_1 & c_2 & c_3 & \cdots \\
 & \vdots & & \vdots & \vdots & \vdots & \vdots & \\
\text{Row} & 2 & & s^2: & * & * & & \\
\text{Row} & 1 & & s: & * & & & \\
\text{Row} & 0 & & s^0: & * & & &
\end{array}
$$

with

$$b_1 = -\frac{\det\begin{bmatrix} 1 & a_2 \\ a_1 & a_3 \end{bmatrix}}{a_1} = \frac{a_1 a_2 - a_3}{a_1}, \qquad c_1 = -\frac{\det\begin{bmatrix} a_1 & a_3 \\ b_1 & b_2 \end{bmatrix}}{b_1} = \frac{b_1 a_3 - a_1 b_2}{b_1},$$

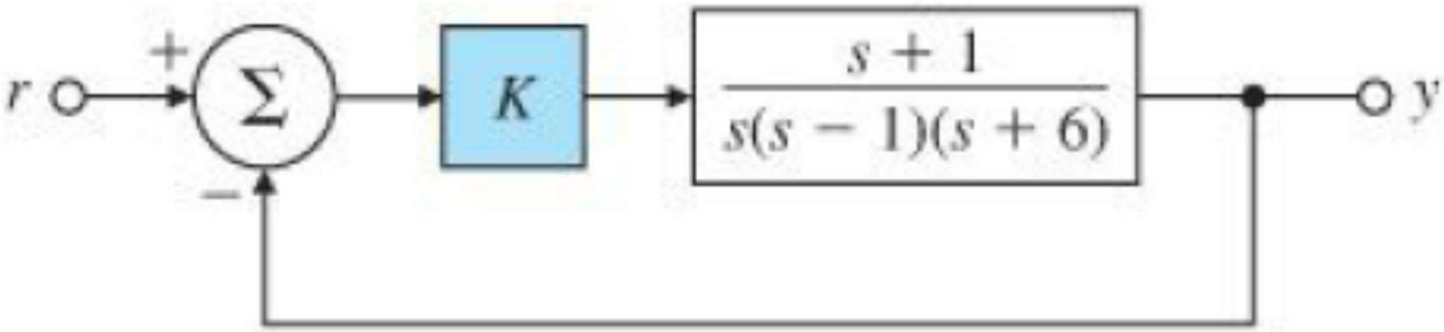
$$b_2 = -\frac{\det\begin{bmatrix} 1 & a_4 \\ a_1 & a_5 \end{bmatrix}}{a_1} = \frac{a_1 a_4 - a_5}{a_1}, \qquad c_2 = -\frac{\det\begin{bmatrix} a_1 & a_5 \\ b_1 & b_3 \end{bmatrix}}{b_1} = \frac{b_1 a_5 - a_1 b_3}{b_1},$$

$$b_3 = -\frac{\det\begin{bmatrix} 1 & a_6 \\ a_1 & a_7 \end{bmatrix}}{a_1} = \frac{a_1 a_6 - a_7}{a_1}, \qquad c_3 = -\frac{\det\begin{bmatrix} a_1 & a_7 \\ b_1 & b_4 \end{bmatrix}}{b_1} = \frac{b_1 a_7 - a_1 b_4}{b_1}.$$

If the elements of 1st column of Routh array are not all positive, then the number of unstable roots (poles in RHP) equals the number of sign changes in the column.

- Routh stability criterion: use for parameters/gains variation

Illustrative Example 17 *source: [4]*



Which feedback gain is required? (note that one pole is already unstable)

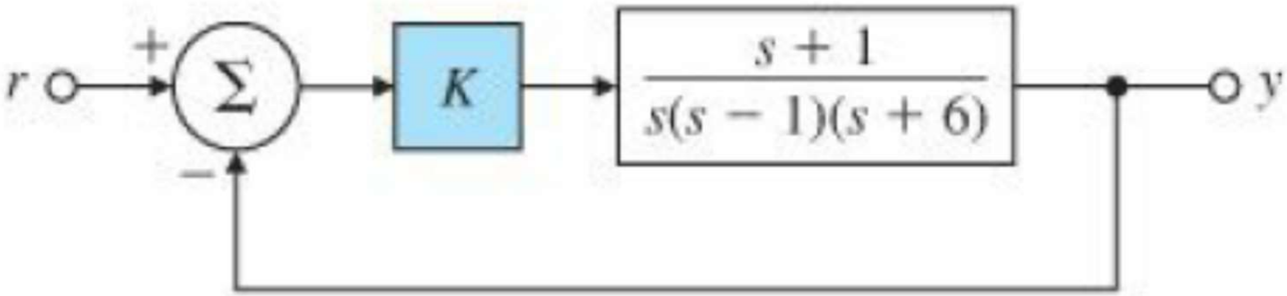
The characteristic equation

$$1 + K\frac{s+1}{s(s-1)(s+6)} = 0, \quad \Rightarrow \quad s^3 + 5s^2 + (K-6)s + K = 0.$$

The corresponding Routh array is

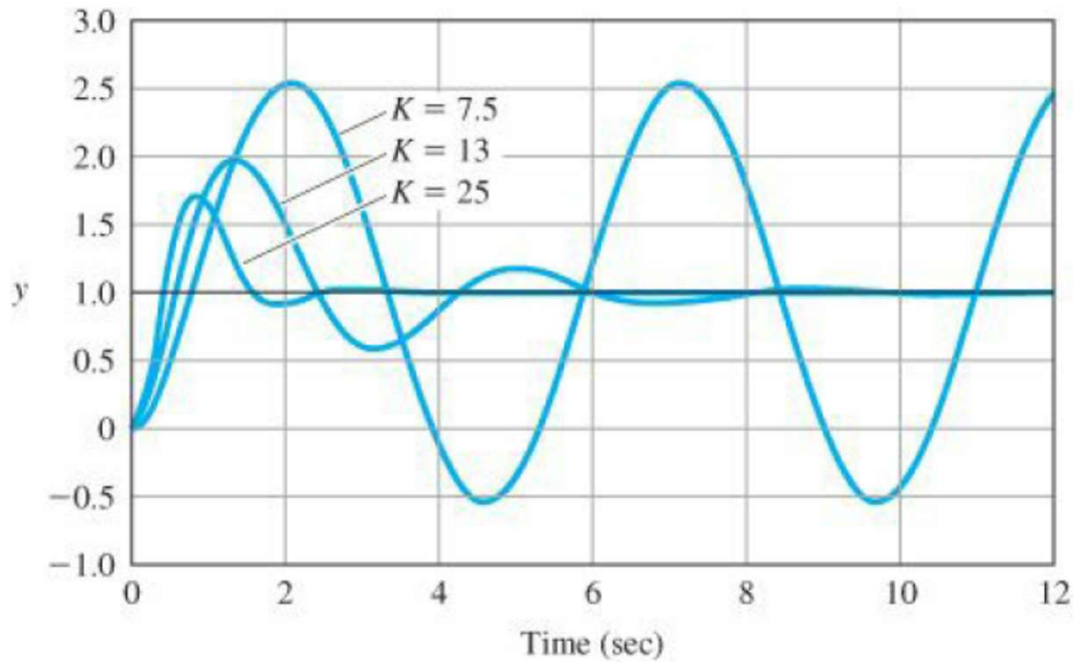
$$\begin{array}{lcc} s^3: & 1 & K-6 \\ s^2: & 5 & K \\ s\;: & (4K-30)/5 & \\ s^0: & K. & \end{array}$$

Illustrative Example 17, cont.



For the closed-loop system is stable (i.e. for Routh criterion is fulfilled):

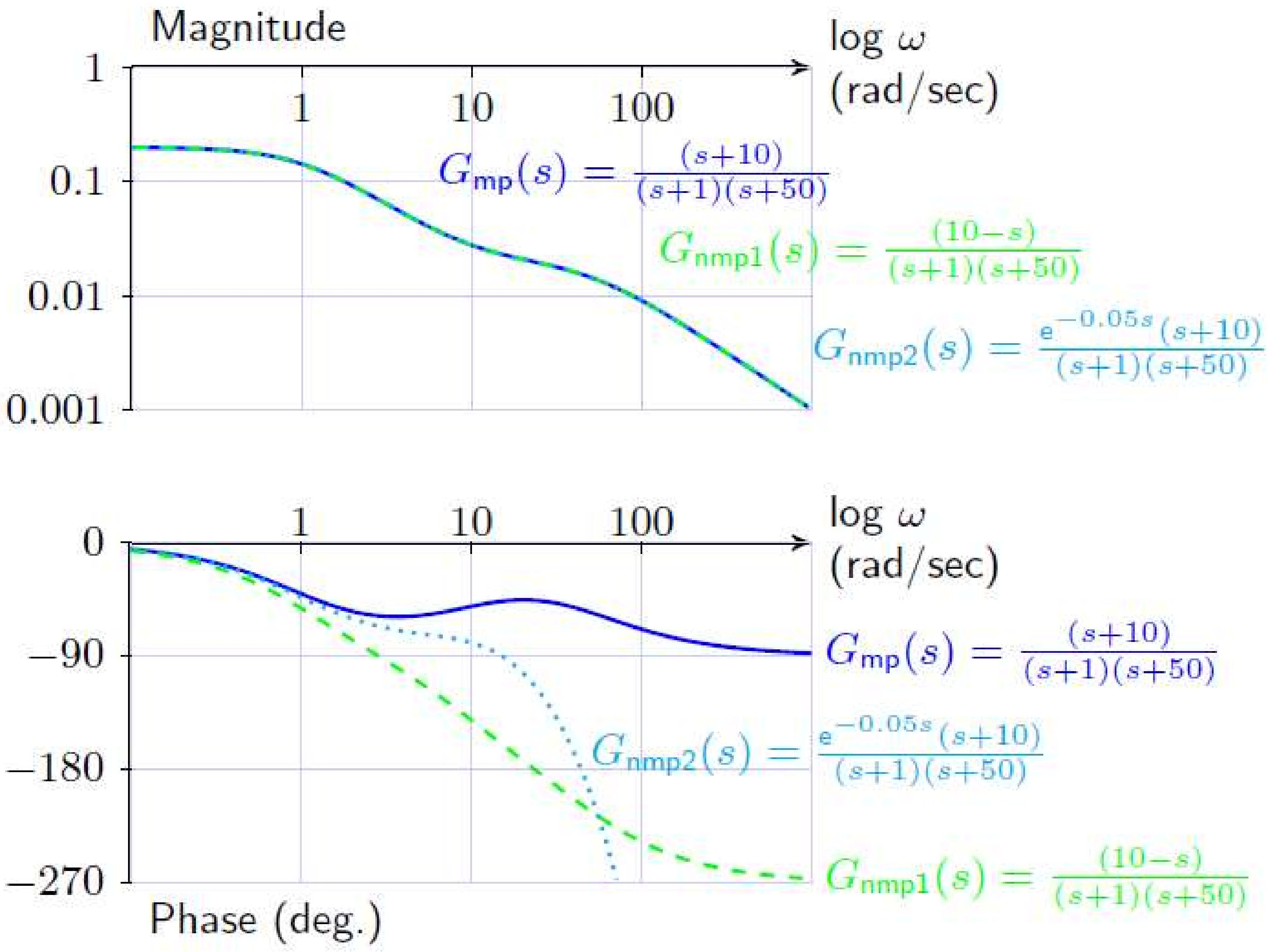
$$\frac{4K - 30}{5} > 0 \quad \text{and} \quad K > 0, \qquad \Rightarrow \quad \mathrm{K} > 7.5 \text{ and } \mathrm{K} > 0.$$



source: [4]

- Why phase response is as important as magnitude response?

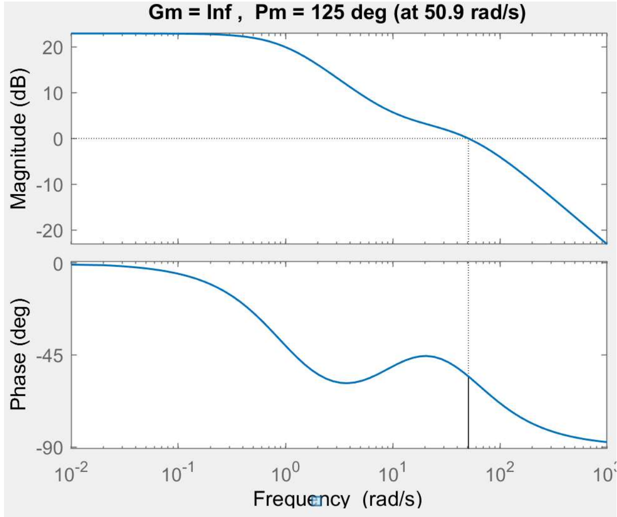
Example 18: compare Bode diagrams of the 3 transfer functions



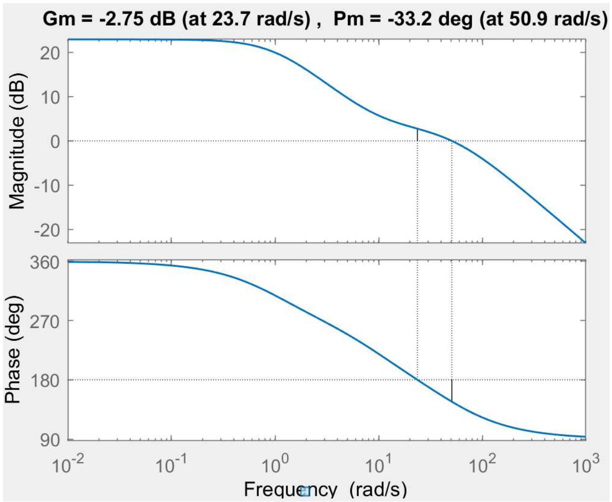
Example 18: compare Bode diagrams of the 3 transfer functions, cont.

Now, consider these loop transfer functions but with additional gain K=70

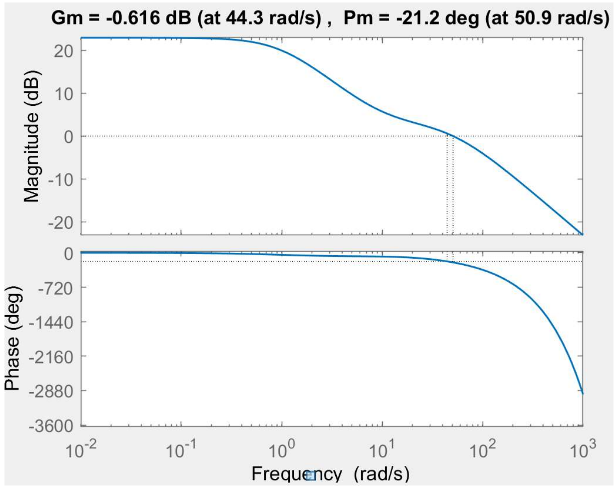
$$G_1(s) = \frac{70\,(s+10)}{(s+1)(s+50)}$$



$$G_2(s) = \frac{70\,(-s+10)}{(s+1)(s+50)}$$



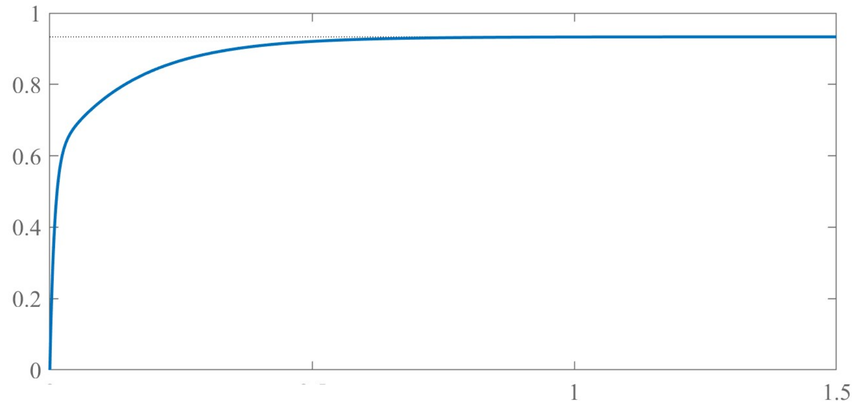
$$G_3(s) = \frac{70\,(s+10)}{(s+1)(s+50)}\,e^{-0.05s}$$



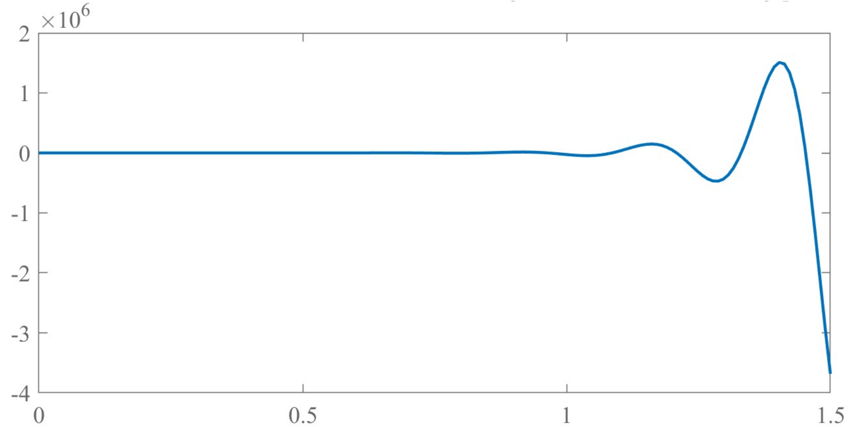
Example 18: compare Bode diagrams of the 3 transfer functions, cont.

