Sliding Mode Controllers: Stages of Development

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Outline

Section 1

Preliminaries

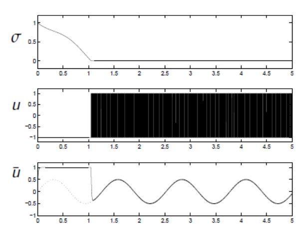
The simplest example

$\dot{\sigma} = \alpha + u = \alpha - \text{sign}(\sigma), \quad \sigma(0) = 1$

with $\alpha \in (-1, 1)$.

- $\sigma > 0 \Rightarrow \dot{\sigma} =< 0$
- $\sigma < 0 \Rightarrow \dot{\sigma} => 0$

and $\sigma(t) \equiv 0, \forall t \geq T$.



Remark

- $0 = \alpha - \text{sign}(0)$?
- The right-hand side is discontinuous.
- After arriving to $\sigma = 0$, sliding along $\sigma \equiv 0$.
- Finite-time convergence.
- Differential inclusion.

$$\dot{\sigma} \in [-\alpha, \alpha] - \text{sign}(\sigma)$$

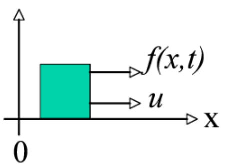
Mechanical system

A generic mechanic system

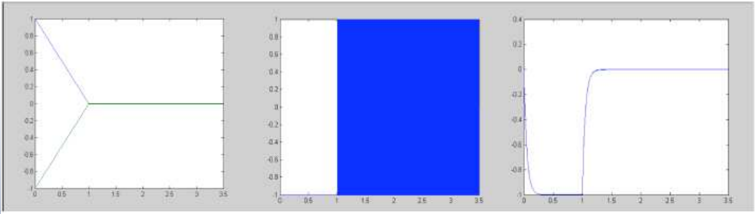
$$\begin{cases} \dot{x}_1 = & x_2 \\ \dot{x}_2 = & u + f(t,x) \\ \sigma = & x_2, \ |f(t,x)| < 1 \end{cases}$$

with σ as output and select

$$u = -\text{sign}(\sigma) = \text{dry friction}$$



x_1 : position.
x_2 : velocity.
σ : measurement



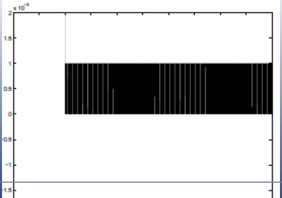
Summary: Late 50 th

Mathematics

- Theory of the differential equations with the discontinuous right hand site w.r.t. the state variables was needed:
- Specially for engineers: definition of solution on the discontinuity surface

Engineering

- Certainly we stopped, but where?
- No control over x_1 (position)
- Can we manipulate both x_1 and x_2 at the same time?
- High frequency discontinuous (switching) control
- Chattering



Filippov's solution of an ODE with discontinuous right-hand side(1960)

$$\dot{x} = f(x), \quad x(0) = x_0$$

with $\|f(x)\| \leq L, \forall x \in D$.

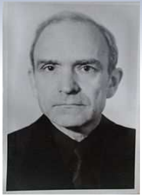


Figure: Prof. Filippov

$x(t)$ is a solution of the initial value problem on $[0, T]$ if it is absolutely continuous on $[0, T]$, $x(0) = x_0$ and

$$\dot{x}(t) \in K[f](x(t)) \quad \text{a. e. on } [0, T]$$

where

$$K[f](x) := \bigcap_{\delta > 0} \bigcap_{\mu(N)=0} co\{\ f(B(x,\delta) \backslash N)\}$$

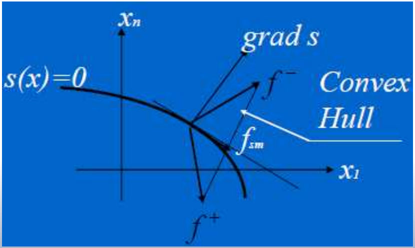
Filippov's definition of solution, 1960

$$\dot{x} = f(x),$$

- A sliding motion exists if the projections of the vectors $f^+ = f(x)^+, f^- = f(x)^-$ on grad(s) are of opposite signs
- The motion on the surface is $\dot{x} = f^0 := \mu f^+ + (1-\mu)f^-$ with μ computed to satisfy

$$\langle \text{grad}(s), f^0 \rangle = 0$$

- The surface characterizes the equivalent dynamics f_0.



Section 2

Stage 1: First Order Sliding Modes

Two Main Concepts of First Order Sliding Mode Control


Figure: Prof. Utkin and Prof. Emel'yanov. IFAC Sensitivity Conference, Dubronovik 1964

Equivalent control

$$\dot{x} = f(x,t) + B(x,t)u$$

with u discontinuous as previously defined.
To find the value of control u allowing to slide on the given the surface $s(x) = 0$ and given dynamics on s:

$$\dot{s} = Gf + GBu = 0, \;\; G = grad \;\; s$$

If GB is not singular $\forall(x,t)$ than an "equivalent control"

$$u_{eq}(x,t) := -[G(x)B(x,t)]^{-1}G(x)f(x,t)$$

The sliding mode dynamics

$$\dot{x} = f - B(GB)^{-1}Gf$$

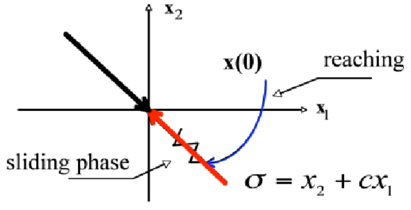
Sliding surface

Desired error dynamics

$$\sigma := x_2 + c x_1 = 0 \Longrightarrow x_1(t) = x_{10} e^{-ct}, x_2(t) = c x_{20} e^{-ct}$$