The closed-loop response for $G_{1,2,3}$ (from the previous page)

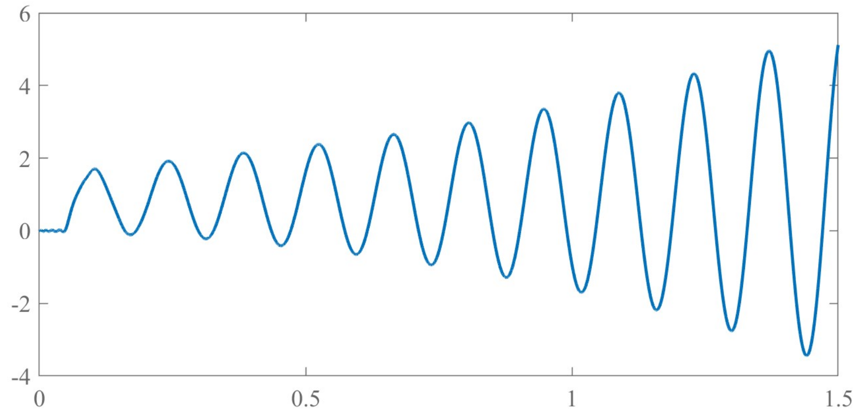
$$G_{cl,1}(s) = \frac{G_1(s)}{1+G_1(s)}$$



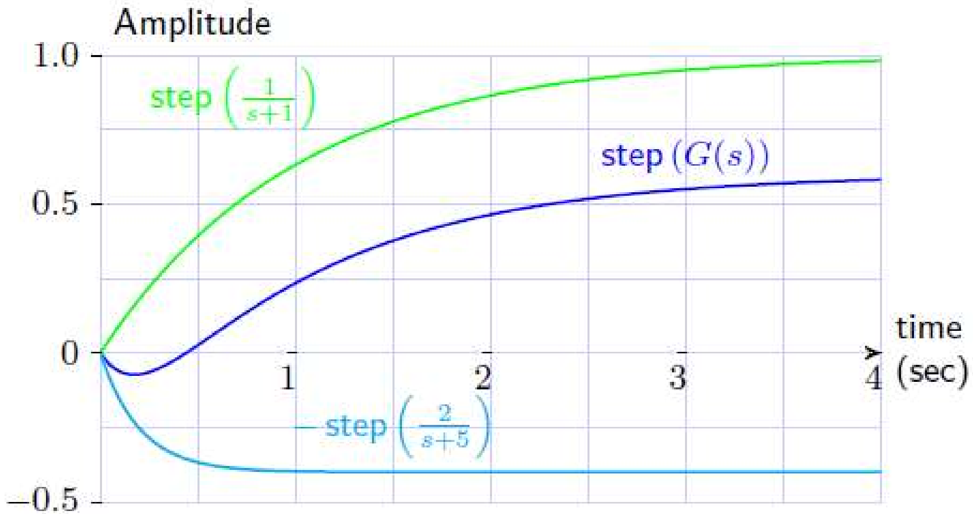
$$G_{cl,2}(s) = \frac{G_2(s)}{1+G_2(s)}$$



$$G_{cl,3}(s) = \frac{G_1(s)\,\mathrm{e}^{-0.05s}}{1+G_1(s)\,\mathrm{e}^{-0.05s}}$$

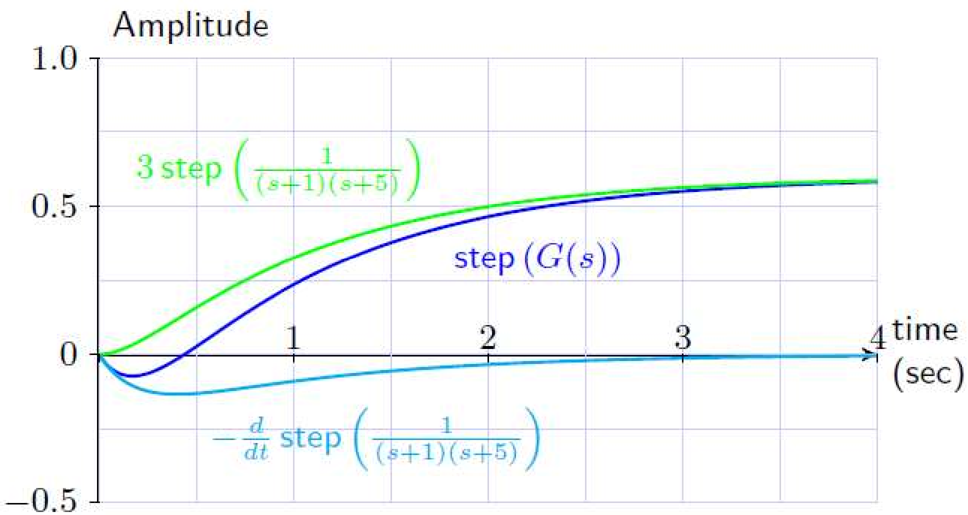


- Non-minimum-phase systems, i.e. with unstable zero(s)



Superposition of two PT_1 elements, one with positive and other with negative gain

$$G(s) = \frac{1}{s+1} - \frac{2}{s+5} = \boxed{\frac{3-s}{(s+1)(s+5)}}$$



It can equally be interpreted as a negative derivative response

$$G(s) = \frac{3}{(s+1)(s+5)} - \frac{s}{(s+1)(s+5)}$$

$$g(t) = 3\left(\frac{1}{4}e^{-t} + \frac{-1}{4}e^{-5t}\right) - \frac{d}{dt}\left(\frac{1}{4}e^{-t} + \frac{-1}{4}e^{-5t}\right)$$

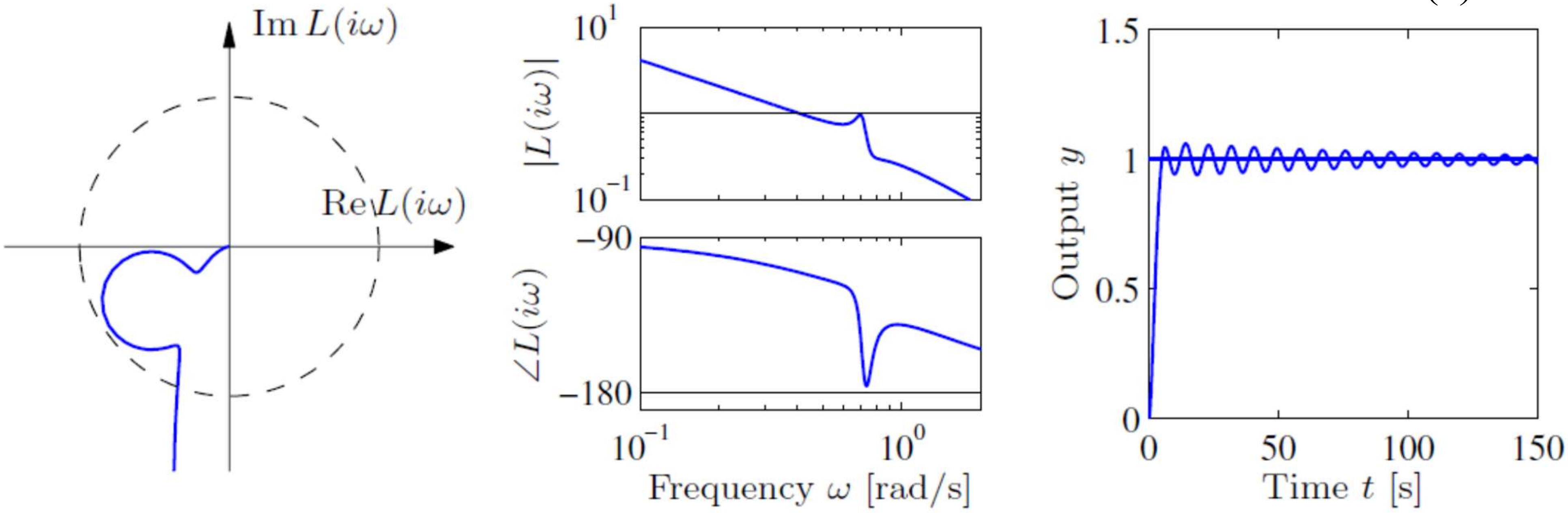
- Some shortcomings of gain and phase margins

Example 19: sufficient gain and phase margins, but (!) poor stability

$$L(s) = \frac{0.38(s^2 + 0.1s + 0.55)}{s(s+1)(s^2 + 0.06s + 0.5)}$$

source: [20]

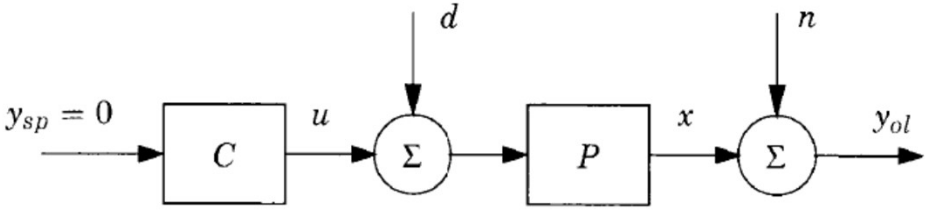
$$G_{cl}(s) = \frac{L(s)}{1 + L(s)}$$

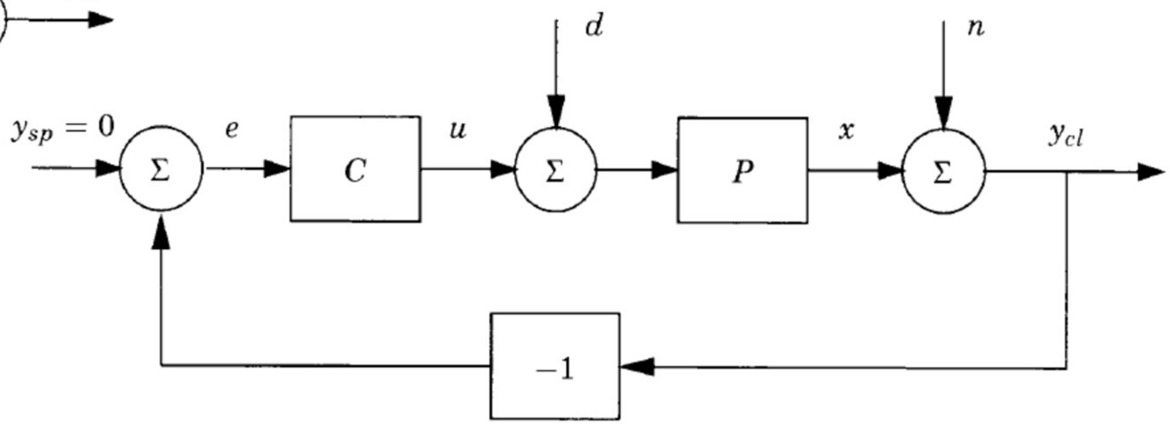


(a) Nyquist plot (b) Bode plot (c) Step response

- Why sensitivity function S is then so important?

Consider the open- and closed-loop systems with the same disturbance d



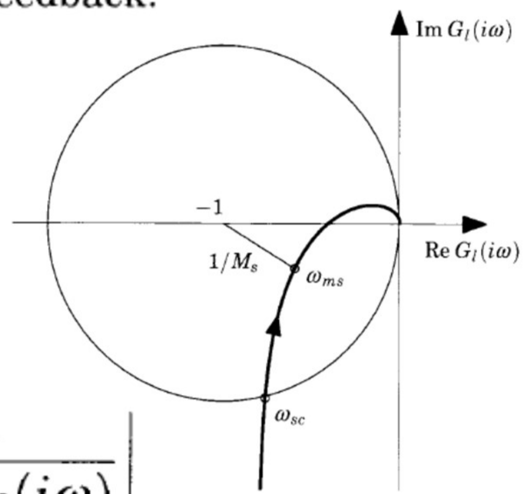


$$\underbrace{\frac{Y_{cl}(s)}{Y_{ol}(s)} = S(s) = \frac{1}{1+C(s)P(s)}}$$

$$\Downarrow$$

Disturbances with frequencies ω such that $|S(i\omega)| < 1$ are thus attenuated by feedback, but disturbances such that $|S(i\omega)| > 1$ are amplified by the feedback.

- Maximum of sensitivity function magnitude $|\,S\,|$ can be seen as a stability margin (M_s)

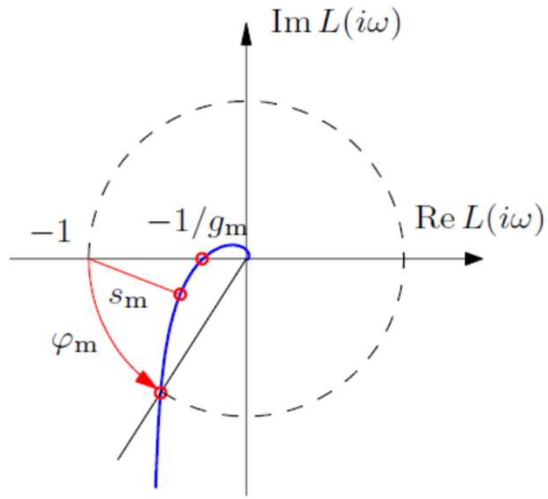


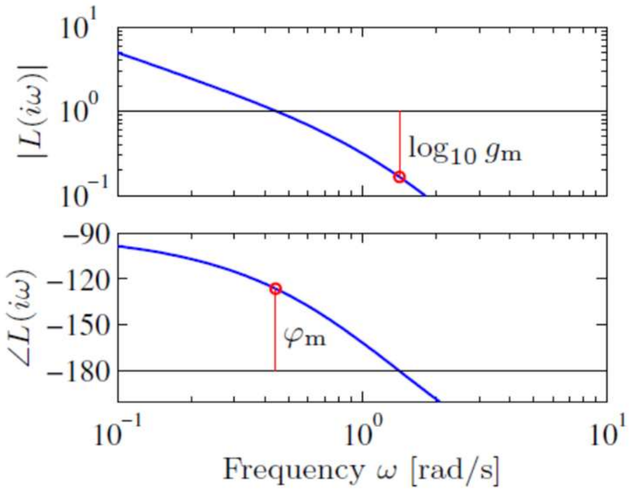
$$M_s = \max_\omega |S(i\omega)| = \max_\omega \left| \frac{1}{1+P(i\omega)C(i\omega)} \right| = \max_\omega \left| \frac{1}{1+G_l(i\omega)} \right|$$

• Stability margin

Consider <u>shortest distance</u> s_m from the Nyquist curve to the critical point -1

Remark: stability margin frequency lies between the gain-crossover and phase-crossover frequencies

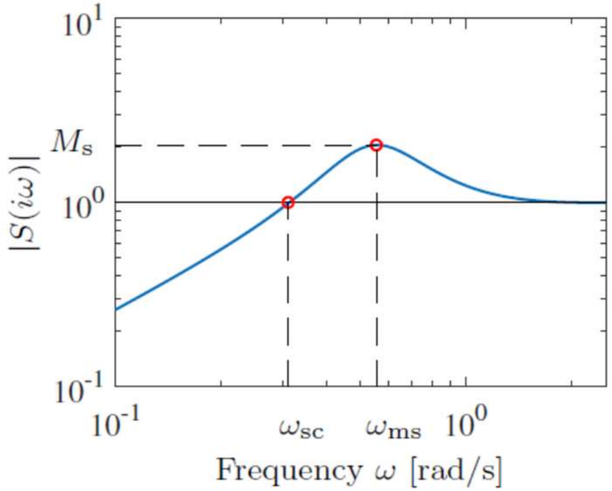


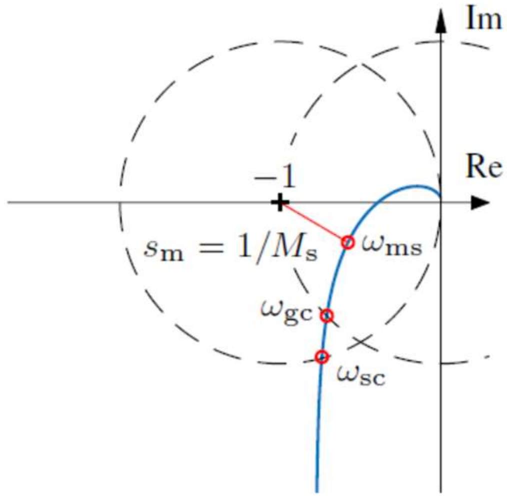


source: [20]

From the sensitivity function's viewpoint

$$S(s) = \frac{1}{1 + L(s)}$$





Recall that the control error has:

$$E_r(s) = S(s) \cdot \mathit{reference}(s)$$

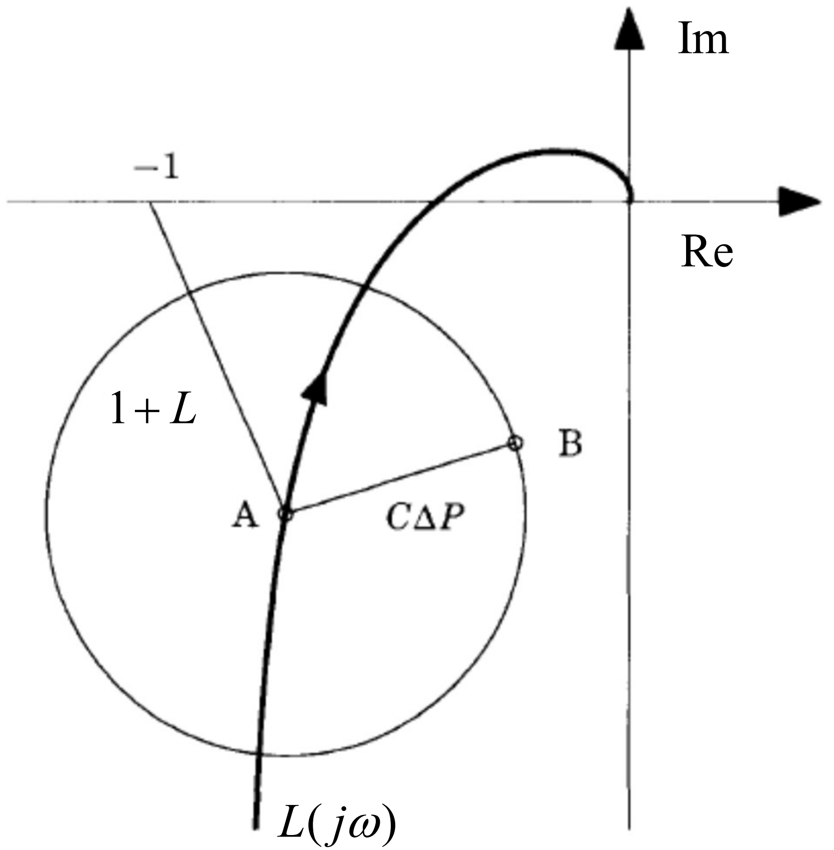
$$E_d(s) = S(s)G(s) \cdot \mathit{disturbance}(s)$$

- Stability margin for plant variations

If within nominal loop function $L = CP$ the plant P is varying as $P + \Delta P$

Then, the open-loop transfer function changes to $CP + C\Delta P$

For the plant variations ΔP, each point A on the Nyquist plot changes to a circle of the points B with the radius $C\Delta P$

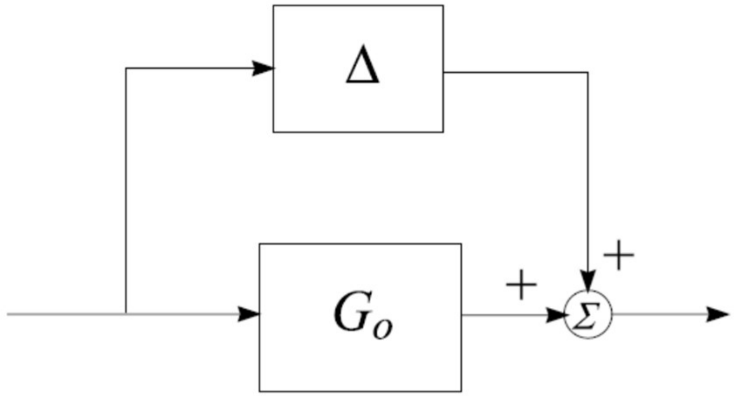


Then, for not violating the critical point −1, i.e. for not destabilizing the closed-loop control system, one needs to ensure

$$\left|C\Delta P\right| < \left|1+L\right| \qquad \Rightarrow \qquad \left|\Delta P\right| < \left|\frac{1+L}{C}\right|$$

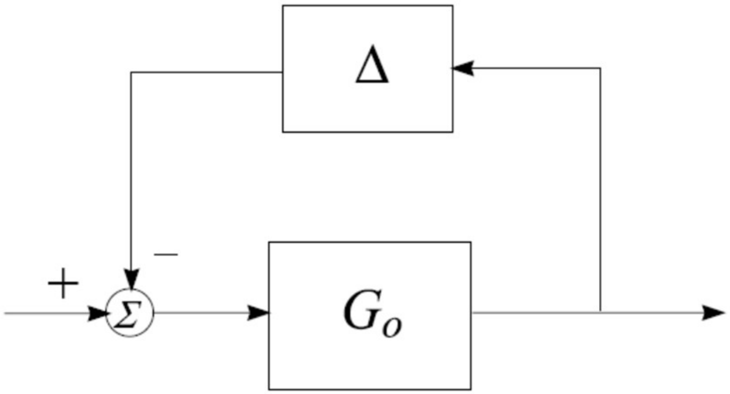
- Unstructured uncertainties of the plant transfer function are equivalent to the perturbations in frequency domain

  – Additive perturbations



$$G_p(s) = G_o(s) + \Delta(s)$$

  – Inverse additive perturbations



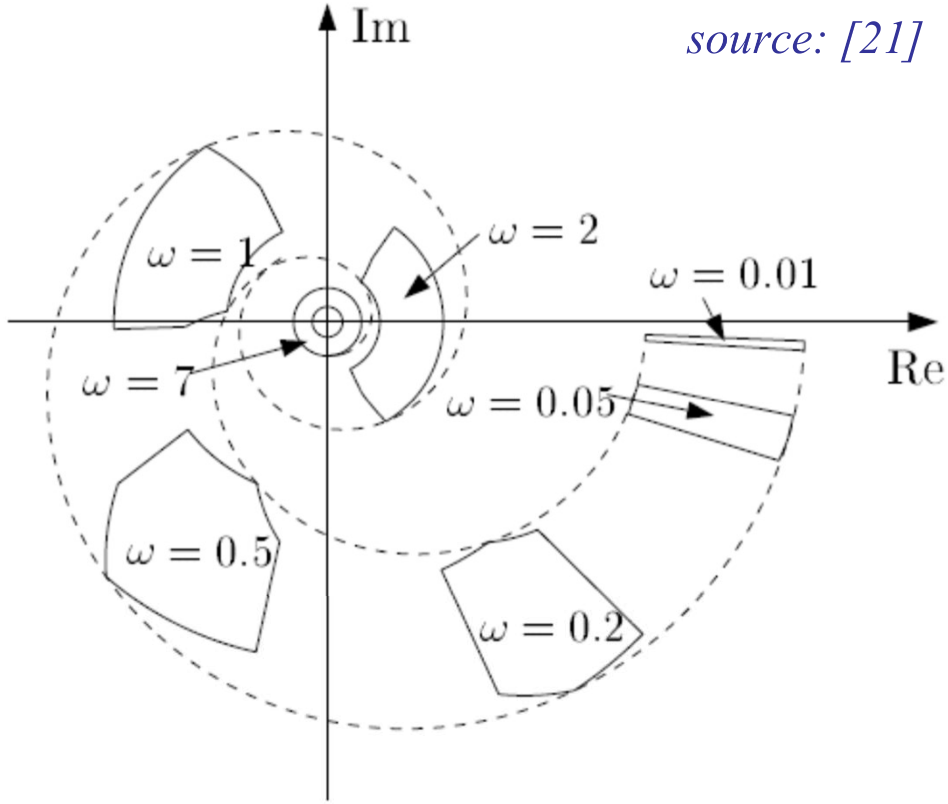
$$(G_p(s))^{-1} = (G_o(s))^{-1} + \Delta(s)$$

• Additive uncertainties in frequency domain

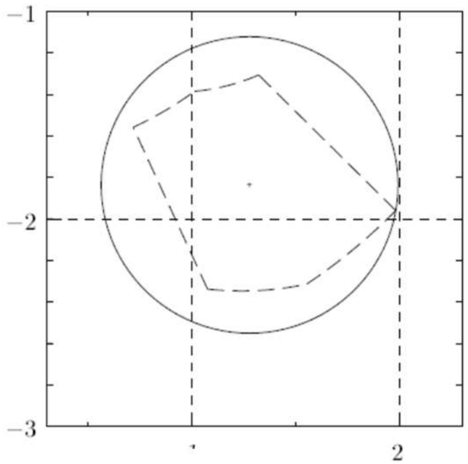
Example 20: Nyquist plot of the open-loop G_p with uncertain parameters

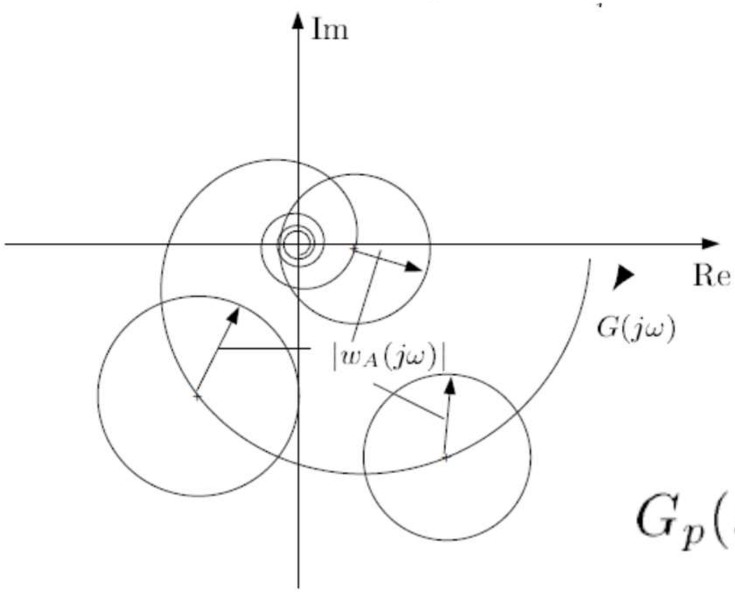
$$G_p(s) = \frac{k}{\tau s + 1} e^{-\theta s}, \quad 2 \le k, \theta, \tau \le 3$$

source: [21]



With circles of additive uncertainty





One can approximate such regions by circles, resulting in a (complex) <u>additive uncertainty</u>

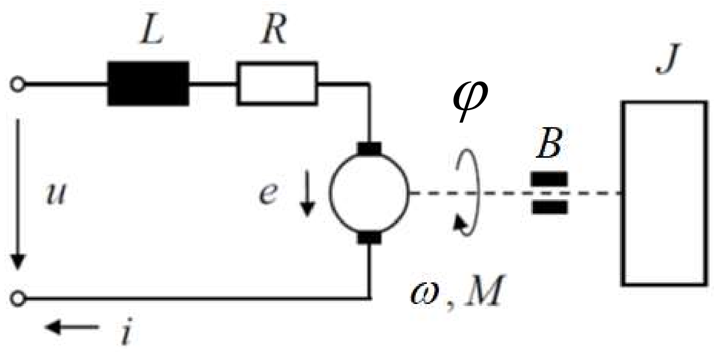
$$G_p(s) = G(s) + w_A(s)\Delta_A(s); \quad |\Delta_A(j\omega)| \le 1 \, \forall \omega$$

• Practical (hand on) session 8

Consider DC motor with position (!) output φ (from page 49)

Assume the following parameter values:

$$R = 1, \quad L = 0.0002, \quad \Psi = 0.04,$$
$$B = 0.0001, \quad J = 0.00005$$



1. Assuming the proportional feedback control (with K_p gain) of position, determine the loop transfer function, while $K_p > 0$ is first unknown.
2. Use the Routh stability criterion for determining the range of possible K_p.
3. Now, the range of possible K_p and, therefore, the stability of the closed-loop control system must be analyzed by using the root locus.

Feedback control systems for mechatronics and robotics

Michael Ruderman
michael.ruderman@uia.no

Motion-, force- and impedance-control in mechatronics and robotics

- Relation between motion- and force-control

Totally imposed force (Σ all forces) in a mechanical controlled system

$$F = g(x, \dot{x}, t)$$

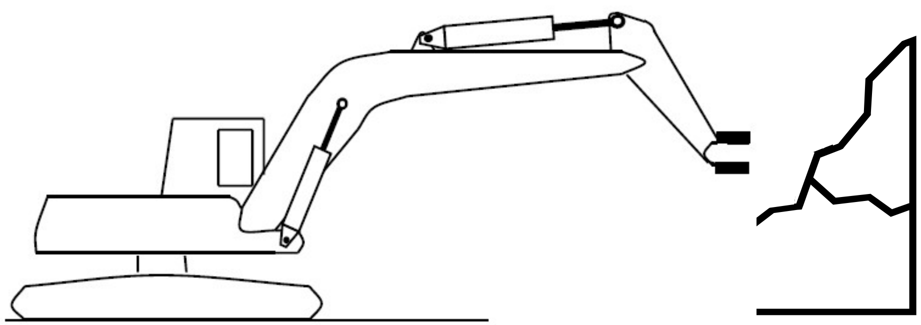
Stiffness of the controlled motion system $K = \dfrac{\partial F}{\partial x}$

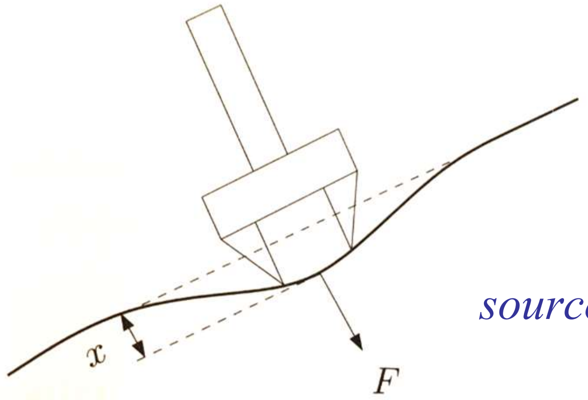
Ideal **motion** control $K \to \infty$

Ideal **force** control $K \to 0$

(varying) impedance control as a ‘trade-off’ in-between

Often used in applications to switch between motion- and force-control





source: [17]

• Relation between motion- and force-control, cont.

Stiffness of motion as one of the main features of the controlled system

$$K = \frac{\partial F}{\partial X}$$

Motion-Control Techniques of Today and Tomorrow

MICHAEL RUDERMAN, MAKOTO IWASAKI, and WEN-HUA CHEN

A Review and Discussion of the Challenges of Controlled Motion

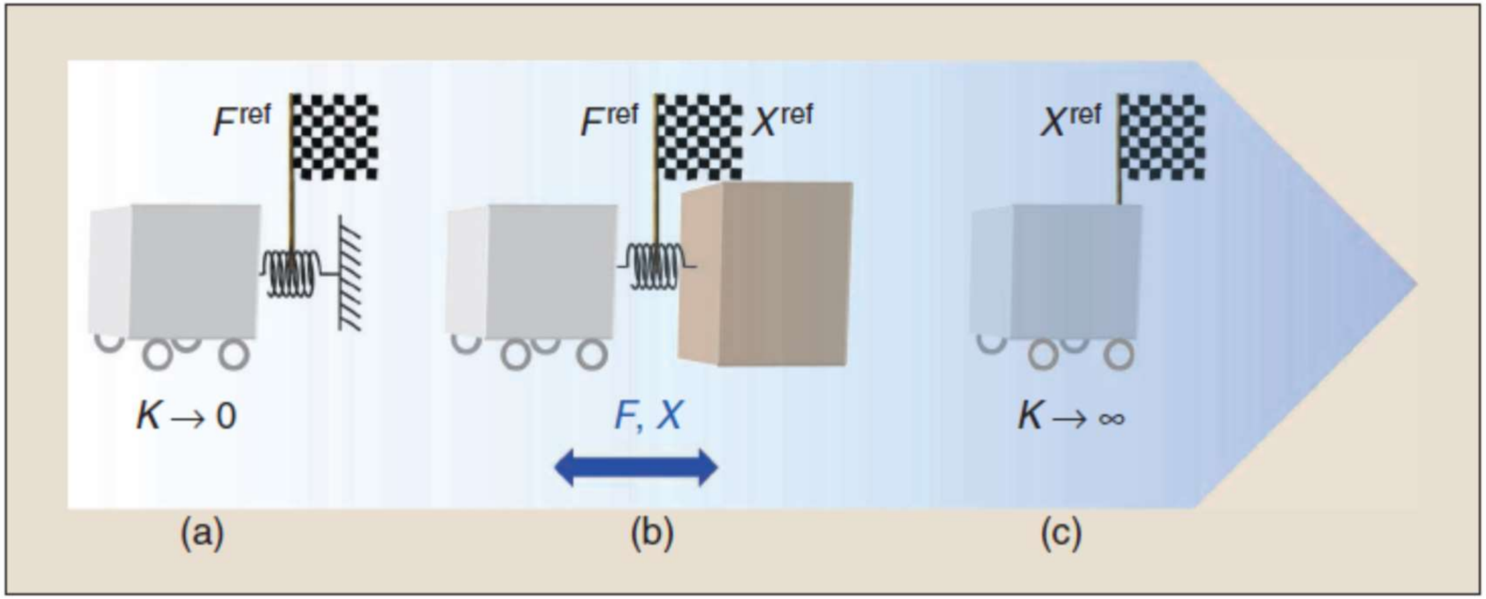


FIGURE 1 – The stiffness of controlled motion for the (a) force, (b) impedance, and (c) position control. The motion stiffness increases from the left to the right, while the flag represents the reference set value.

source: [24]

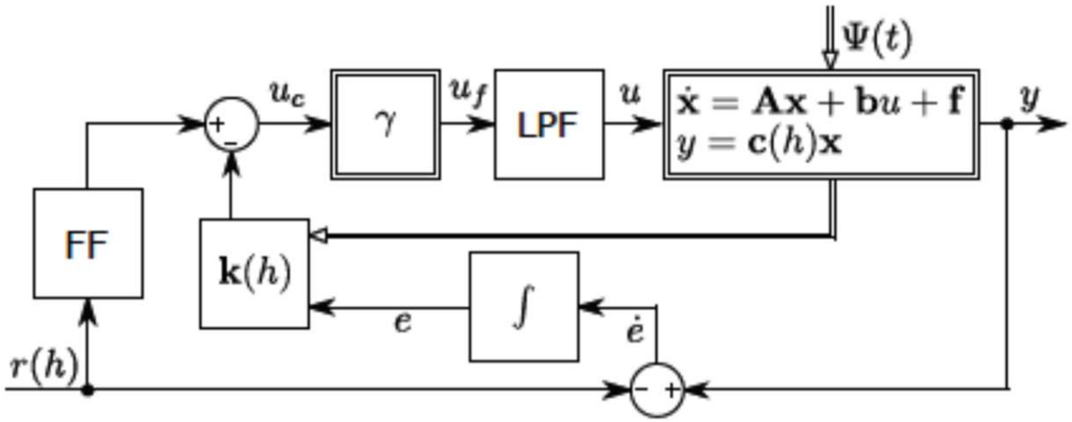
- Motion and force control example in mechatronics

  - Hybrid control framework

  - Control law

$$u_c = \mathrm{FF}\, r(h) - \mathbf{k}(h)\mathbf{x}_e$$

$$\mathbf{x}_e = (\mathbf{x}^T, e)^T$$



source: [26]

$$\mathbf{x} \;=\; (x, \dot{x}, P_L, F_L)^T \qquad \mathbf{k}(h) \;=\; (K_1, K_2, K_3, K_4, K_i)$$

  - Overall hybrid control system (with affine term due to system linearization)

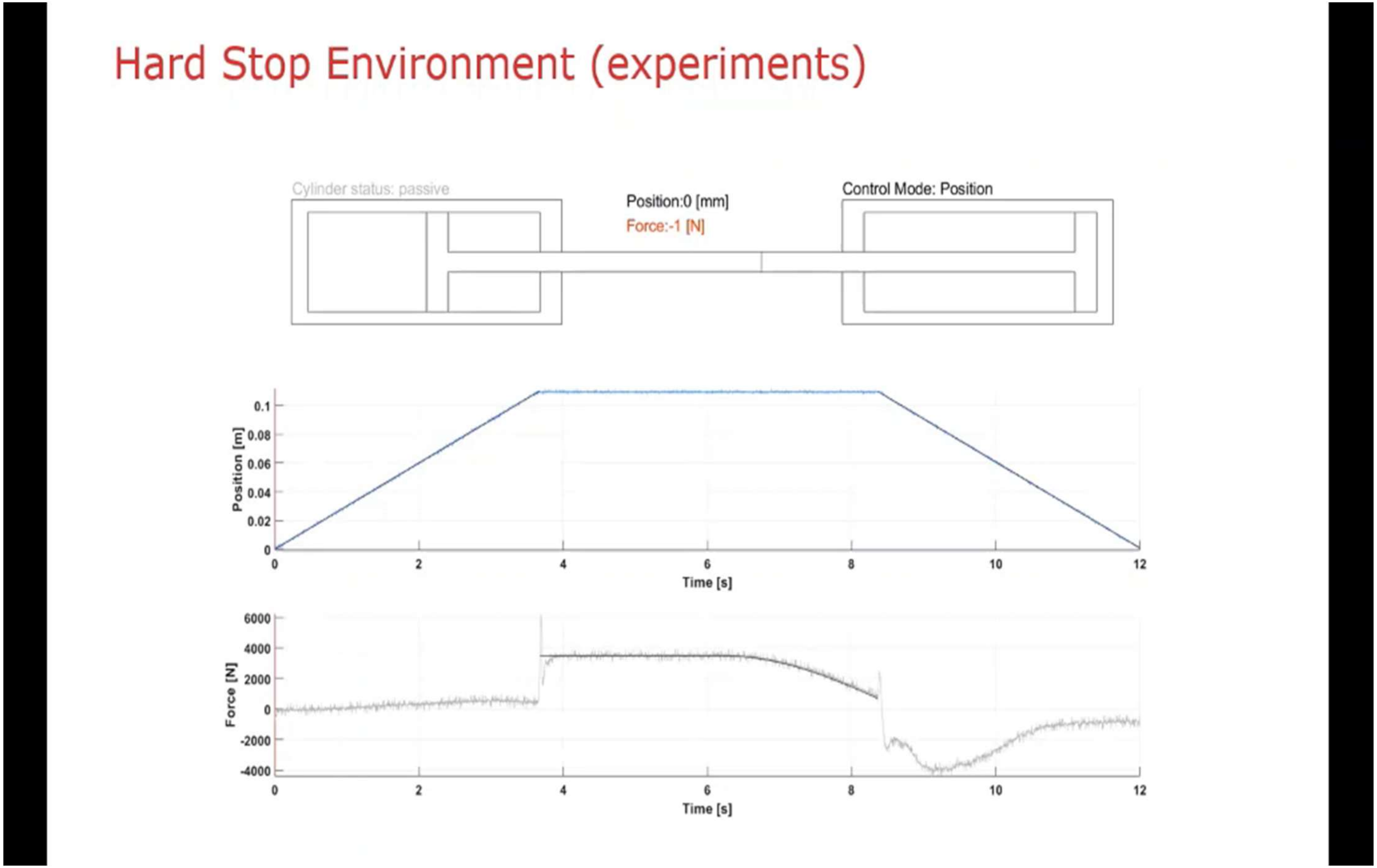
$$\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}(h)\bar{\mathbf{x}} + \bar{\mathbf{b}}(h)r(h), \qquad \bar{\mathbf{x}} = [\mathbf{x}_e^T, 1]^T \qquad \bar{\mathbf{c}} = [\mathbf{c}_e, 0]$$

$$y = \bar{\mathbf{c}}(h)\bar{\mathbf{x}}.$$

$$\dot{\bar{\mathbf{x}}} = \begin{bmatrix} \dot{\mathbf{x}}_e \\ 0 \end{bmatrix} = \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{b}}\, r = \begin{bmatrix} \mathbf{A}_e & \mathbf{f}_e \\ \mathbf{0} & 0 \end{bmatrix} \bar{\mathbf{x}} + \begin{bmatrix} \mathbf{b}_e \\ 0 \end{bmatrix} r$$

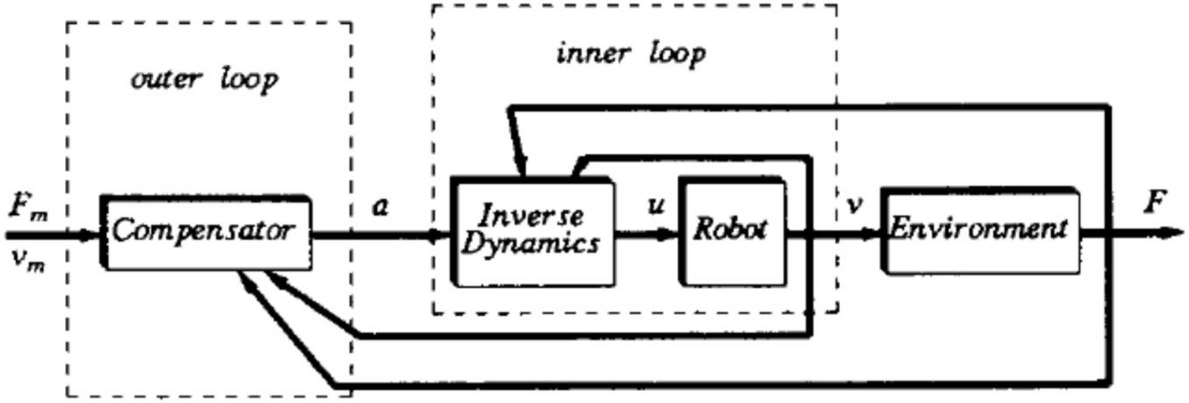
  - Switching state for changing between position and force $\quad h = [-1, 1]$

- Motion and force control example in mechatronics, cont.



source: [26]

- Principles of impedance control



source: [24]

Fig. 1. Inner/outer loop control.

Feedback linearization resulting in a double-integrator plant

$$D(x)\ddot{x} + h(x, \dot{x}) - F = J^{-T}u$$

$$u = J^{T}(D(x)a + h(x, \dot{x}) - F)$$

$$\boxed{\ddot{x} = a = \dot{v}}$$

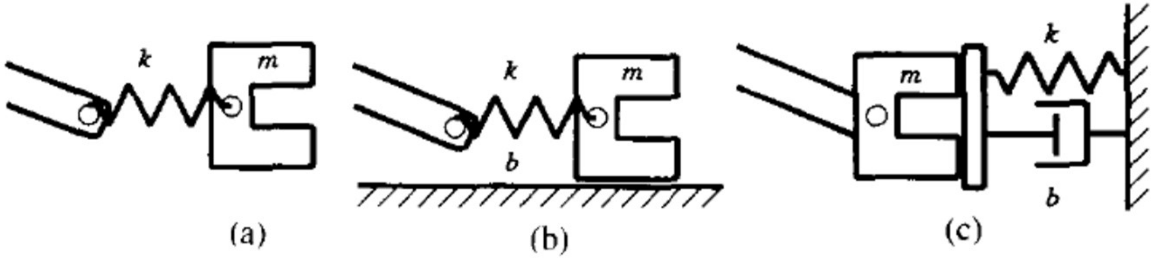


Fig. 2. Environment types. (a) Inertial. (b) Resistive. (c) Capacitive.

source: [24]

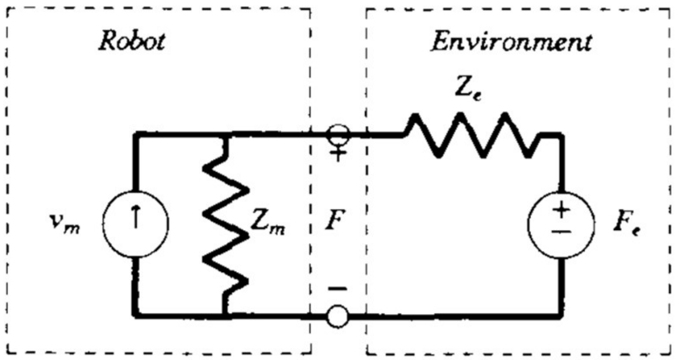


Fig. 3. Position-control model.