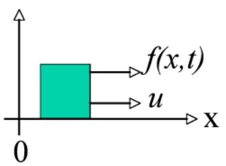
then:

- The manifold $\sigma = 0$ is known as the sliding surface
- The surface characterizes the desired dynamics
- The control objective of sliding mode control is to reach $\sigma = 0$ in finite time
- Once on the surface, the control must keep the trajectories "sliding" on the surface: sliding mode



Problem Formulation

$$\begin{cases} \dot{x}_1 = & x_2 \\ \dot{x}_2 = & u + f(t,x) \end{cases}$$



Problem formulation

Design $u(t)$ such that $\lim_{t\to\infty} x_1(t) = \lim_{t\to\infty} x_2(t) = 0$ and $\exists T > 0$ such that

$$\sigma(t) = 0, \forall t > T,$$

considering bounded uncertainty, i.e.

$$|f(x,t)| \leq L$$

that represents modeling imperfections and external perturbations.

Invariance of sliding-modes [B. Drazenovic]

B. Drazenovic. "**The invariance conditions in variable structure systems**", Automatica, v.5, No.3, Pergamon Press, 1969.

$$\dot{x} = f(x,t) + B(x,t)u + h(t,x)$$

with $h(t,x)$ as uncertainty. A sliding-mode is insensitive against uncertainty satisfying

$$h(t,x) \in \text{span}\{B(x,t)\}$$

(matched perturbations). Under this condition $\exists \lambda \in \mathbb{R}^m | h = B\lambda$ and then

$$\dot{x} = f(x,t) + B(x,t)[u + \lambda]$$

Figure: Prof. Drazenovic

Design in the regular form [Louk'yanov, 1981]

A. Loukyanov, V. Utkin. "**Reducing dynamic systems to the regular form**". Automation and Remote Control, No 3, pp, 5-13., 1981.

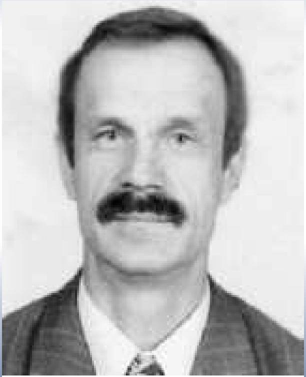


Figure: Prof. Louk'yanov

Design in the regular form [Louk'yanov, 1981]

$$
\begin{aligned}
\dot{\bar{x}}_1 &= \bar{f}_1(\bar{x}_1, \bar{x}_2) \\
\dot{\bar{x}}_2 &= \bar{f}_2(\bar{x}_1, \bar{x}_2) + \bar{B}(\bar{x}_1, \bar{x}_2)[u + \lambda]
\end{aligned}
$$

- Fictitious control: $\bar{x}_2 = -s_0(\bar{x}_1)$.
- Sliding surface: $\sigma(\bar{x}_1, \bar{x}_2) = \bar{x}_2 + s_0(\bar{x}_1) = 0$
- Equations on sliding

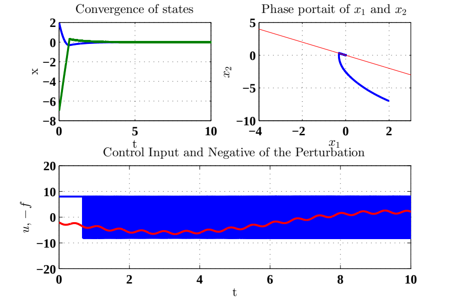
$$\dot{\bar{x}}_1 = \bar{f}_1\left(\bar{x}_1, -s_0(\bar{x}_1)\right)$$

that does not depend on $f_2(\cdot)$ nor $B_2(\cdot)$.

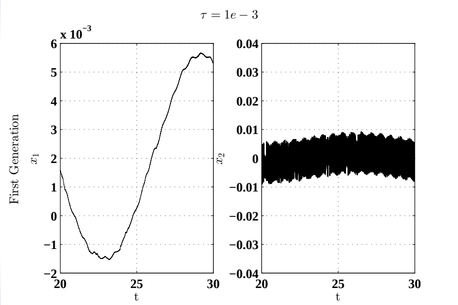
Example: First Order Sliding Mode

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u + f$$

- x_1, x_2 are the states
- u is the control
- $f = 2 + 4sin(t/2) + 0.6sin(10t)$.
- $\sigma = x_1 + x_2$



First Order Sliding Mode: Precision



Sliding Mode Differentiator

- Signal to differentiate: $f(t)$
- Assume $|\dot{f}(t)| \leq M$
- Find differentiator

$$y = f(t), \quad \dot{y} = \dot{f},$$

- Sliding Mode Differentiator

$$\dot{x} = k\,\mathrm{sign}(e), \;\; e = y - x$$

- $\dot{e} = \dot{f} - k\,\mathrm{sign}(e) \;\; =>$
- in finite time $\dot{e} = 0 \quad => \quad \dot{f} =$ filtered $k\,\mathrm{sign}(e)$

SUMMARY: First order sliding modes

Advantages

- Theoretically exact compensation of matched uncertainties it supposed that the states are available
- Reduces SMC design to control selection for two reduced order systems
- Saturated control law
- Ensures finite-time convergence to the sliding surface

Disadvantages

- Chattering
- For SISO systems the dimension of sliding dynamics is reduced just for 1
- State variables converge asymptotically
- High order derivatives are needed to design sliding surfaces The theory was not complete:theoretically exact compensation needs theoretically exact differentiation

Section 3

Stage 2: Second Order Sliding Modes

Chattering as the relative degree problem

Figure: Prof. Levant and Prof. Fridman

Second Order Sliding Modes

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u + f(x,t) \\ \sigma &= x_1 \end{cases}$$

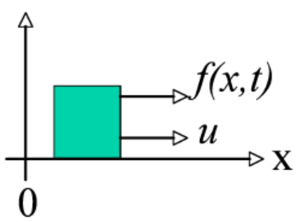


Figure: Prof. Emelyanov,Prof. Korovin and Prof. Levantovsky

- $f(x,t)$ unknown uncertainties/perturbations.
- All the partial derivatives of $f(x,t)$ are bounded on compacts

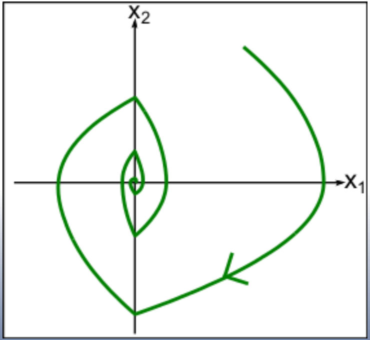
Main Objective

To design a control u such that the origin of system is finite-time stable, in spite of the uncertainties/perturbations $f(x,t)$, with $|f(x,t)| < f^+$ for all t, x

Twisting algorithm

$$u = -a\,\text{sign}(x_2) - b\,\text{sign}(x_1),\;\; b > a + f^+, a > f^+.$$

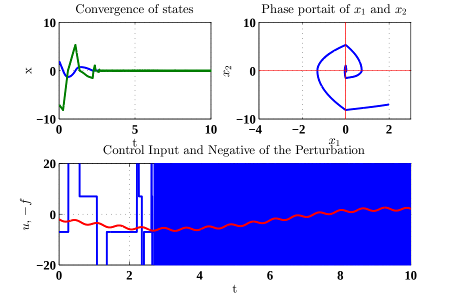
- Known bounds f^+
- a and b chosen appropriately (Emelyanov et al. 86),
- Ensures finite-time exact convergence for both x_1 and x_2



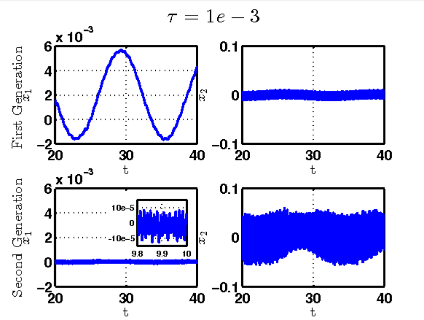
Twisting Algorithm

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u + f$$

- x_1, x_2 are the states
- u is the control
- $f = 2 + 4sin(t/2) + 0.6sin(10t)$.



Comparison First Stage vs Second Stage:Precision



Anti-chattering Strategy

$$\dot{X} = F(t, X) + G(t, X)u, X \in R^n, u \in R, |F| < F^+,$$

The switching variable $\sigma(X)$: $\dot{\sigma} = f(\sigma, t) + g(\sigma, t)u.$

Anti-chattering strategy:

Add an Integrator in control input:

If $\dot{u} = v = -a\,\mathrm{sign}(\dot{\sigma}(t)) - b\,\mathrm{sign}(\sigma(t))$, so u is a Lipschitz continuous control signal ensuring finite-time convergence to $\sigma = 0$

Criticism(1987) **If it is possible to measure $\dot{\sigma} = f(t, \sigma) + g(t, \sigma)u$, then the uncertainty $f(t, \sigma) = \dot{\sigma} - g(t, \sigma)u$ is also known and can be compensated without any discontinuous control!**

Counter-argument

If g is uncertain so $\ddot{\sigma}$ depends on u through uncertainty! The anti-chattering strategy is reasonable for the case of uncertain control gains.

Discussion about SOSM

Advantages of SOSM

1. Allows to compensate bounded matched uncertainties for the systems with relative degree two with discontinuous control signal
2. Allows to compensate Lipschitz matched uncertainties with continuous control signal using the first derivative of sliding inputs
3. Ensures quadratic precision of convergence with respect to the sliding output
4. For one degree of freedom mechanical systems: the sliding surface design is no longer needed.
5. For systems with relative degree r: the order of the sliding dynamics is reduced up to $(r-2)$. The design of the sliding surface of order $(r-2)$ is still necessary!

OPEN PROBLEMS:EARLY 90th

- To reduce the chattering substituting **discontinuous control signal with continuous one** the derivative of the sliding input still needed!
- The problem of exact finite-time stabilization and exact disturbance compensation for SISO systems with arbitrary relative degree remains open. More deep decomposition is still needed
- Theoretically exact differentiators are needed to realize theoretically exact compensation of the Lipschitz matched uncertainties

First Stage vs Second Stage

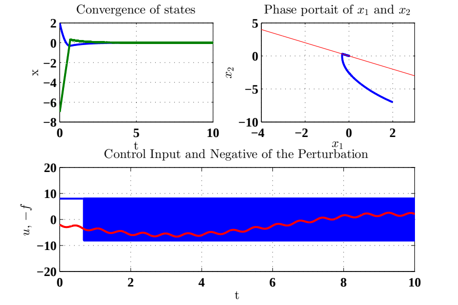


Figure: First Stage

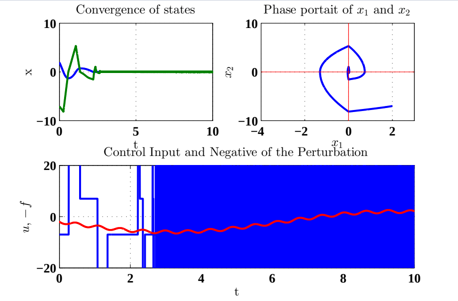


Figure: Second Stage

Terminal Algorithm

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u(x),$$
$$u(x) = -\alpha \,\mathrm{sign}(s(x)),$$
$$s(x) = x_2 + \beta\sqrt{|x_1|}\,\mathrm{sign}(x_1).$$