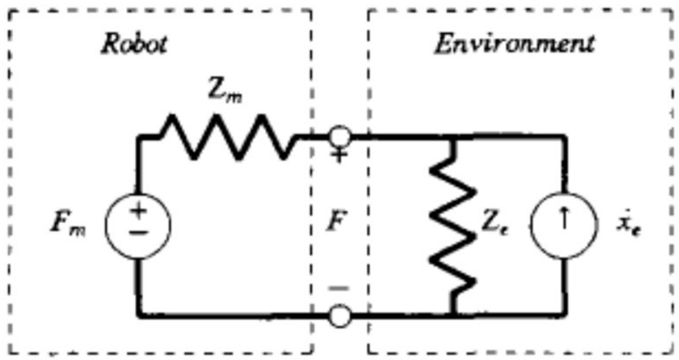


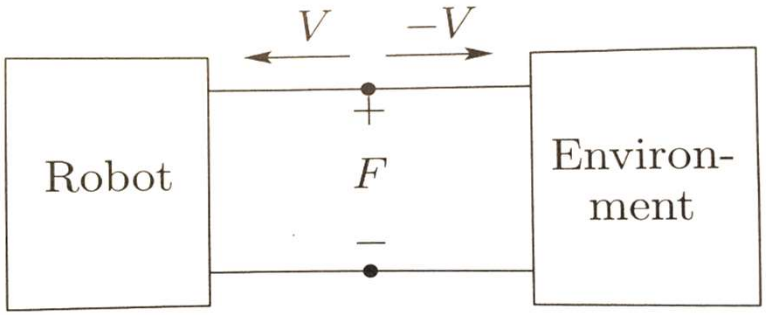
Fig. 4. Force-control model.

• Principles of impedance control, cont.

Robot and environment are coupled through <u>interaction ports</u>

Product of port variables, $V^T F$, constitutes the instantaneous power. The integral of this is the energy stored or dissipated in network

$$\int_0^t V^T(\sigma) F(\sigma) d\sigma$$



source: [17]

Relationship of <u>effort</u> and <u>flow</u> variables in network can be determined by <u>impedance operator</u>

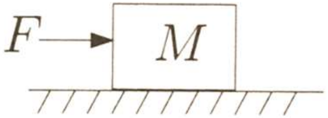
$$Z(s) = \frac{F(s)}{V(s)}$$

<u>Example</u>: mass-spring-damper system

$$M\ddot{x} + B\dot{x} + Kx = F$$

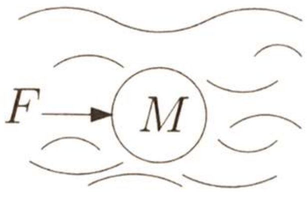
$$Z(s) = F(s)/V(s) = Ms + B + K/s$$

$|Z(0)| = 0$



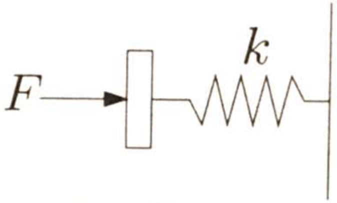
(a) Inertial

$|Z(0)| = B$ with $0 < B < \infty$



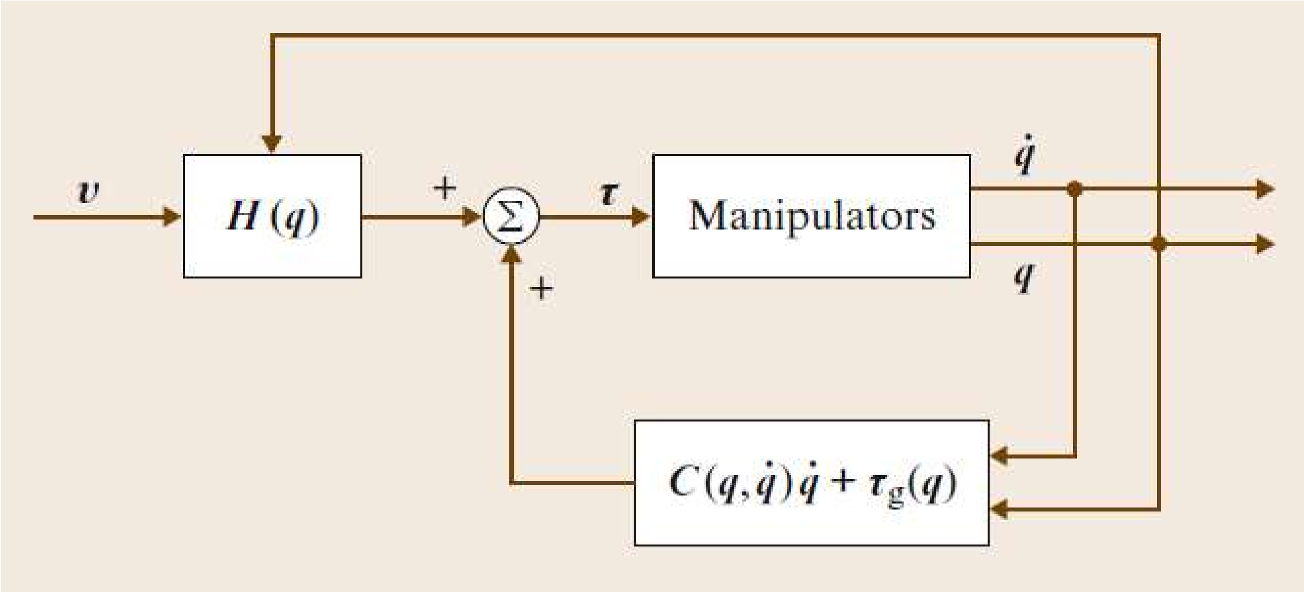
(b) Resistive

$|Z(0)| = \infty$



(c) Capacitive

- Inverse dynamics control in robotics



source: [5]

Consider the control input

$$\tau = H(q)v + C(q, \dot{q})\dot{q} + \tau_g(q)$$

Substituting it into manipulator dynamics yields a double-integrator plant

$$\ddot{q} = v$$

Then, the outer loop can be arbitrary shaped with simple (e.g. PD) control

$$v = \ddot{q}_d + K_V \dot{e}_q + K_P e_q$$

• Inverse dynamics control in robotics, cont.

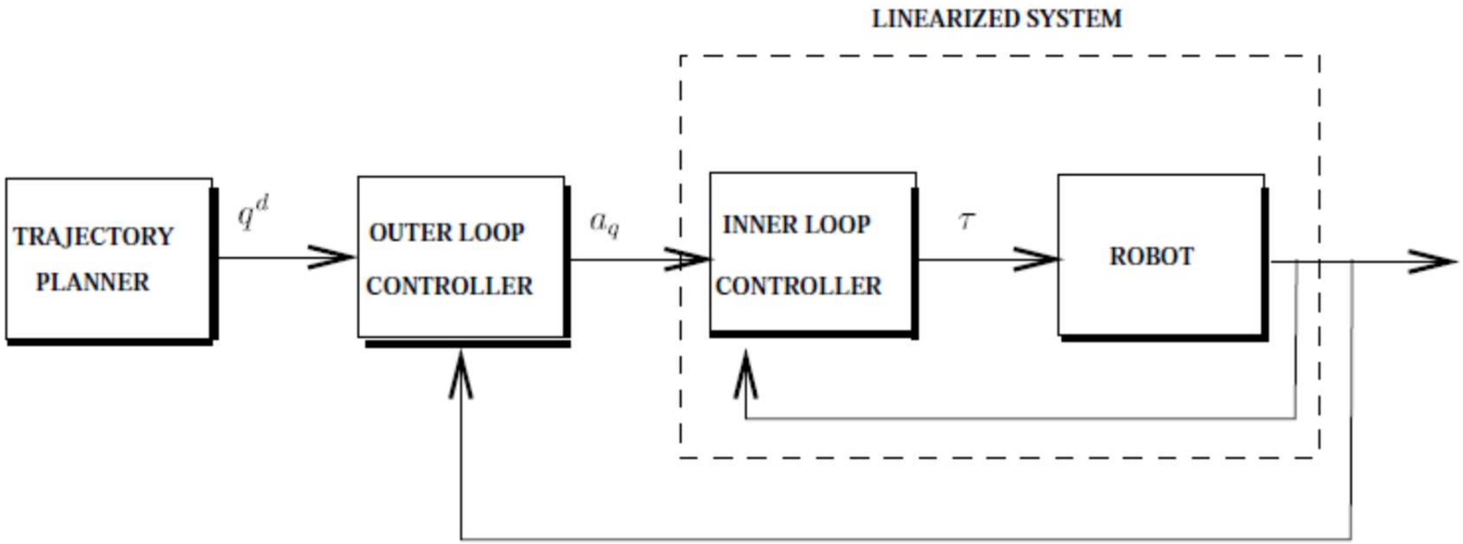
With the position control error $e_q = q_d - q$, the error dynamics is

$$\ddot{e}_{\mathrm{q}} + K_{\mathrm{V}}\dot{e}_{\mathrm{q}} + K_{\mathrm{P}}e_{\mathrm{q}} = 0$$

The total control law is then

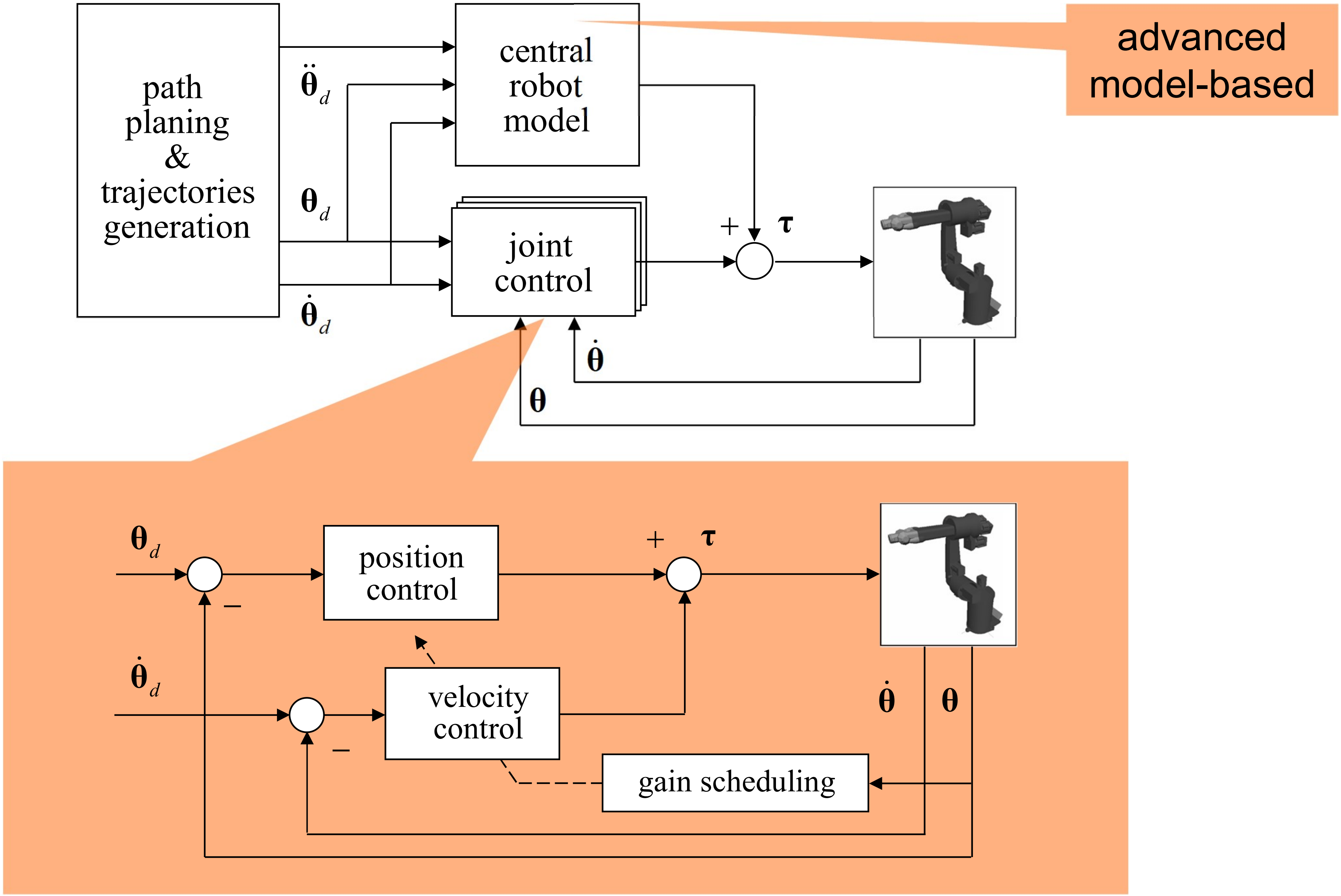
$$\tau = H(q)(\ddot{q}_{\mathrm{d}} + K_{\mathrm{V}}\dot{e}_{\mathrm{q}} + K_{\mathrm{P}}\,\mathrm{e}_{\mathrm{q}}) + C(q,\dot{q})\dot{q} + \tau_{\mathrm{g}}(q)$$

Achieved is the system linearization through: (i) feedback of nonlinearities and (ii) control input transformation (via state-dependent inertia matrix)



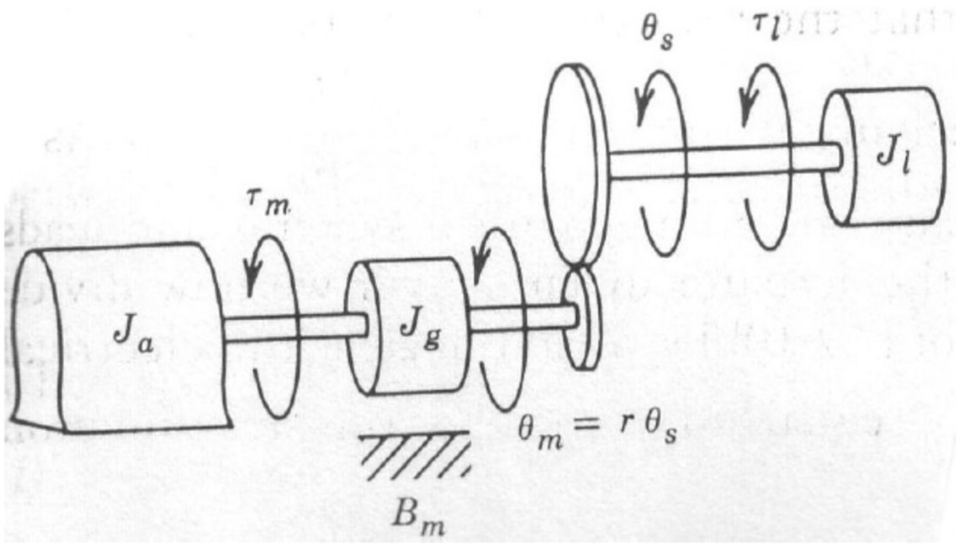
$$\tau = H(q)a_q + C(q,\dot{q})\dot{q} + \tau_{\mathrm{g}}(q)$$

- Robot control with feed-forwarding: example architecture



- # Independent joint control

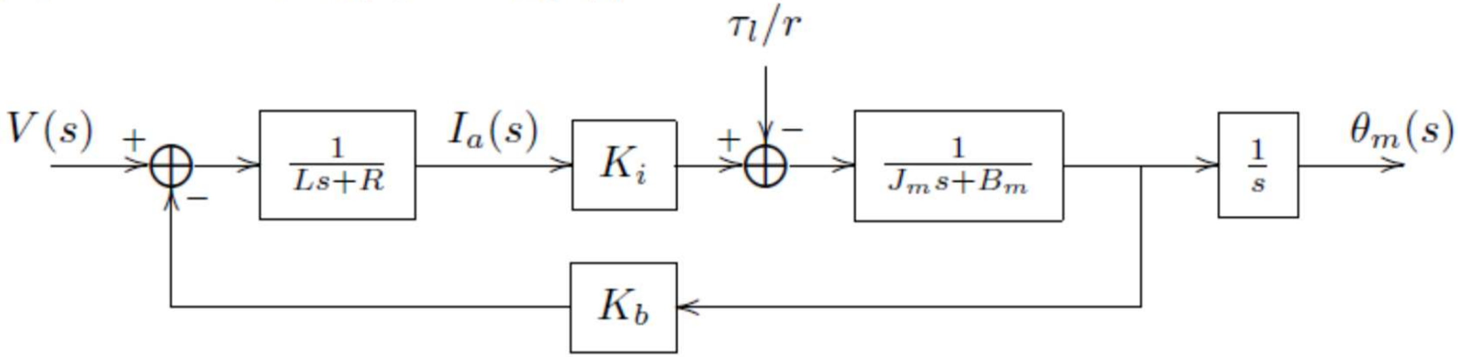
Rigid drive chain (lumped inertia) with DC motor as joint's actuator



source: [17]

Corresponding (extended) electro-mechanical model

$$
\begin{aligned}
(Ls + R)I_a(s) &= V(s) - K_b s\Theta_m(s) \\
(J_m s^2 + B_m s)\Theta_m(s) &= K_i I_a(s) - \tau_\ell(s)/r
\end{aligned}
$$



Transfer function without external load

$$
\frac{\Theta_m(s)}{V(s)} = \frac{K_m}{s\left[(Ls+R)(J_m s + B_m) + K_b K_m\right]}
$$

$$K_m \equiv K_i$$

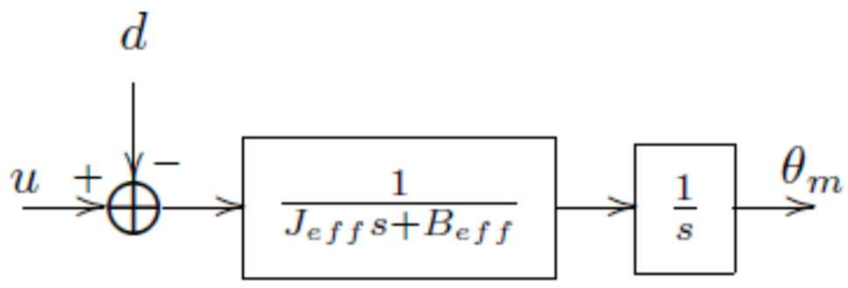
- Independent joint control, cont.

– Reduced-order model with (already regulated) torque as input

$$J_{eff_k}\ddot{\theta}_{m_k} + B_{eff_k}\dot{\theta}_{m_k} = u_k - d_k$$

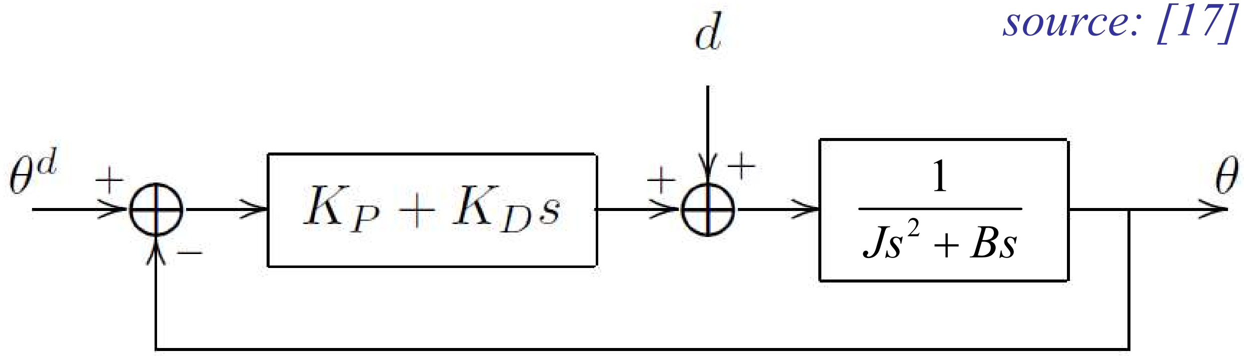
$$B_{eff_k} = B_{m_k} + K_{b_k}K_{m_k}/R_k$$

$$u_k = K_{m_k}/R_k V_k$$



source: [17]

– **PD control**



$$U(s) = (K_P + K_D s)(\Theta^d(s) - \Theta(s))$$

$$\Theta(s) = \frac{K_P + K_D s}{\Omega(s)}\Theta^d(s) - \frac{1}{\Omega(s)}D(s) \qquad D \equiv d$$

$$\Omega(s) = Js^2 + (B + K_D)s + K_P$$

– **PD control**, cont.