Figure: Prof. Z. Man

Relative Degree of Terminal Sliding Variable

Time derivative of the switching surface

$$\dot{s}(x) = \dot{x}_2 + \beta \frac{x_2}{2\sqrt{|x_1|}} = -\alpha \operatorname{sign}(s(x)) + \beta \frac{x_2}{2\sqrt{|x_1|}}.$$

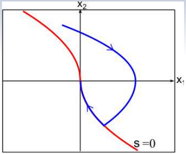
- $s(x)$ is singular for $x_1 = 0$, and **the relative degree of the switching surface does not exist**
- On $x_2 = -\beta\sqrt{|x_1|}\operatorname{sign}(x_1)$

$$\dot{s} = -\alpha \operatorname{sign}(s(x)) - \frac{\beta^2}{2} \operatorname{sign}(x_1).$$

- Two types of behavior for the solution of the system are possible

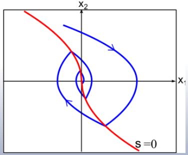
Terminal mode:

- $\beta^2 < 2\alpha$,
- Trajectories of the system reach the surface $s(x) = 0$ and remain there.



Twisting mode

- $\beta^2 > 2\alpha$
- Trajectories do not slide on the surface $s(x) = 0$



Section 4

Stage 3: Super-Twisting Algorithm

The Super–Twisting Algorithm (STA)

Emalyanov, Korovin, Levantovsky, 1990, Levantovsky 1993

$$\dot{x} = f(t) + g(t)u,$$

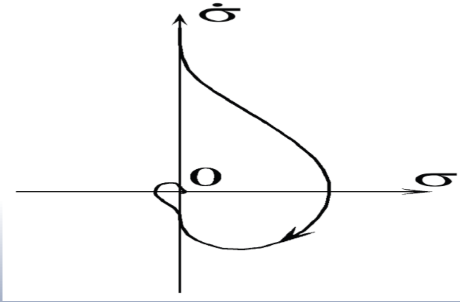
$$u = -k_1|x|^{\frac{1}{2}}\operatorname{sign}(x) + v,$$

Integral extension

$$\dot{v} = -k_2\operatorname{sign}(x),$$

$f(x(t), t)$ is Lipschitz disturbance

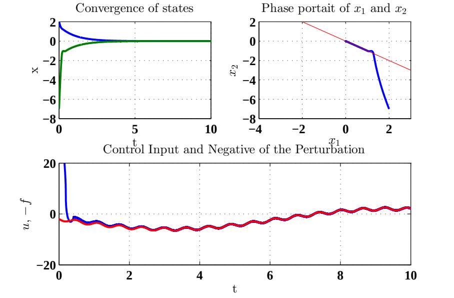
- Continuous control signal
- Exact finite time convergence to $x(t) = \dot{x}(t) = 0, \; \forall t \geq T$
- The derivative of x is not used!!!
- If x is measured, the STA is an output-feedback controller



Example Super-Twisting Algorithm

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u + f$$

- x_1, x_2 are the states
- u is the control
- $f = 2 + 4sin(t/2) + 0.6sin(10t)$.
- $\sigma = x_1 + x_2$



Second Stage vs Third Stage

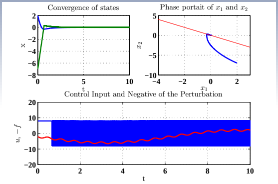


Figure: First Stage

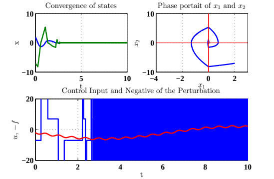


Figure: Second Stage

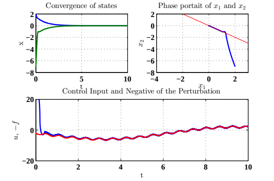
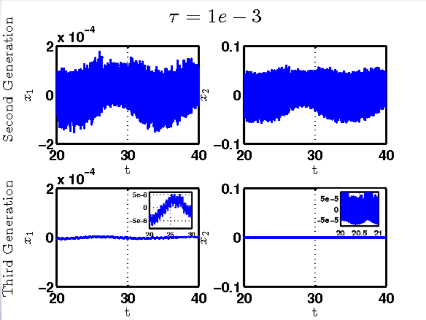


Figure: Third Stage

Comparison Second Stage to Third Stage:Precision



Robust Exact Differentiator, Levant(1998)

- Signal to differentiate: $f(t)$
- Assume $|\ddot{f}(t)| \leq L$
- Find an observer for

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = \ddot{f}, \quad y = x_1,$$

- $\ddot{f}(t)$ bounded perturbation.
- STA observer

$$\dot{\hat{x}}_1 = k_1 |y - \hat{x}_1|^{\frac{1}{2}} \operatorname{sign}(y - \hat{x}_1) + \hat{x}_2,$$
$$\dot{\hat{x}}_2 = k_2 \operatorname{sign}(y - \hat{x}_1), \; k_2 > L$$

- Convergence of STA assures:$(f - \hat{x}_1) = (\dot{f} - \hat{x}_2) = 0$ after finite time without filtration!

SUMMARY

Advantages

1. Continuous control signal compensating Lipschitz uncertainties
2. Chattering attenuation but not its complete removal! (Boiko, Fridman 2005)
3. Differentiator obtained using the STA:
   - Finite-time exact estimation of derivatives in the absence of both noise and sampling;
   - Best possible asymptotic approximation in the sense of Kolmogorov 62.