Control error transfer function

$$\begin{aligned} E(s) &= \Theta^d(s) - \Theta(s) \\ &= \frac{Js^2 + Bs}{\Omega(s)}\Theta^d(s) + \frac{1}{\Omega(s)}D(s) \end{aligned}$$

Characteristic polynomial of the closed-loop transfer function

$$s^2 + \frac{(B_{eff} + K_D)}{J_{eff}}s + \frac{K_p}{J_{eff}} \quad = \quad s^2 + 2\zeta\omega s + \omega^2$$

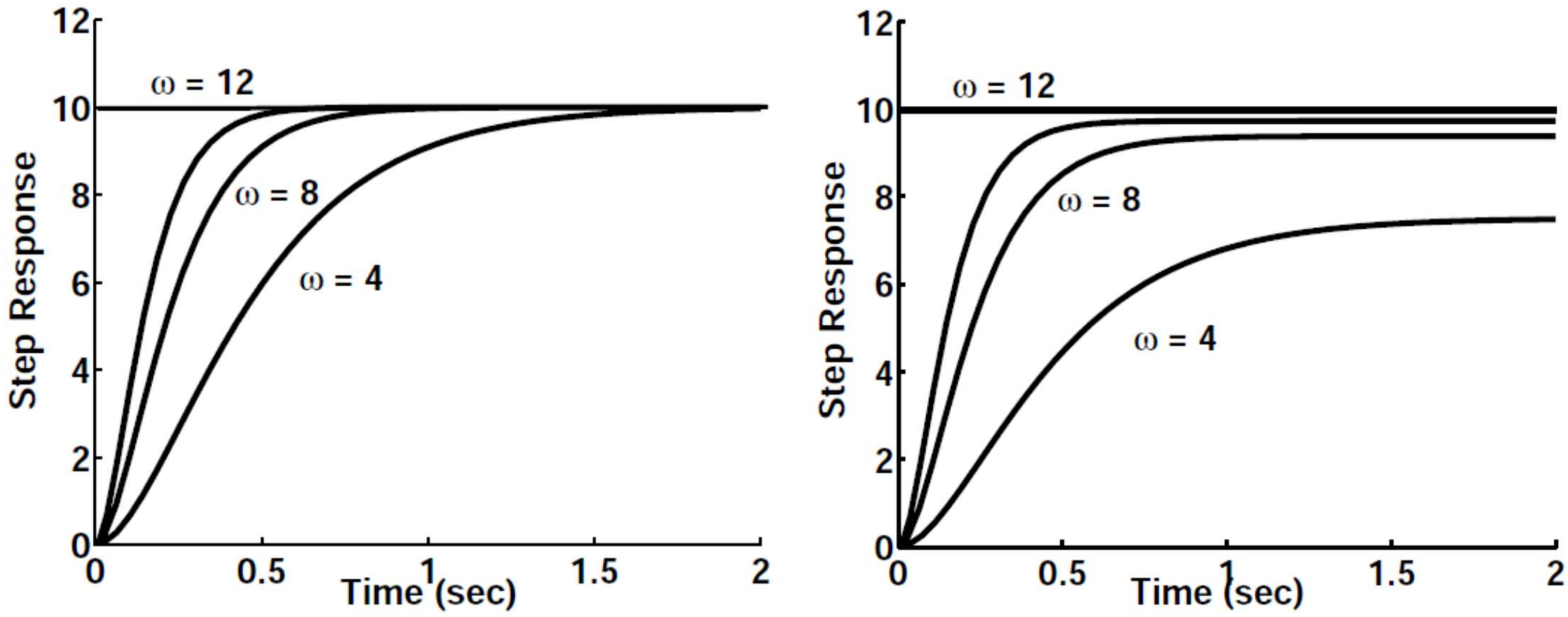
Exemplary values for critically damped (i.e. ζ=1) control design

Natural Frequency (ω)	Proportional Gain K_P	Derivative Gain K_D
4	16	7
8	64	15
12	144	23

– **PD control**, cont.

Control performance

without disturbances, i.e. d=0 | with disturbances, e.g. d=const

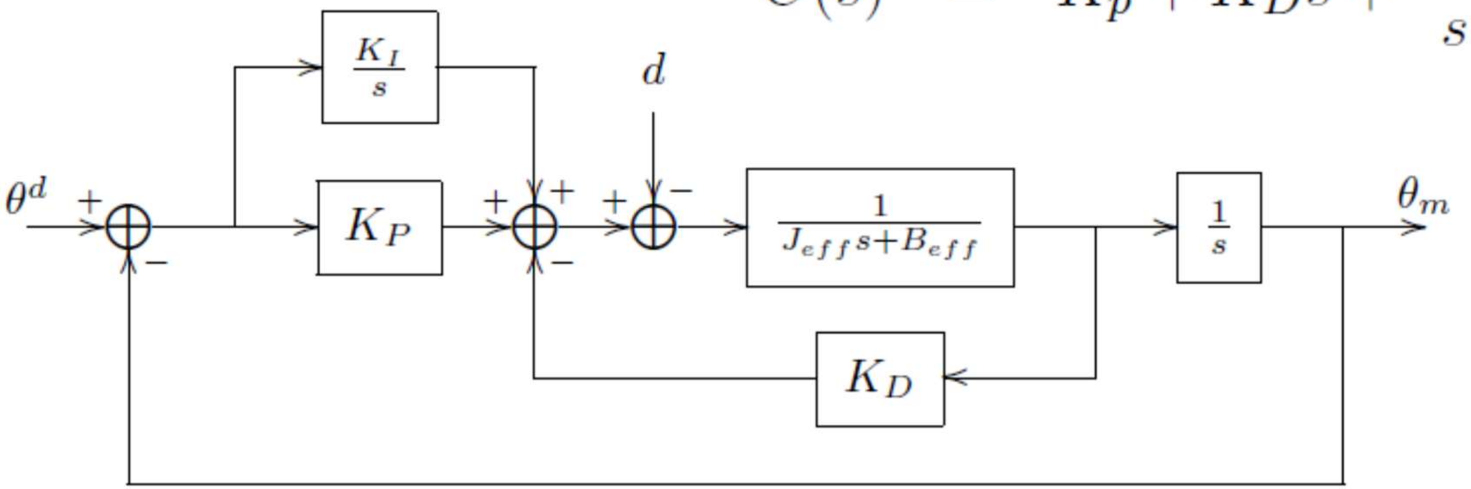


source: [17]

There is a need for an explicit compensation of disturbances

- Applying PID control

$$C(s) = K_p + K_D s + \frac{K_I}{s}$$

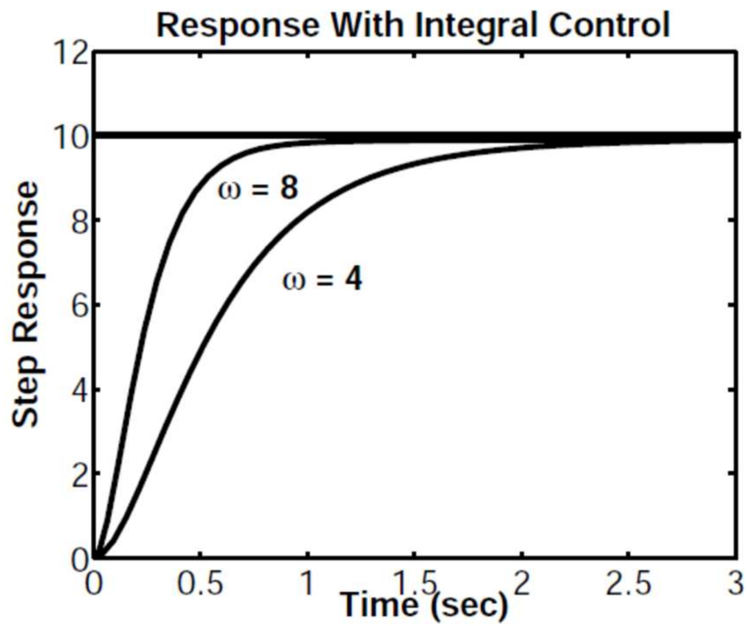


source: [17]

Motor output response

$$\Theta_m(s) = \frac{(K_D s^2 + K_p s + K_I)}{\Omega_2(s)}\Theta^d(s) - \frac{s}{\Omega_2(s)}D(s)$$

$$\Omega_2 = J_{eff}s^3 + (B_{eff} + K_D)s^2 + K_p s + K_I$$

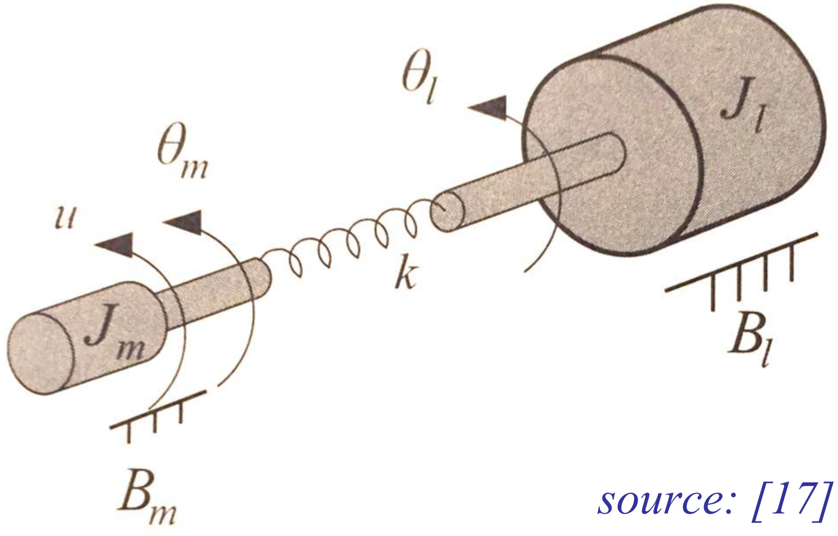


Routh stability criterion for the integral gain

$$K_I < \frac{(B_{eff} + K_D)K_p}{J_{eff}}$$

- # Effect of joint elasticities

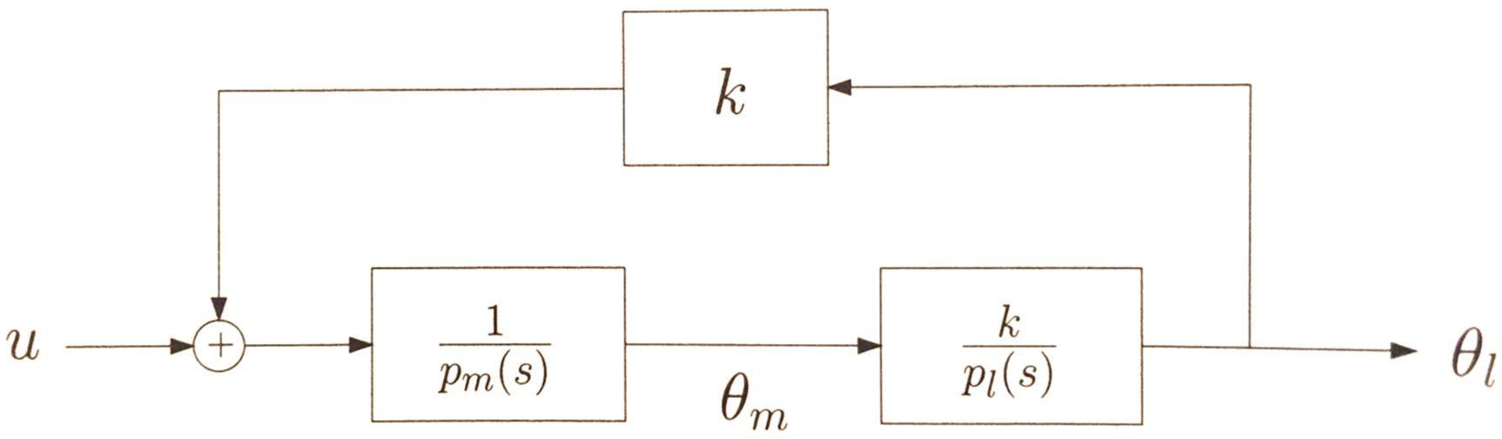
$$
\begin{aligned}
J_\ell \ddot{\theta}_\ell + B_\ell \dot{\theta}_\ell + k(\theta_\ell - \theta_m) &= 0 \\
J_m \ddot{\theta}_m + B_m \dot{\theta}_m - k(\theta_\ell - \theta_m) &= u
\end{aligned}
$$



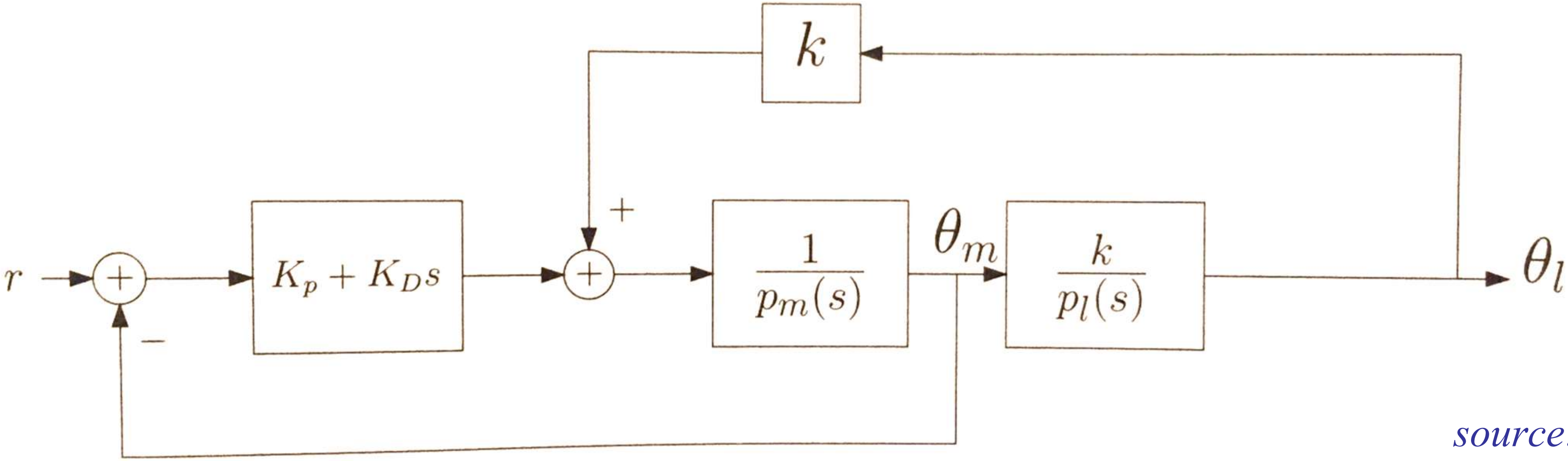
source: [17]

Transfer functions and block diagram

$$
\begin{aligned}
p_\ell(s)\Theta_\ell(s) &= k\Theta_m(s) \\
p_m(s)\Theta_m(s) &= k\Theta_\ell(s) + U(s)
\end{aligned}
\qquad
\begin{aligned}
p_\ell(s) &= J_\ell s^2 + B_\ell s + k \\
p_m(s) &= J_m s^2 + B_m s + k
\end{aligned}
$$

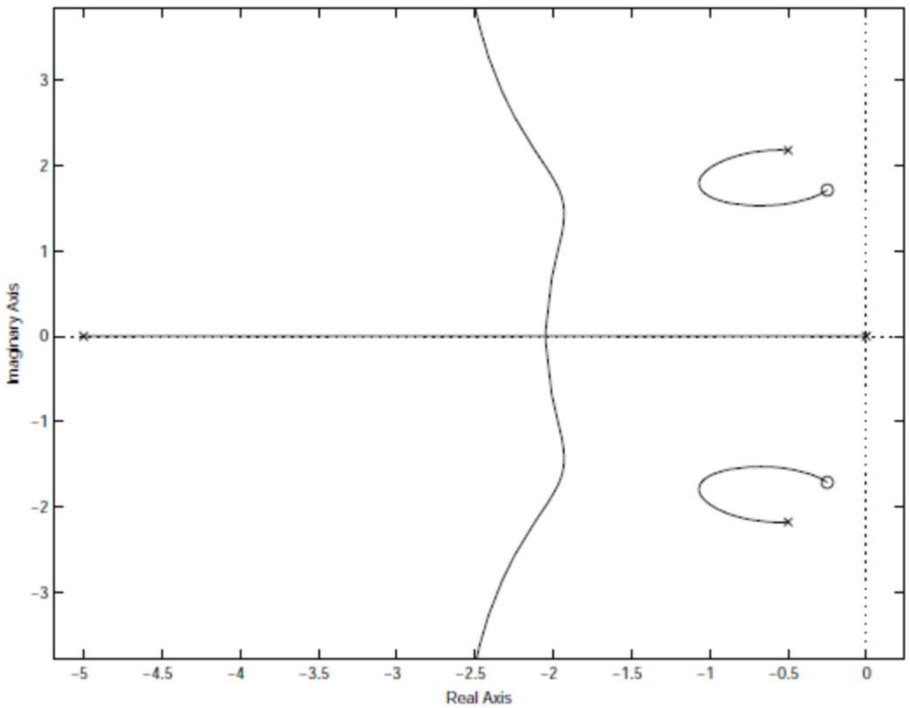


- Effect of joint elasticities, cont.



source: [17]

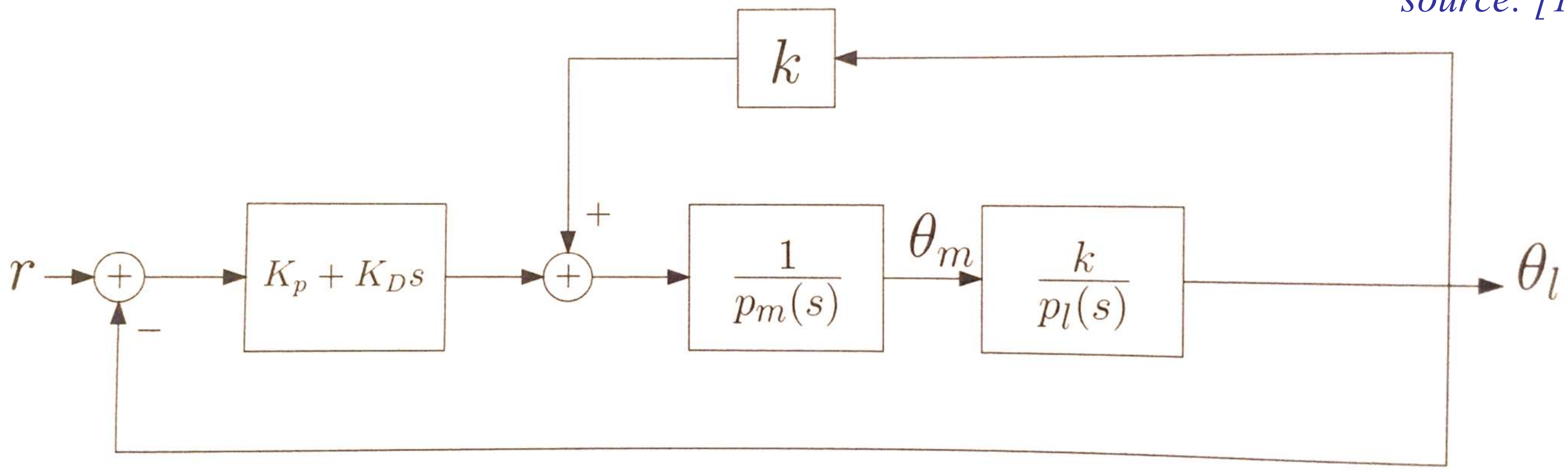
When using **motor feedback** for control