Open problems: End of 20th century

Open problems: End of 20th century

1. Relative degree $r \geq 2$: Need sliding surface. Consequently the states converge to the origin asymptotically. Deeper decomposition is needed!
2. STA based differentiator for the sliding surface design is not enough: he can not provide the best possible precision for highest derivatives
3. STA signal can grow together with perturbation! Saturation is needed
4. Direct application of STA together with the first order differentiator for control of mechanical system can not be done because it is necessary to form sliding surface with Lipschitz derivatives

Section 5

Stage 4:Arbitrary Order Sliding Mode Controllers

Arbitrary Order Sliding Mode Controllers

$$\dot{X} = F(t, X) + G(t, X)u, X \in R^n, u \in R$$
$$\sigma = \sigma(X, t), \in R.$$

- σ has a fixed and known relative degree r.
- Control problem is transformed into the finite-time stabilization of an uncertain differential equation

$$\sigma^{(r)} = f(t, X) + g(t, X)u, \quad (1)$$

and corresponding differential inclusion

$$\sigma^{(r)} \in [-C, C] + [K_m, K_M]u, \quad (2)$$

where C, K_m and K_M are known constants.

Nested arbitrary order sliding-mode controllers

- 2001: Nested arbitrary order SM controller
- Solve the finite-time exact stabilization problem for an output with an arbitrary relative degree.
- Bounded Lebesgue measurable uncertainties.
- "Nested" higher order sliding-mode(HOSM) controllers are constructed using a recursion

Figure: Prof. Levantovsky

Nested Third Order Singular Terminal Algorithm

- Third Order

$$u = -\alpha \,\mathrm{sign}\left(\ddot{\sigma} + 2(|\dot{\sigma}|^3 + |\sigma|^2)^{\frac{1}{6}} \times \mathrm{sign}(\dot{\sigma} + |\sigma|^{\frac{2}{3}}\,\mathrm{sign}(\sigma))\right)$$

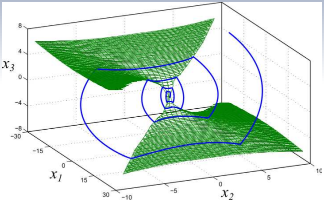


Figure: 3rd Order Nested SM

- Fourth Order

$$u = -\alpha \,\mathrm{sign}\left(\dddot{\sigma} + 3(\ddot{\sigma}^6 + \dot{\sigma}^4 + \sigma^3)^{\frac{1}{12}} \times \mathrm{sign}\left(\ddot{\sigma} + (\dot{\sigma}^4 + |\sigma|^3)^{\frac{1}{6}}\,\mathrm{sign}(\dot{\sigma} + 0.5|\sigma|^{\frac{3}{4}}\,\mathrm{sign}(\sigma))\right)\right)$$

- Finite-time stabilization of $\sigma = 0$ and its successive derivatives up to $r-1$.

HOSM Differentiator

- The Nested Controller needs the output and its successive derivatives
- Instrument: HOSM arbitrary order differentiator
- Let $\sigma(t)$ signal to be differentiated $k-1$ times
- Assume that $|\sigma^{(k)}| \leq L$.
- 3-th order HOSM differentiator

$$\begin{aligned}
\dot{z}_0 &= v_0 = -3L^{\frac{1}{4}} \left|z_0 - \sigma\right|^{\frac{3}{4}} \operatorname{sign}(z_0 - \sigma) + z_1, \\
\dot{z}_1 &= v_1 = -2L^{\frac{1}{3}} \left|z_1 - v_0\right|^{\frac{2}{3}} \operatorname{sign}(z_1 - v_0) + z_2, \\
\dot{z}_2 &= v_2 = -1.5L^{\frac{1}{2}} \left|z_2 - v_1\right|^{\frac{1}{2}} \operatorname{sign}(z_2 - v_1) + z_3 \\
\dot{z}_3 &= -1.1L \operatorname{sign}(z_3 - v_2)
\end{aligned} \tag{3}$$

- z_i true derivative $\sigma^{(i)}(t)$.

Black Box Control Concept for HOSM

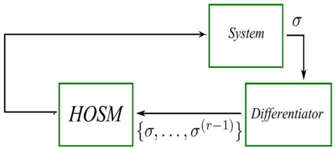


Figure: Black Box Control Concept for HOSM

Advantages of nested HOSM for SISO systems with relative degree r

- Theoretically exact disturbance compensation basing on output information only
- Full dynamical collapse: ensures $\sigma = \dot{\sigma} = \ddot{\sigma} = \cdots = \sigma^{(r-1)} = 0$ in finite-time
- Ensures the r-th order precision for the sliding output with respect to the discretization step and fast parasitic dynamics
- The sliding surface design is no longer needed

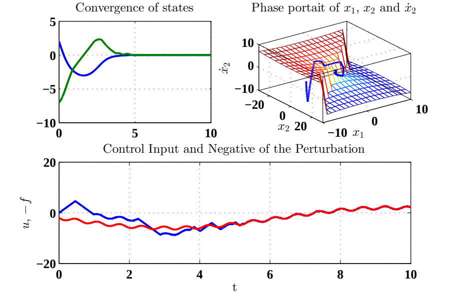
Open problems: After 2005

- For SISO systems with relative degree r still produces a discontinuous control signal
- Anti-chattering strategy: the information about $\sigma^{(r)}$ containing perturbations is needed

Example Nested 3rd Order Singular Terminal Controller with anti-chattering strategy

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u + f$$
$$\dot{u} = u_2$$

- x_1, x_2 are the states
- u_2 is the control
- $f = 2 + 4sin(t/2) + 0.6sin(10t)$.
- $\sigma = x_1$



First Stage to Fourth Stage

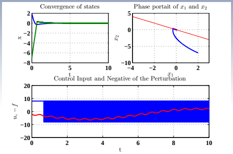


Figure: First Stage

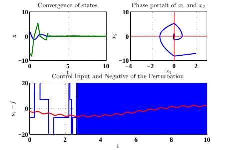


Figure: Second Stage

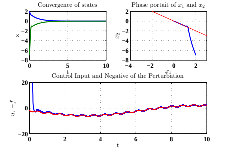


Figure: Third Stage

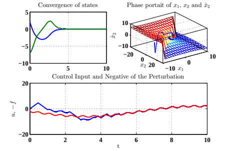
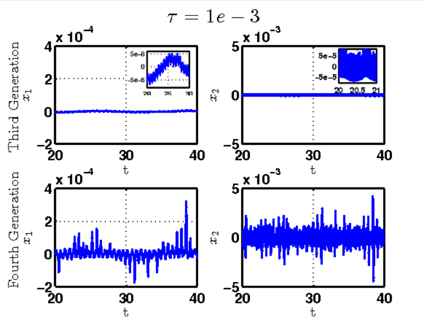