Root locus (i.e. poles location / trajectories) depending on the K_D control gain

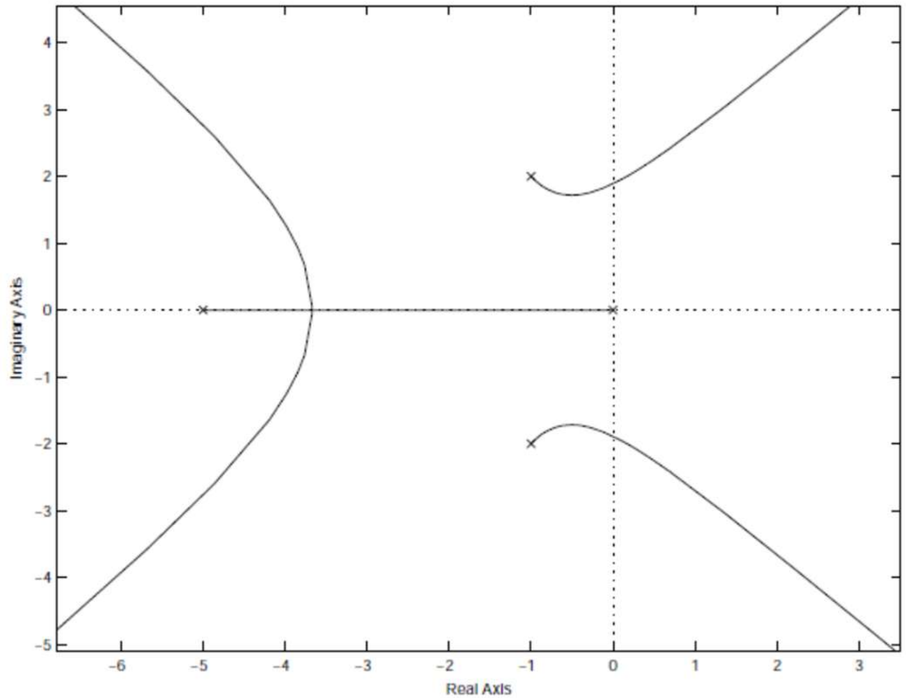
- Effect of joint elasticities, cont.

source: [17]

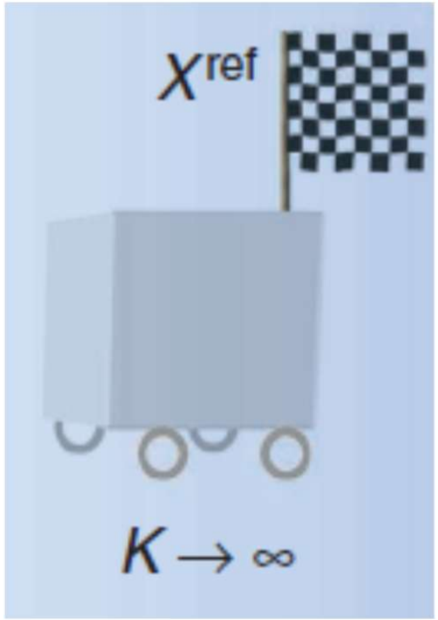


When using **load feedback** for control

Root locus (i.e. poles location / trajectories) depending on the K_D control gain



- Position control with motion stiffness



Actuator dynamics with 1DOF

$$J\,\ddot{x}(t) + B\,\dot{x}(t) = u(t)$$

Assuming PD-control with K_p-gain as 'stiffness'

$$u(t) = K_p\left(X^{ref}(t) - x(t)\right) - K_d\dot{x}(t)$$

Closed-loop system dynamics

$$J\,\ddot{x}(t) + \left(B + K_d\right)\dot{x}(t) + K_p\,x(t) = K_p X^{ref}(t)$$

Equivalent natural behavior

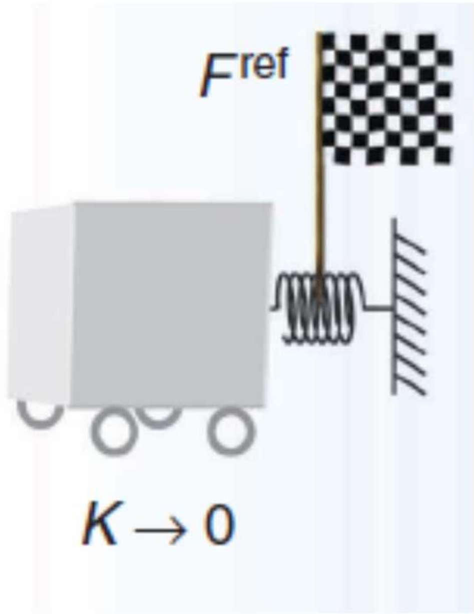
$$\ddot{x}(t) + 2\delta\omega_0\dot{x}(t) + \omega_0^2 x(t) = 0 \quad \Rightarrow \quad \frac{x(s)}{X^{ref}(s)} = \frac{\omega_0^2}{s^2 + 2\delta\omega_0 s + \omega_0^2}$$

with $\quad 2\delta\omega_0 = \dfrac{B + K_d}{J}, \quad \omega_0 = \sqrt{\dfrac{K_p}{J}}$

- Force control with interface stiffness

Same actuator dynamics, but after stiff contact

$$J\,\ddot{x}(t) + B\,\dot{x}(t) + \underbrace{K\,x(t)}_{\text{environmental counter-force}} = u(t)$$



Assuming force control with the measured F

$$u(t) = F^{ref}(t) - F(t) = F^{ref}(t) - \underbrace{\left[-K x(t)\right]}_{\text{'stiff' sensor on contact interface}}$$

Closed-loop system dynamics

$$J\,\ddot{x}(t) + B\,\dot{x}(t) = F^{ref}(t)$$

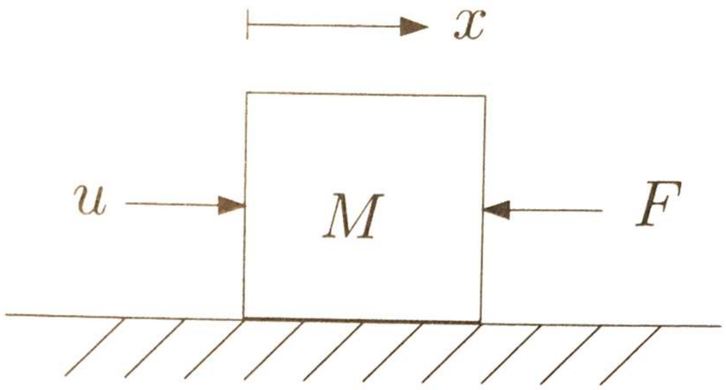
At steady-state: $\text{contact force} = F^{ref}(t)$

At impact: $\text{velocity} = F^{ref}(t) B^{-1}$

• Position and force control in common impedance structure

Example 20: Simplest 1DOF dynamics

$$M\ddot{x} = u - F$$



source: [17]

If $u=0$, then a pure inertia with mass M

If control u as force feedback term $u=-mF$,

$$M\ddot{x} = -(1+m)F \implies \frac{M}{1+m}\ddot{x} = -F$$

This results in changing the **apparent inertia** in system from M to $M/(m+1)$

- Advantage of separating the position and force control:
  a_x as a function of position and velocity only, and a_f as a function of force only

- Idea behind impedance control (as acceleration a-control) is to change the apparent inertia, stiffness, damping through the assigned feedback parameters

$$\begin{aligned} a_x &= \ddot{x}^d - M_d^{-1}(B_d\dot{e} + K_d e + F_e) \\ a_f &= F_e \end{aligned}$$

e is position error,
F_e is the force error

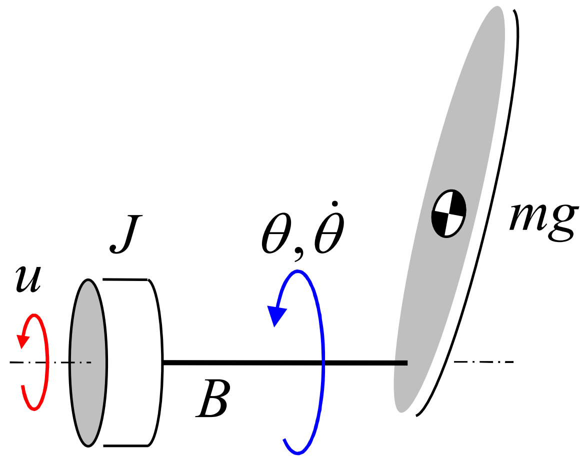
- Practical (hand on) session 9

Consider the single rigid robotic joint

The system dynamics is described by

$$J\ddot{\theta} + B\dot{\theta} + m\,g\,L\sin(\theta) = u,$$

with the driving torque u, total lumped inertia J, mass of the link m, distance to COG (center of gravity) L, and damping coefficient B. The gravity acceleration constant is g=9.8.



Further assume the parameter values: $J = 0.1, \quad B = 0.05, \quad m = 5, \quad L = 1.$

1. Draw the block diagram of the system. Make the corresponding Simulink model and show the open-loop step for a steady-state close to $\theta \approx 90$ deg.
2. Design a feed-forward control for the gravity and damping terms. Then, design a PD feedback control, so that the closed-loop system has a critical damping δ=1 and natural frequency ω_0=6 rad/s. Show the controlled response for 0 deg < θ_{ref}= 45 t < 90 deg. How will it change for double m?

Feedback control systems for mechatronics and robotics

Michael Ruderman
michael.ruderman@uia.no

Use of observers and estimators in feedback control systems

Luenberger state observer

Observing the State of a Linear System

DAVID G. LUENBERGER, STUDENT MEMBER, IEEE

Summary—In much of modern control theory designs are based on the assumption that the state vector of the system to be controlled is available for measurement. In many practical situations only a few output quantities are available. Application of theories which assume that the state vector is known is severely limited in these cases. In this paper it is shown that the state vector of a linear system can be reconstructed from observations of the system inputs and outputs.

It is shown that the observer, which reconstructs the state vector, is itself a linear system whose complexity decreases as the number of output quantities available increases. The observer may be incorporated in the control of a system which does not have its state vector available for measurement. The observer supplies the state vector, but at the expense of adding poles to the over-all system.

Received November 2, 1963. This research was partially supported by a grant from Westinghouse Electric Corporation.
The author is with the Department of Electrical Engineering, Stanford University, Stanford, Calif.

techniques have been developed to find the function F for special classes of control problems. These techniques include dynamic programming [8]–[10], Pontryagin's maximum principle [11], and methods based on Lyapunov's theory [2], [12].

In most control situations, however, the state vector is not available for direct measurement. This means that it is not possible to evaluate the function $F[y(t), t]$. In these cases either the method must be abandoned or a reasonable substitute for the state vector must be found.

In this paper it is shown how the available system in-

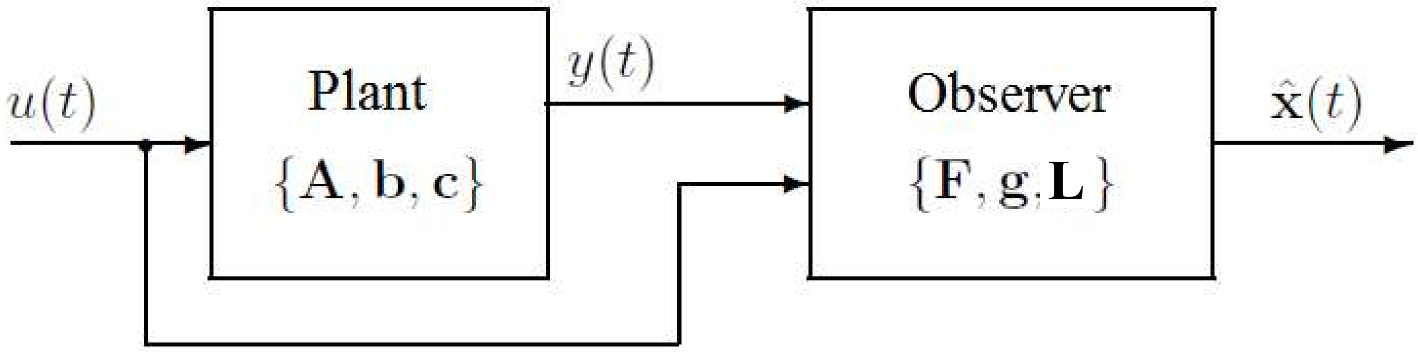
596 IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. AC-16, NO. 6, DECEMBER 1971

An Introduction to Observers

DAVID G. LUENBERGER, SENIOR MEMBER, IEEE

source: [27]

- Asymptotic observation principle



$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{F}\hat{\mathbf{x}}(t) + \mathbf{g}u(t) + \mathbf{L}y(t)$$

– We require that

$$\frac{\hat{X}_i(s)}{U(s)} = \frac{X_i(s)}{U(s)}, \qquad i = 1, \ldots, n,$$

with

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}U(s)$$

$$\hat{\mathbf{X}}(s) = (s\mathbf{I} - \mathbf{F})^{-1}[\mathbf{g} + \mathbf{L}\mathbf{c}^T(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}]U(s)$$

- Asymptotic observation principle, cont.

$$(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}=(s\mathbf{I}-\mathbf{F})^{-1}[\mathbf{g}+\mathbf{L}\mathbf{c}^T(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}]$$

$$[\mathbf{I}-(s\mathbf{I}-\mathbf{F})^{-1}\mathbf{L}\mathbf{c}^T](s\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}=(s\mathbf{I}-\mathbf{F})^{-1}\mathbf{g},$$

$$\vdots$$

$$(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}=(s\mathbf{I}-\mathbf{F}-\mathbf{L}\mathbf{c}^T)^{-1}\mathbf{g}$$

– In case that $\mathbf{g}=\mathbf{b}$ one obtains:

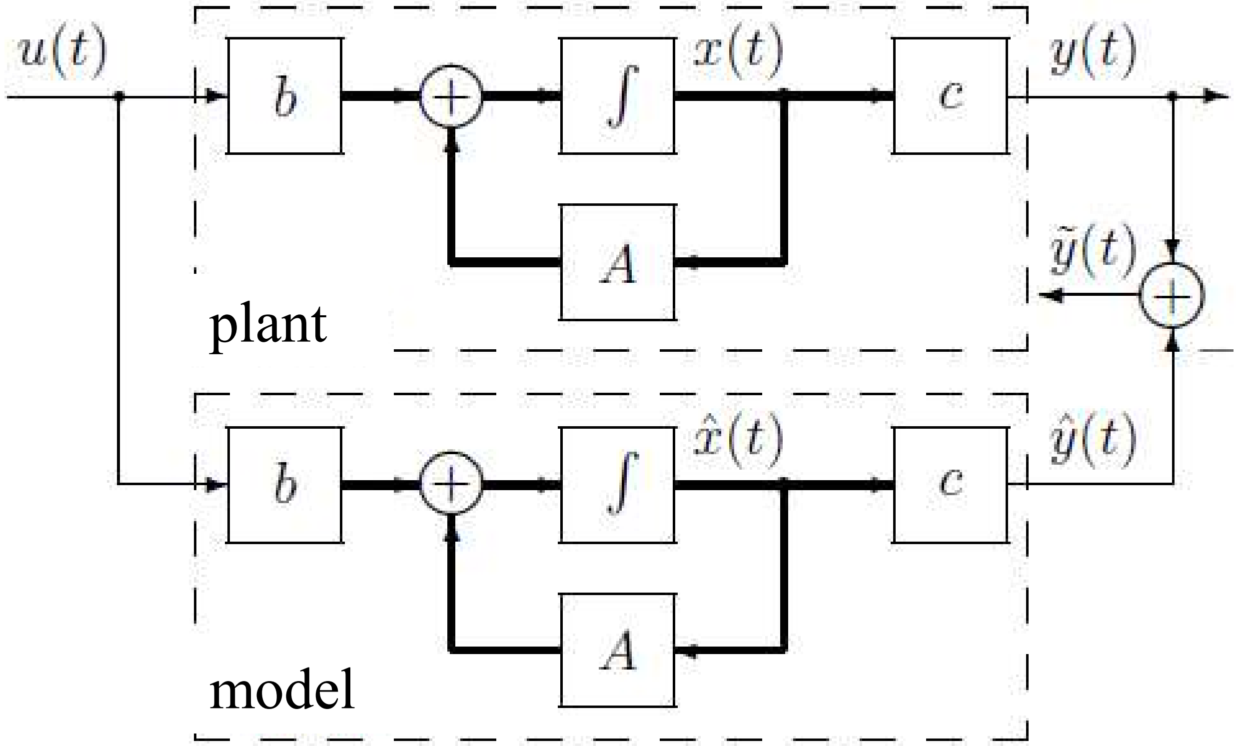
$$\mathbf{F}=\underbrace{\mathbf{A}-\mathbf{L}\mathbf{c}^T}_{\text{system matrix of observer}}$$

– The resulted observer dynamics

$$\begin{aligned}\dot{\hat{\mathbf{x}}}(t) &= (\mathbf{A}-\mathbf{L}\mathbf{c}^T)\hat{\mathbf{x}}(t)+\mathbf{b}u(t)+\mathbf{L}y(t),\\ &= \mathbf{A}\hat{\mathbf{x}}(t)+\mathbf{b}u(t)+\mathbf{L}(y(t)-\hat{y}(t)),\end{aligned}$$

- # Luenberger observer structure

– Block diagram and signal flow when designing observer

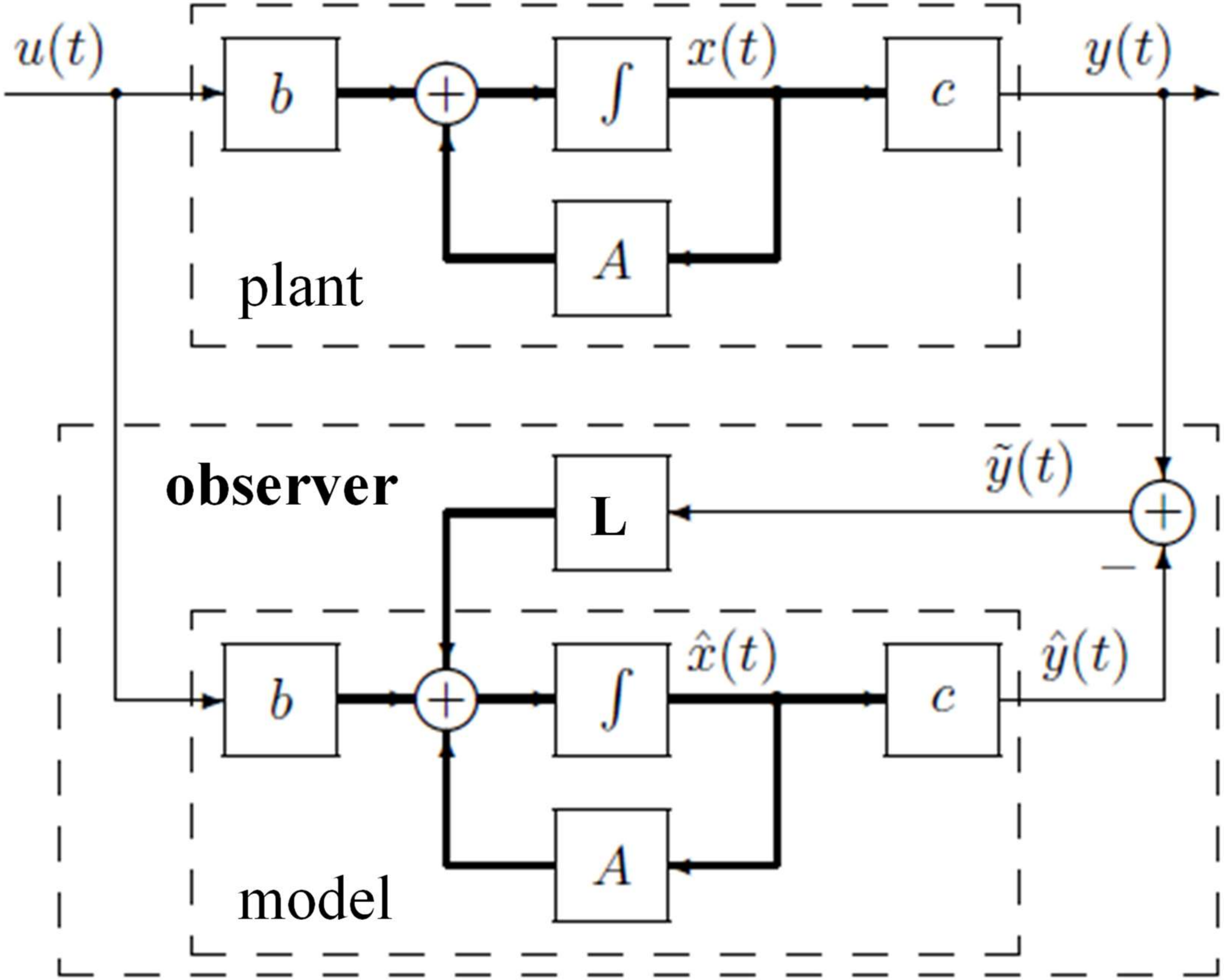


– Then, shape the observer dynamics (via pole placement) by feedback **L** (observer ~ model + feedback of observation error $\tilde{y}(t)$)

$$a_L(s) = |s\mathbf{I} - \mathbf{A} + \mathbf{L}\mathbf{c}^T| = 0, \qquad a_L(s) = \alpha(s)$$

: desired observer poles (design specification)

- Luenberger observer structure, cont.