Figure: Fourth Stage

Comparison Third Stage to Fourth Stage:Precision



Section 6

Stage 5: Continuous Arbitrary Order Sliding-Mode Controllers

Continuous Arbitrary Order Sliding-Mode Controllers

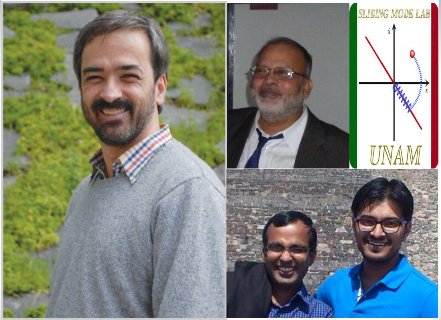


Figure: UNAM: J. Moreno IIT: B. Bandyopadhyay, S. Kamal, A. Chalanga, Shtessel& Edwards

Properties of CHOSM

For the systems with relative degree r

- Continuous control signal
- Finite-time convergence to the $(r+1)$-th order sliding-mode set
- Derivatives of the output up to the $(r-1)$ order

Continuous arbitrary order sliding-mode controllers

- Discontinuous–Integral Algorithm(D-I), (Zamora, Moreno,2013)
- Two versions of the Continuous Terminal Sliding Mode Algorithm(CTSMA) (Mexico- India 2014-16)
  (a) Continuous Singular Terminal Sliding Mode Algorithm (CSTSMA);
  (b) Continuous Nonsingular Terminal Sliding Mode Algorithm (CNTSMA);.
- Continuous Twisting Algorithm(CTA)(Moreno, Fridman et al 2015-18)
- Arbitrary Order Continuous Sliding Mode Controller Laghrouche et al(2017)

Discontinuous - Integral(D-I) Algorithm

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u + f(t) \\ \sigma &= x_1 \end{cases}$$

$$u = -k_1 \lfloor x_1 \rceil^{1/3} - k_2 \lfloor x_2 \rceil^{1/2} - \int_0^t (k_3 \lfloor x_1(\tau) \rceil^0) d\tau, \quad (4)$$

where k_1, k_2, k_3 are appropriate positive gains.
New Notation: $\lfloor z \rceil^p = |z|^p sgn(z)$
NONLINEAR PID!

Continuous Twisting Algorithm (CTA)

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u + f(t) \\ \sigma &= x_1 \end{cases}$$

$$u = -k_1 \lfloor x_1 \rceil^{1/3} - k_2 \lfloor x_2 \rceil^{1/2} - \int_0^t (k_3 \lfloor x_1(\tau) \rceil^0 + k_4 \lfloor x_2(\tau) \rceil^0) d\tau, \qquad (5)$$

where k_1, k_2, k_3, k_4 are appropriate positive gains.
New Notation: $\lfloor z \rceil^p = |z|^p sgn(z)$

Continuous Twisting Algorithm (CTA)

- Closed Loop System

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_1 \lfloor x_1 \rceil^{1/3} - k_2 \lfloor x_2 \rceil^{1/2} + x_3 \\ \dot{x}_3 &= -k_3 \lfloor x_1 \rceil^{0} - k_3 \lfloor x_2 \rceil^{0} + \rho, \end{cases} \tag{6}$$

- $\rho = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial t}$, and $|\rho| \leq \Delta$.
- Twisting structure to reject perturbations.

Simulation: CTA

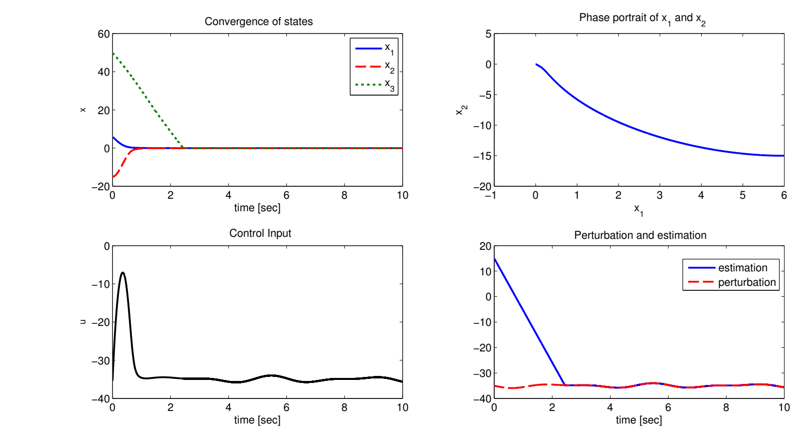


Figure: Numerical results for a double integrator with perturbation

Discussion about the CTA

Advantages

- Algorithms homogeneous of degree $\delta_f = -1$, with weights $\rho = 3, 2, 1$.
- **The only information needed to mantain finite time convergence of all three variables x_1, x_2 and x_3 is the output (x_1) and its derivative (x_2)**
- Precision corresponds to a 3rd order sliding mode

Continuous Singular Terminal Sliding Mode Algorithm (CSTSMA)

$$\begin{cases} \dot{x}_1 & = x_2 \\ \dot{x}_2 & = u + f(t) \\ \sigma & = x_1 \end{cases}$$

$$u = -\,k_1 \lfloor \phi \rceil^{1/2} - k_3 \int_0^t \lfloor \phi \rceil^0 d\tau, \tag{7}$$

where $\phi = \left(x_2 + k_2 \lfloor x_1 \rceil^{2/3}\right)$, and k_1, k_2, k_3 are appropriate positive gains.