– For the observer's state equation

$$\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{LC})\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{L}y(t)$$

evaluate the **state error** dynamics, i.e. dynamics of $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$

$$\dot{\mathbf{e}}(t) = \frac{d}{dt}\left(\mathbf{x}(t) - \hat{\mathbf{x}}(t)\right) =$$

$$= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) - \mathbf{A}\hat{\mathbf{x}}(t) - \mathbf{B}u(t) - \mathbf{LC}\left(\mathbf{x}(t) - \hat{\mathbf{x}}(t)\right) =$$

$$= \left(\mathbf{A} - \mathbf{LC}\right)\left(\mathbf{x}(t) - \hat{\mathbf{x}}(t)\right) \quad \Rightarrow$$

$$! \quad \dot{\mathbf{e}}(t) = \left(\mathbf{A} - \mathbf{LC}\right)\mathbf{e}(t) \quad \forall \quad \mathbf{e}(0) = \mathbf{x}_0 - \hat{\mathbf{x}}_0$$

$$\Rightarrow \quad \lim_{t \to \infty} \left\| \mathbf{e}(t) \right\| = 0$$

This is valid for all initial values under one and the single condition: **all eigenvalues** of (**A-LC**) < 0

- Observer design by pole placement

  – **Consider observation problem as a state feedback problem**

  – Take the transpose ("^{T}") matrices (**A**, **C**, **L**) and transformed ("x_T") states

$$(\mathbf{A}-\mathbf{LC})^T = \mathbf{A}^T - \mathbf{C}^T\mathbf{L}^T \quad \Rightarrow$$

$$\dot{\mathbf{x}}_T(t) = \mathbf{A}^T\mathbf{x}_T(t) + \mathbf{C}^T u_T(t)$$

$$u_T(t) = -\mathbf{L}^T\mathbf{x}_T(t)$$

  – Determine the natural behavior of (**A–LC**) through **L**-matrix assignment

  – Consider the given system in observable canonical form:

$$\dot{\mathbf{x}}_O(t) = \mathbf{A}_O\mathbf{x}_O(t) + \mathbf{B}_O u(t)$$

$$y(t) = \mathbf{C}_O^T\mathbf{x}_O(t)$$

- Observer design by pole placement, cont.

$$\mathbf{A}_O = \begin{pmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{pmatrix}, \quad \mathbf{C}_O^T = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_{n-1} \\ l_n \end{pmatrix}$$

– Assignment of observer gains

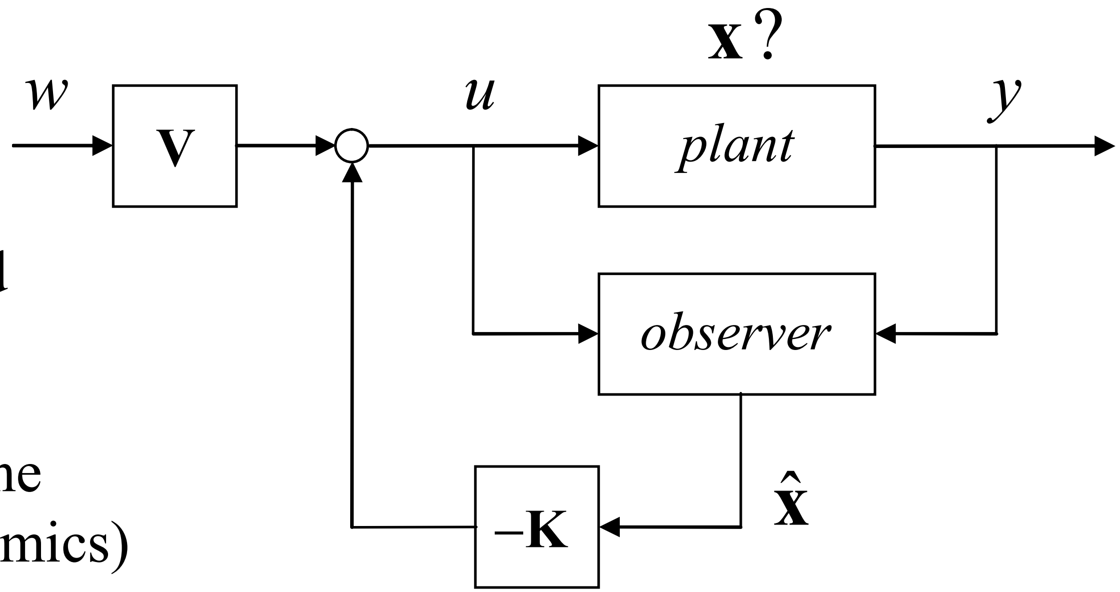
$$\mathbf{A}_O^T - \mathbf{C}_O \mathbf{L}^T = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ \underbrace{-a_0 - l_1}_{=-\alpha_0} & \underbrace{-a_1 - l_2}_{=-\alpha_1} & \dots & \underbrace{-a_{n-2} - l_{n-1}}_{=-\alpha_{n-2}} & \underbrace{-a_{n-1} - l_n}_{=-\alpha_{n-1}} \end{pmatrix}$$

- State feedback control with Luenberger observer

  – If the state vector is not (or only partially) available

  – Use the observed states instead of the real (measured) states

  – Ensure a fast convergence of the observed states (observer dynamics)



  – State feedback control with estimated (observed) states

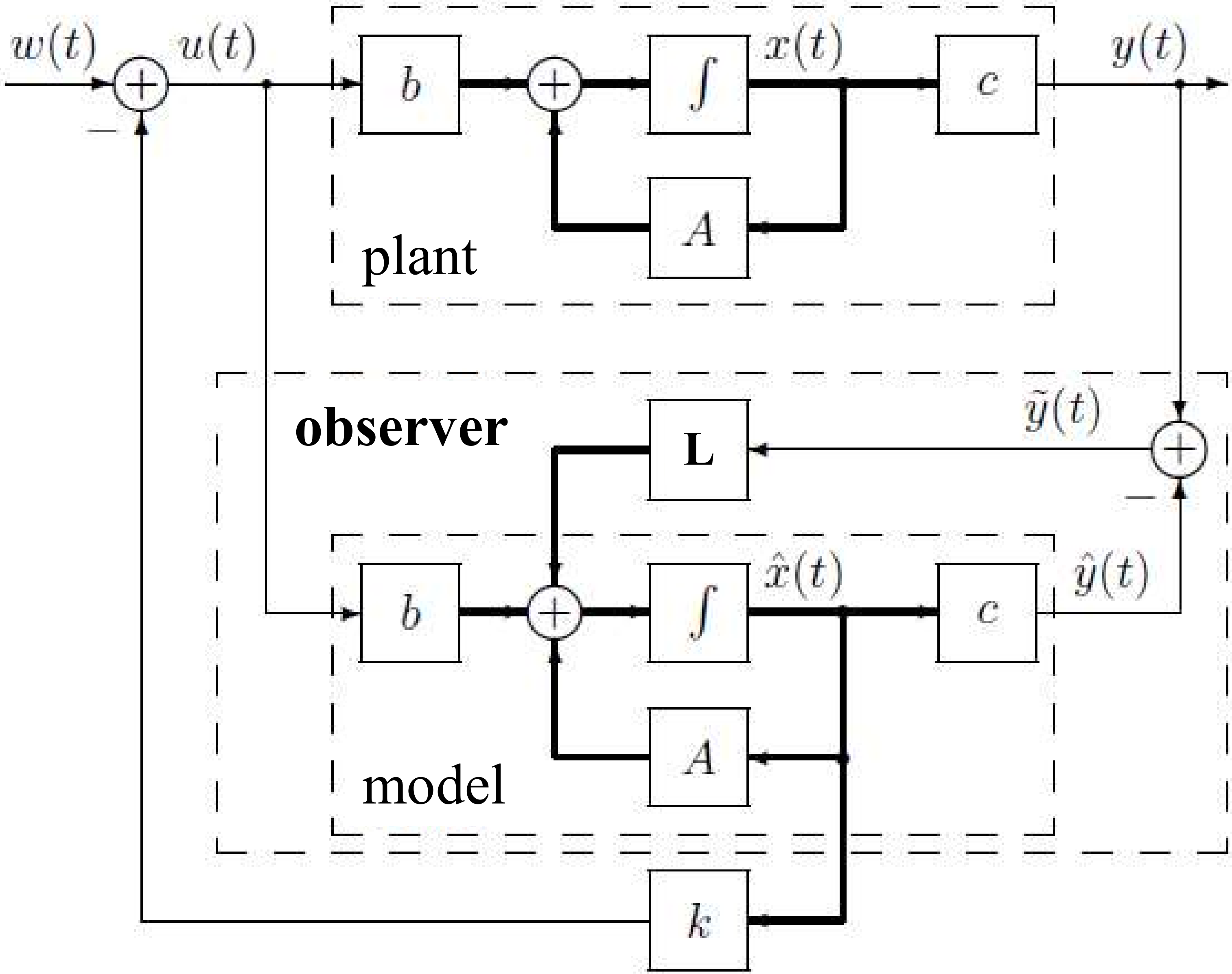
$$u(t) = -\mathbf{K}\mathbf{x}(t) + \mathbf{V}w(t) \quad \Rightarrow \quad u(t) = -\mathbf{K}\hat{\mathbf{x}}(t) + \mathbf{V}w(t)$$

  – Observer dynamics

$$\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{L}\mathbf{C})\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{L}y(t) \quad \Rightarrow$$

$$\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K} - \mathbf{L}\mathbf{C})\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{V}w(t) + \mathbf{L}y(t)$$

- State feedback control with Luenberger observer, cont.



- Separation principle
  - Include observation error as a dynamic state into the state-space model

$$\dot{\mathbf{e}}(t) = \left(\mathbf{A} - \mathbf{LC}\right)\mathbf{e}(t)$$

  - Resulted overall system dynamics

$$\begin{pmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t) \end{pmatrix} = \begin{pmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{BK} \\ \mathbf{0} & \mathbf{A} - \mathbf{LC} \end{pmatrix} \begin{pmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{pmatrix} + \begin{pmatrix} \mathbf{BV} \\ \mathbf{0} \end{pmatrix} w(t)$$

$$y(t) = \begin{pmatrix} \mathbf{C} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} \mathbf{x}(0) \\ \mathbf{e}(0) \end{pmatrix} = \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_0 - \hat{\mathbf{x}}_0 \end{pmatrix}$$

  - {eigenvalues} = {eigenvalues of A–BK} ∪ {eigenvalues of A–LC}
  - **Observer does not change the natural behavior of the state-feedback control loop ⇒ no impact on the control system stability (!)**

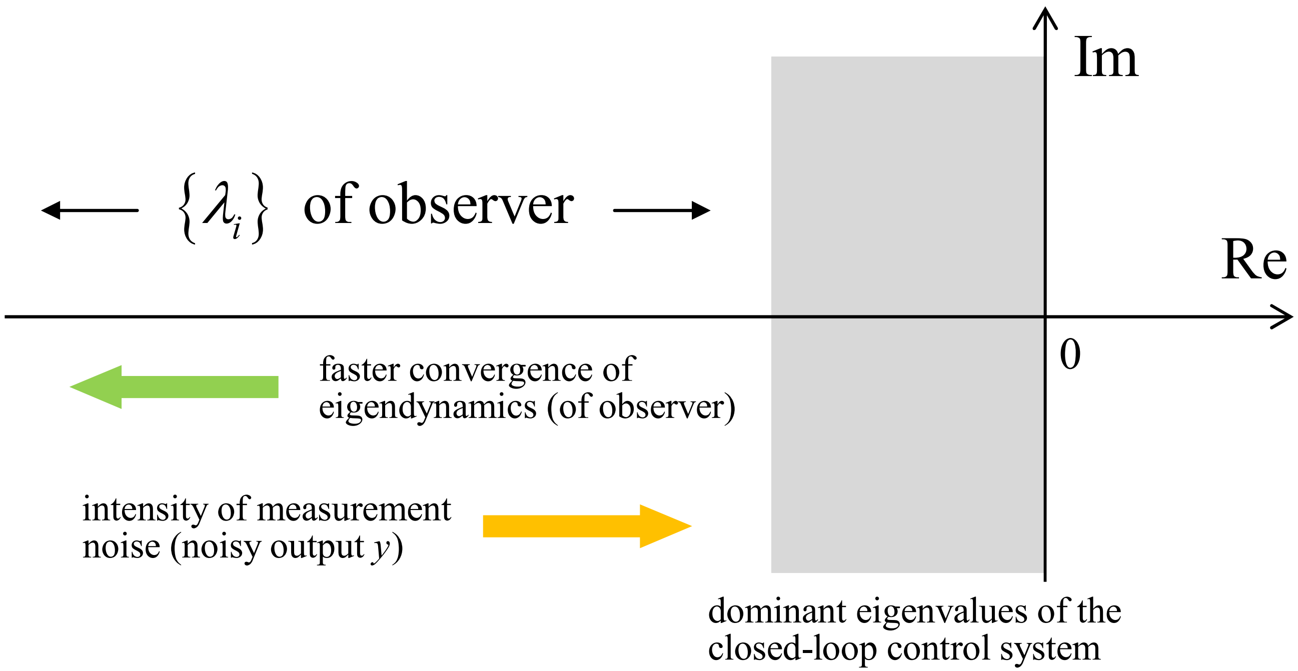
- State-feedback control design with observer – procedure steps

  – Examine the system controllability and observability (e.g. Kalman)

  – Design the state feedback control, i.e. determine the gain matrix **K** for measurable state vector (e.g. pole placement or Riccati / LQR method)

  – Assume the observer poles to be "far" left from those of the closed-loop system, i.e. observer dynamics to be "faster" than the control dynamics

  – Determine the observer feedback **L** (similar as when designing K-feedback)



- Kalman filter

R. E. KALMAN
Research Institute for Advanced Study,[2]
Baltimore, Md.

A New Approach to Linear Filtering and Prediction Problems[1]

The classical filtering and prediction problem is re-examined using the Bode-Shannon representation of random processes and the "state transition" method of analysis of dynamic systems. New results are:

(1) The formulation and methods of solution of the problem apply without modification to stationary and nonstationary statistics and to growing-memory and infinite-memory filters.

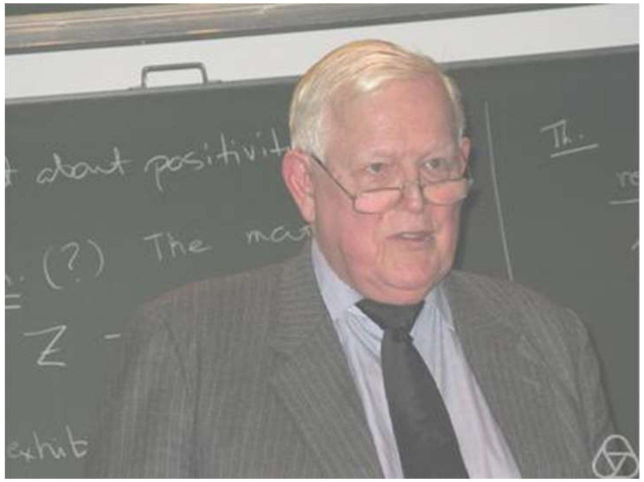
(2) A nonlinear difference (or differential) equation is derived for the covariance matrix of the optimal estimation error. From the solution of this equation the coefficients of the difference (or differential) equation of the optimal linear filter are obtained without further calculations.

(3) The filtering problem is shown to be the dual of the noise-free regulator problem.

The new method developed here is applied to two well-known problems, confirming and extending earlier results.

The discussion is largely self-contained and proceeds from first principles; basic concepts of the theory of random processes are reviewed in the Appendix.

source: [28]



Transactions of the ASME–Journal of Basic Engineering, 82 (Series D): 35-45. Copyright © 1960 by ASME

– Further developed extensions/approaches

- Extended Kalman Filter (nonlinear systems, linearizable)
- Unscented Kalman filter (nonlinear systems, e.g. non-linearizable)
- Kalman-Schmidt Filter (reducing dimensionality of state estimate)
- Kalman-Bucy Filter (time-continuous systems)
- …

- Kalman filter as state estimator

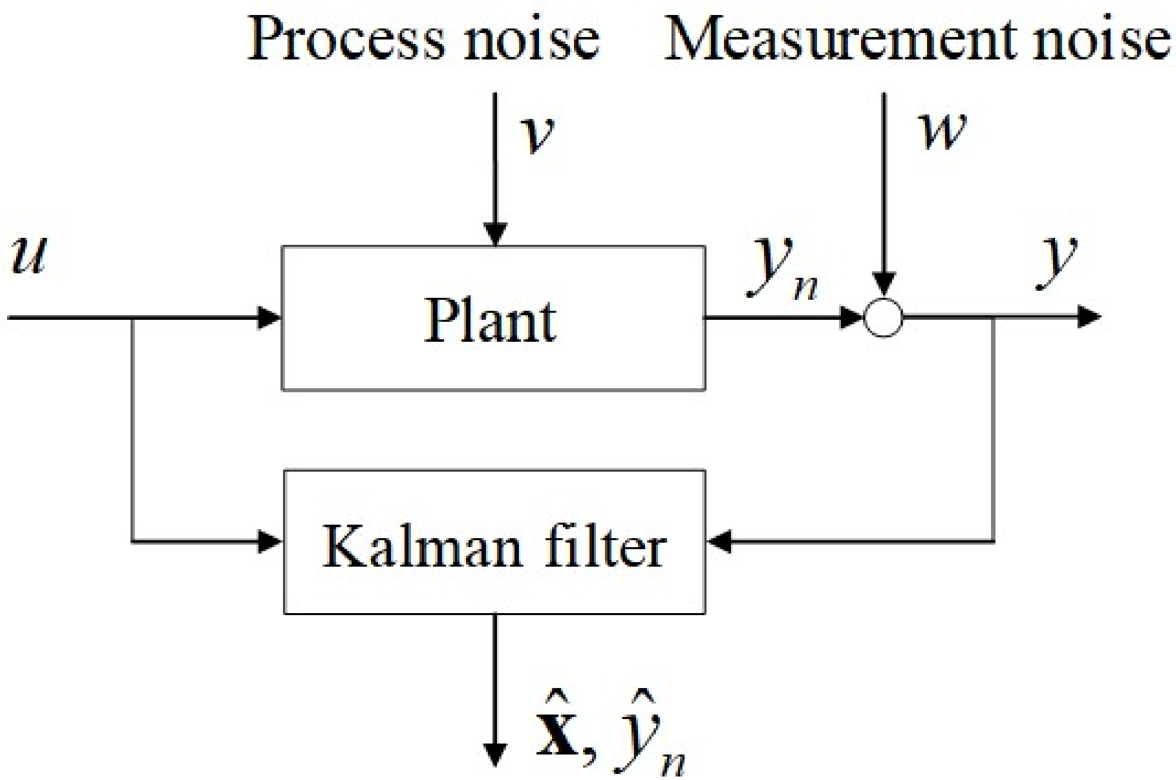
  – State-space representation

$$\dot{x}(t) = Ax(t) + Bu(t) + \boxed{v(t)}$$

process noise affects the dynamic states

$$y(t) = Cx(t) + \boxed{w(t)}$$

measurement noise affects the output value



- Kalman filter versus Luenberger observer

– Luenberger state observer is used for **deterministic** systems

$$\frac{d\hat{x}(t)}{dt} = A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t)) \qquad e(t) = x(t) - \hat{x}(t)$$

with estimation error dynamics and **constant** feedback gain

$$\dot{e}(t) = (A - LC)e(t) \qquad L$$

– Kalman filter (state estimator) is used for **stochastic** disturbances

$$\frac{d\hat{x}(t)}{dt} = A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t)) \qquad e(t) = x(t) - \hat{x}(t)$$

with estimation error dynamics and **varying** feedback gain

$$\dot{e}(t) = (A - LC)e(t) + v(t) - Lw(t) \qquad L(t)$$

where $v(t)$: **process noise**

$w(t)$: **measurement noise**

- Kalman filter as state estimator, cont.

– Noise processes should be Gaussian processes (Gaussian noise), that means

i. zero mean value, $E\left\{ v(t) \right\} = 0$

ii. variance towards infinity, $\mathrm{var}\left\{ v(t) \right\} = E\left\{ \left(v(t) - \bar{v} \right)^2 \right\} = \infty$

iii. sequential values should be **not correlated**

$$\mathrm{cov}\left\{ v(t), v(t) \right\} = E\left\{ v(t) v(t)' \right\} = Q\delta(t-\tau)$$

$$\mathrm{cov}\left\{ w(t), w(t) \right\} = E\left\{ w(t) w(t)' \right\} = R\delta(t-\tau)$$

$$\mathrm{cov}\left\{ w(t), v(t) \right\} = 0$$

Q: covariance matrix of v R: covariance matrix of w δ: impulse (Dirac) function

– Reminder of correlation: $\rho = \pm 1$: fully correlated; $\rho = 0$: fully uncorrelated

correlation coefficient: $\rho_{X,Y} = \dfrac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{var}(X)\,\mathrm{var}(Y)}} = \dfrac{\sigma_{X,Y}^2}{\sigma_X \sigma_Y}, \quad \rho_{X,Y} \in [-1,1].$

- Kalman filter as state estimator, cont.

  – The feedback (Kalman) gain **L** has to be selected so that to minimize the mean square of the state estimation errors e

$$\min_L \sum_{i=1}^{n} E\{e_i^2\} = \min_L \sum_{i=1}^{n} \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e_i^2(t)dt \qquad e(t) = x(t) - \hat{x}(t)$$

  – Optimal feedback (Kalman) gain is given by (similar as LQR design)

$$L = PC'R^{-1}$$

  where P is the positive definite solution of matrix Riccati equation

$$AP + PA' - PC'R^{-1}CP + Q = 0$$

  – Unlike the Luenberger observer where $\boldsymbol{L}$ is determined once and remains constant, the Kalman filter <u>updates $\boldsymbol{L}$ at each iteration step</u>

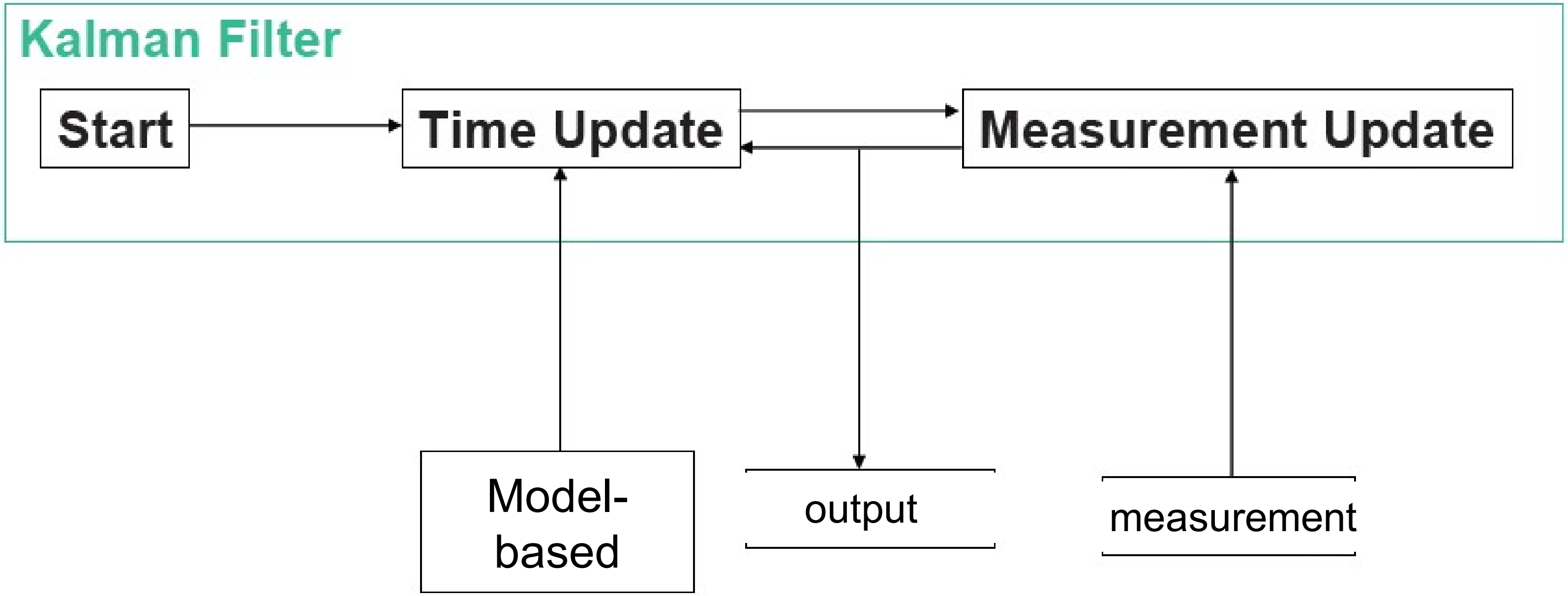
  – Matrices $\boldsymbol{Q}$ and $\boldsymbol{R}$ describe the dispersion (i.e. variance) of the stochastic disturbances v and w

- Kalman filter: two-steps scheme

– Iterative **predictor-corrector** algorithm

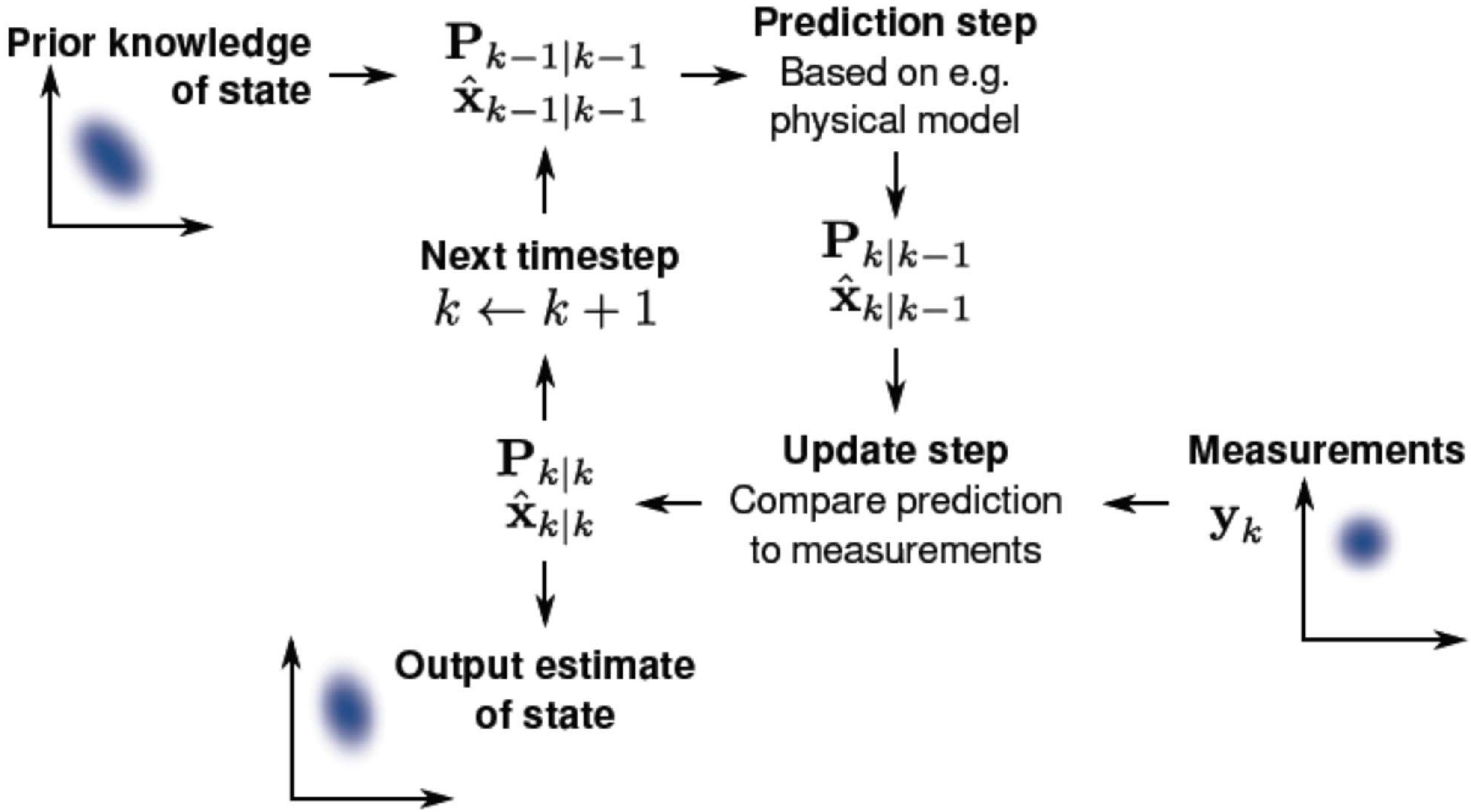
  - A-priori state estimation based on the previous state value and model
  - Measurement of the state (that contains errors)
  - A-posteriori state estimation based on measurement & a-priori estimate

1) Predictor phase denoted as **Time-Update**
2) Corrector phase denoted as **Measurement-Update**



- Kalman filter: two-steps scheme, cont.

  – Iterative evaluation of the estimated state, and of the variance (which is then equivalent to uncertainty) of the states estimate

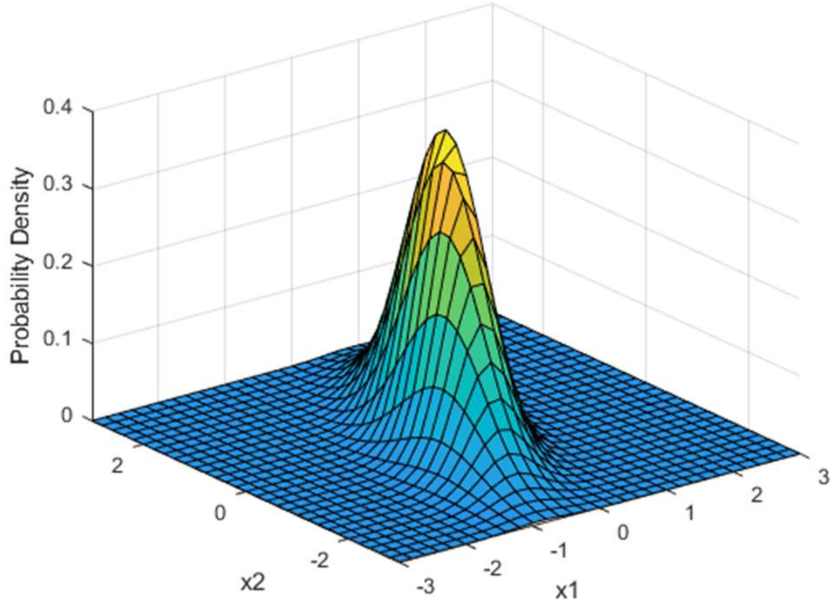


• Kalman filter: assumptions

Stochastic error vectors

– Process noise v_k :

- Transformation error of the transition $x_{k-1} \rightarrow x_k$
- $p(v) \sim N(0,Q)$: normal (μ, σ^2) i.e. Gaussian distributed
- Q is covariance matrix of v_k



– Measurement error (noise) w_k :

- Deviation of measurement y_k from the "true" system output
- $p(w) \sim N(0,R)$: normal (Gaussian) distributed
- R is covariance matrix of w_k

– Simplification:

- Q and R (as design parameters) can be assumed as constant

- Kalman filter: equations

– State vector in **discrete time** notation

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k + \mathbf{B}u_k + v_k$$

$\mathbf{x}_{k+1}$: next state (time step $k+1$)

$\mathbf{x}_k$: previous state (time step k-1)

$\mathbf{A}$: transformation $x_k \rightarrow x_{k+1}$ (like the system matrix for t)

$\mathbf{B}$: transformation $u_k \rightarrow x_k$ (input coupling vector)

v_k : process noise

– Measurement (in discrete time notation):

$$y_k = \mathbf{C}\mathbf{x}_k + w_k$$

y_k : recent measurement (at time step k)

$\mathbf{C}$: transformation $x_k \rightarrow y_k$ (like the output coupling vector C)

w_k : measurement noise

- Kalman filter: equations, cont.

– Covariance prediction (per definition)

$$
\begin{aligned}
\boldsymbol{P}_{k+1|k} &= E\left((\boldsymbol{x}_{k+1} - \hat{\boldsymbol{x}}_{k+1|k})(\boldsymbol{x}_{k+1} - \hat{\boldsymbol{x}}_{k+1|k})^T\right) \\
&= E\left((\boldsymbol{A}_k(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_k) + \boldsymbol{v}_k)(\boldsymbol{A}_k(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_k) + \boldsymbol{v}_k)^T\right) \\
&= E\left(\boldsymbol{A}_k(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_k)(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_k)^T\boldsymbol{A}_k^T\right) + \\
&\quad \underbrace{E\left(\boldsymbol{A}_k(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_k)\boldsymbol{v}_k^T + \boldsymbol{v}_k(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_k)^T\boldsymbol{A}_k^T\right)}_{\equiv 0} + E\left(\boldsymbol{v}_k\boldsymbol{v}_k^T\right)
\end{aligned}
$$

– Due to linearity of E and independence of v_k from x_k and $\hat{x}_k$, 2nd term is zero

– Resulted covariance prediction

$$
\begin{aligned}
\boldsymbol{P}_{k+1|k} &= \boldsymbol{A}_k E\left((\boldsymbol{x}_k - \hat{\boldsymbol{x}}_k)(\boldsymbol{x}_k - \hat{\boldsymbol{x}}_k)^T\right)\boldsymbol{A}_k^T + E\left(\boldsymbol{v}_k\boldsymbol{v}_k^T\right) \\
&= \boldsymbol{A}_k\boldsymbol{P}_k\boldsymbol{A}_k^T + \boldsymbol{Q}_k
\end{aligned}
$$

- Kalman filter: equations, cont.

Time-Update

– A-priori estimate of state vector ***x*** and error covariance matrix ***P***

$$\hat{\boldsymbol{x}}_{k+1|k} = \boldsymbol{A}_k \hat{\boldsymbol{x}}_k + \boldsymbol{B}_k \boldsymbol{u}_k$$

$$\boldsymbol{P}_{k+1|k} = \boldsymbol{A}_k \boldsymbol{P}_k \boldsymbol{A}_k^T + \boldsymbol{Q}_k$$

Measurement-Update

– Kalman gain ***L*** and a-posteriori estimate of the state vector and error covariance matrix (***P***)

$$\boldsymbol{L}_{k+1} = \boldsymbol{P}_{k+1|k} \boldsymbol{C}_{k+1}^T (\boldsymbol{C}_{k+1} \boldsymbol{P}_{k+1|k} \boldsymbol{C}_{k+1}^T + \boldsymbol{R}_{k+1})^{-1}$$

$$\hat{\mathbf{x}}_{k+1} = \hat{\mathbf{x}}_{k+1|k} + \boldsymbol{L}_{k+1} (y_{k+1} - \boldsymbol{C}\, \hat{\mathbf{x}}_{k+1|k})$$

$$\boldsymbol{P}_{k+1} = \boldsymbol{P}_{k+1|k} - \boldsymbol{L}_{k+1} \boldsymbol{C}_{k+1} \boldsymbol{P}_{k+1|k}$$

• Practical (hand on) session 10

Consider DC motor without load M_L (from page 49)

Assume the following parameter values: $R = 1, \quad L = 0.0002, \quad \Psi = 0.04,$

$$B = 0.0001, \quad J = 0.00005$$

1. Using the state-space model of the system, design the Luenberger state observer for estimating online the motor current state. The observer poles can be placed within a range $[-\lambda_I, \ldots, -0.5\lambda_I]$, where $-\lambda_I$ is the system pole which corresponds to the eigenvalue of the motor current.
2. Implement in Simulink the system plant and the designed Luenberger state observer. Compare the trajectories of the system state and estimated state for different initial values and, also, when there is some motor torque disturbance.

List of references

1. Modern control systems. R.C. Dorf, R.H. Bishop. 2017, Pearson
2. On governors. J.C. Maxwell, 1868, Proceedings of the Royal Society of London
3. Negative feedback, amplifiers, governors, and more [historical]. M. Guarnieri. IEEE Indust. Elect. Mag., 2017
4. Feedback control of dynamic systems. G.F. Franklin, J. Powell, A.F. Emami-Naeini. 2015, Pearson
5. Springer handbook of robotics. Eds. B. Siciliano, O. Khatib. 2016, Springer
6. Respect the unstable. G. Stein. IEEE Control Systems Magazine, 2003
7. Nonlinear systems and controls. J. Adamy. 2022, Springer
8. On stability of linear dynamic systems with hysteresis feedback. M. Ruderman. Math. Model. Nat. Phenom. 2020
9. Feedback control theory. J.C. Doyle, B.A. Francis, A.R. Tannenbaum. 2009, Dover
10. Identification of dynamic systems: an introduction with applications. R. Isermann, M. Münchhof. 2010, Springer
11. Nonlinear systems. H. Khalil, 2002, Prentice Hall
12. Asymptotic observer of the link states of flexible joint robots with motor-side sensing. M. Ruderman. IEEE IECON conf., 2015
13. Identification and control design for path tracking of hydraulic loader crane. M.H. Rudolfsen, T.N. Aune, O. Auklend, L.T. Aarland, M. Ruderman. IEEE/ASME AIM conf., 2017
14. Optimal state space control of DC motor. M. Ruderman, J. Krettek, F. Hoffmann, T. Bertram. IFAC World Congress, 2008
15. Mechatronic systems: fundamentals. R. Isermann. 2007, Springer
16. Full-and reduced-order model of hydraulic cylinder for motion control. M. Ruderman. IEEE IECON conf., 2017

- # List of references, cont.

17. Robot modeling and control. M.W. Spong, S. Hutchinson, M. Vidyasagar. 2005, Wiley & Sons
18. Elementary differential equations. W. Kohler, L. Johnson. 2006, Pearson
19. A linear systems primer. P.J. Antsaklis, A.N. Michel. 2007, Birkhäuser
20. Feedback systems: an introduction for scientists and engineers. K.J. Åström, R.M. Murray. 2010, Princeton
21. Multivariable feedback control: analysis and design. S. Skogestad, I. Postlethwaite. 2005, Wiley & Sons
22. Linear feedback control: analysis and design with MATLAB. D. Xue, Y.Q. Chen, D.P. Atherton. 2007, SIAM
23. Extended SDRE control of 1-DOF robotic manipulator with nonlinearities. M. Ruderman, D. Weigel, F. Hoffmann, T. Bertram. IFAC World Congress, 2011
24. Motion-control techniques of today and tomorrow: a review and discussion of the challenges of controlled motion. M. Ruderman, M. Iwasaki, W.H. Chen. Industrial Electronics Magazine, 2020
25. Hybrid impedance control of robotic manipulators. R.J. Anderson, M.W. Spong. IEEE Journal of Robotics and Automation, 1988
26. Hybrid position/force control for hydraulic actuators. P. Pasolli, M. Ruderman. IEEE MED20 conf., 2020
27. An introduction to observers. D. Luenberger. IEEE Transactions on Automatic Control, 1971
28. A new approach to linear filtering and prediction problems. R.E. Kalman. Transactions of the ASME–Journal of Basic Engineering, 1960