Continuous Singular Terminal Sliding Mode Algorithm (CSTSMA)

- Closed Loop System

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_1 \lfloor \phi \rceil^{1/2} + x_3 \\ \dot{x}_3 &= -k_3 \lfloor \phi \rceil^{0} + \rho, \end{cases} \tag{8}$$

- $\rho = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial t}$, and $|\rho| \leq \Delta$.
- Combination of the Super-Twisting algorithm with the Singular Terminal Sliding mode.

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Continuous Nonsingular Terminal Sliding Mode Algorithm (CNTSMA)

$$\begin{cases} \dot{x}_1 & = x_2 \\ \dot{x}_2 & = u + f(t) \\ \sigma & = x_1 \end{cases}$$

$$u = -k_1 \lfloor \phi_N \rceil^{1/3} - k_3 \int_0^t \lfloor \phi_N \rceil^0 d\tau, \tag{9}$$

where $\phi_N = \left(x_1 + k_2 \lfloor x_2 \rceil^{3/2} \right)$, and $k_1, k_2,\ k_3$ are appropriate positive gains.

Continuous Nonsingular Terminal Sliding Mode Algorithm (CNTSMA)

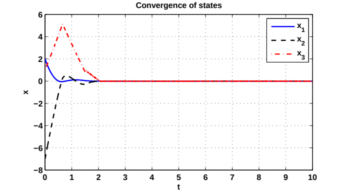
- Closed Loop System

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_1 \lfloor \phi_N \rceil^{1/3} + x_3 \\ \dot{x}_3 &= -k_3 \lfloor \phi_N \rceil^{0} + \rho, \end{cases} \tag{10}$$

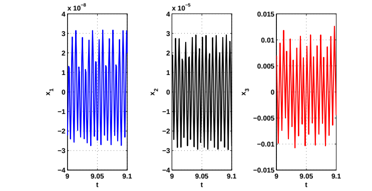
- $\rho = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial t}$ and $|\rho| \leq \Delta$.
- Combination of the Super-Twisting algorithm with the Nonsingular Terminal Sliding Mode algorithm.

Stage 5: Continuous Arbitrary Order Sliding-Mode Controllers

Simulation:CSTSMA



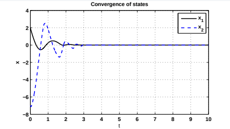
(a)



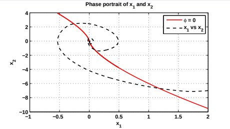
(b)

Continuous Singular Terminal Sliding Mode Control(CSTSMC)

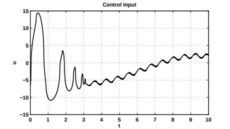
Twisting controller-like Behavior.



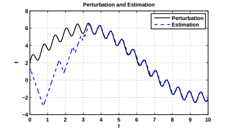
(a)



(b)



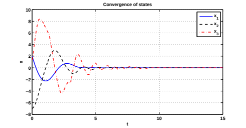
(c)



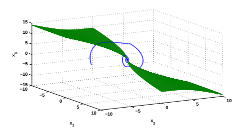
(d)

Figure: Numerical example uncertain double integrator

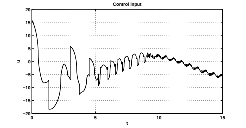
Continuous Nonsingular Terminal Sliding Mode Control(CNTSMC)



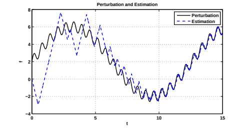
(a)



(b)



(c)



(d)

Figure: Numerical example uncertain triple integrator

Sliding-Like Behavior of CNTSMC

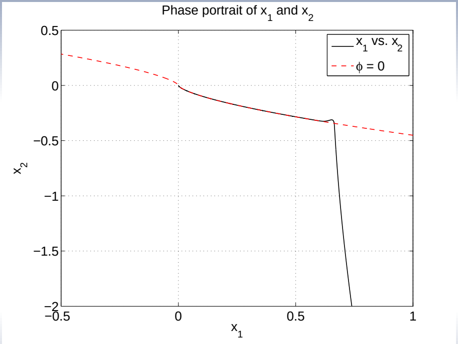


Figure: Phase portrait of Plant's states x_1 and x_2, and locus of the switching curve $\phi = \phi_N = 0$, showing a Sliding-Like behavior of the CNTSMC

Twisting-Like Behavior of CNTSMC

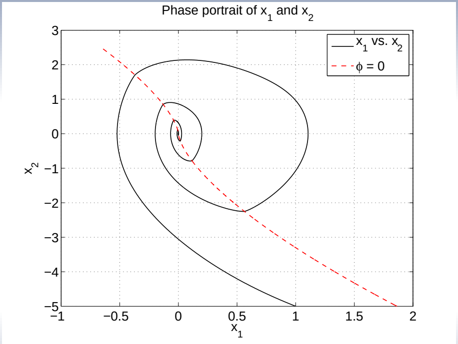


Figure: Phase portrait of Plant's states x_1 and x_2, and locus of the switching curve $\phi = \phi_N = 0$, showing a Twisting-Like behavior of the CNTSM controller.

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Stage 5: Continuous Arbitrary Order Sliding-Mode Controllers

All five Stages

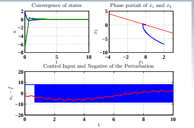


Figure: First Stage

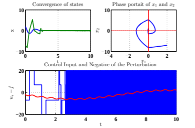


Figure: Second Stage

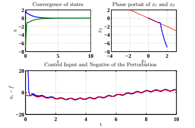


Figure: Third Stage

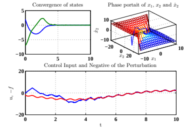


Figure: Fourth Stage

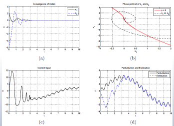


Figure: Fifth Stage:CSTSMC

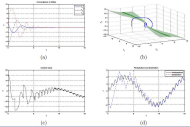
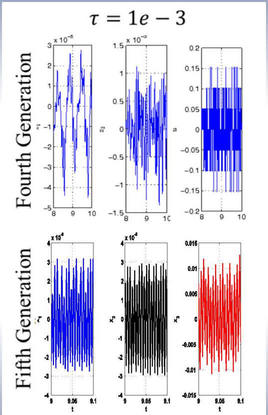
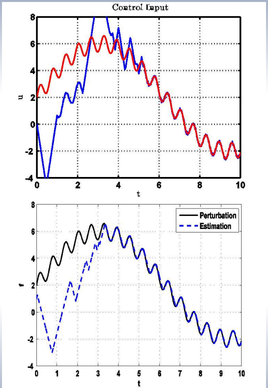


Figure: Fifth Stage:CNTSMC

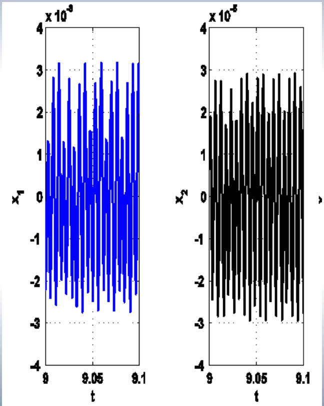
Fourth Stage vs Fifth Stage:Precision

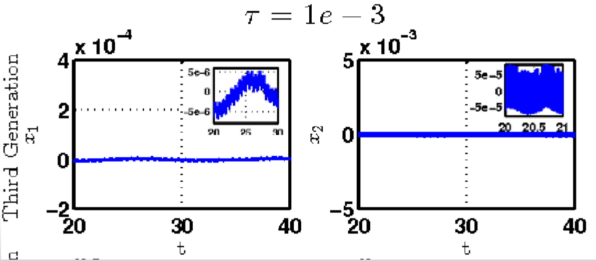




Same precision and smoothness of control without using $\ddot{\sigma}$

Third Stage vs Fifth Stage:Precision





Comparison 5 Stages

Algorithm	Control Signal	Information	Stability	Precision w.r.t. sampling
First SMC	Discontinuous	$\sigma, \dot{\sigma}$	Asymptotic	$O(h)$
2SMC	Discontinuous	$\sigma, \dot{\sigma}$	Finite time	$0(h^2)$
Super-twisting	Continuous	$\sigma, \dot{\sigma}$	Asymptotic	$0(h^2)$
3SMC + anti-chattering	Continuous	$\sigma, \dot{\sigma}, \ddot{\sigma}$	Finite time	$0(h^3)$
Continuous 2SMC	**Continuous**	$\sigma, \dot{\sigma}$	**Finite time**	$0(h^3)$

Third Order CTA (Mendoza, Fridman, Moreno,2017)

- 3-CTA

$$\dot{x}_1 = x_2 \tag{11}$$

$$\dot{x}_2 = x_3 \tag{12}$$

$$\dot{x}_3 = -k_1 \lceil x_1 \rfloor^{\frac{1}{4}} - k_2 \lceil x_2 \rfloor^{\frac{1}{3}} - k_3 \lceil x_2 \rfloor^{\frac{1}{2}} + x_4 \tag{13}$$

$$\dot{x}_4 = -k_4 \lceil x_1 \rfloor^{0} - k_5 \lceil x_2 \rfloor^{0} \tag{14}$$

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3-CTA

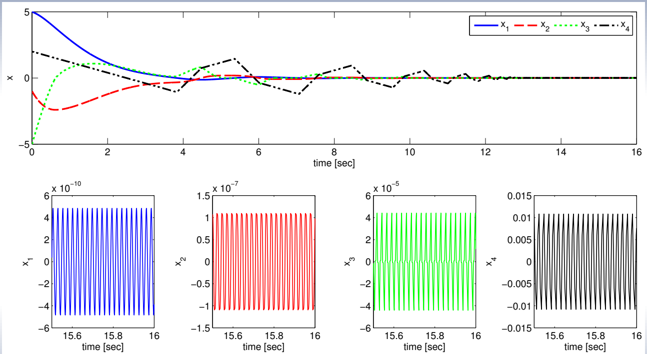

Figure: 3-CTA and states precision with $\tau = 0.001$

Discussion about C2SM

C2SM

- Homogeneous of degree $\delta_f = -1$, with weights $\rho = 3, 2, 1$.
- The only information that needed to mantain finite time convergence of all three variables x_1, x_2 and x_3 is the output (x_1) and its derivative (x_2)
- Can work for an uncertain system with relative degree 2 with respect to its output.
- Can compensate bounded Lebesgue measurable perturbations.

r-CHOSM

- Homogeneous of degree $\delta_f = -1$, with weights $\rho = r, r-1, \cdots, 2, 1$.
- Can be used for uncertain system with relative degree $r-1$ with respect to output.
- **Insensible to perturbations whose time derivative is bounded ! (can not grow faster than a linear function of time!)**

Section 7

Conclusions

Conclusions

- Last three decades new generations of controllers:
  - second order slding mode controllers(1985);
  - super-twisting controllers(1993);
  - arbitrary order sliding-mode controllers(2001,2005).
- We have presented the next generation: two families of continuous nested sliding-mode controllers, that can be used on Lipschitz systems with relative degree r, providing a continuous control signal.
- New controllers ensure a finite-time convergence of the sliding output to the $(r+1)-th$-order sliding set using information on the sliding output and its derivatives up to the order $(r-1)$.