Sliding Mode Controllers: Stages of Development

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Outline

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3. Stage 2: Second Order Sliding Modes
4. Stage 3: Super-Twisting Algorithm
5. Stage 4: Arbitrary Order Sliding Mode Controllers
6. Stage 5: Continuous Arbitrary Order Sliding-Mode Controllers
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Section 1

Preliminaries
The simplest example

\[ \dot{\sigma} = \alpha + u = \alpha - \text{sign}(\sigma), \quad \sigma(0) = 1 \]

with \( \alpha \in (-1, 1) \).

- \( \sigma > 0 \Rightarrow \dot{\sigma} = < 0 \)
- \( \sigma < 0 \Rightarrow \dot{\sigma} = > 0 \)

and \( \sigma(t) \equiv 0, \forall t \geq T \).

Remark

- \( 0 = \alpha - \text{sign}(0) \)?
- The right-hand side is discontinuous.
- After arriving to \( \sigma = 0 \), sliding along \( \sigma \equiv 0 \).

- Finite-time convergence.
- Differential inclusion.

\[ \dot{\sigma} \in [\!-\alpha, \alpha \!] - \text{sign}(\sigma) \]
Mechanical system

A generic mechanic system

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u + f(t, x) \\
\sigma &= x_2, \quad |f(t, x)| < 1
\end{align*}
\]

with \(\sigma\) as output and select

\[u = -\text{sign} (\sigma) = \text{dry friction}\]
Summary: Late 50th

Mathematics
- Theory of the differential equations with the discontinuous right hand site w.r.t. the state variables was needed:
  - Specially for engineers: definition of solution on the discontinuity surface

Engineering
- Certainly we stopped, but where?
- No control over $x_1$ (position)
- Can we manipulate both $x_1$ and $x_2$ at the same time?
- High frequency discontinuous (switching) control
- Chattering
Filippov’s solution of an ODE with discontinuous right-hand side (1960)

\[ \dot{x} = f(x), \quad x(0) = x_0 \]

with \( \|f(x)\| \leq L, \forall x \in D \).

\( x(t) \) is a solution of the initial value problem on \([0, T]\) if it is absolutely continuous on \([0, T]\), \( x(0) = x_0 \) and

\[ \dot{x}(t) \in K[f](x(t)) \quad \text{a. e. on} \ [0, T] \]

where

\[ K[f](x) := \bigcap_{\delta > 0} \bigcap_{\mu(N) = 0} co \{ f(B(x, \delta) \setminus N) \} \]
Filippov’s definition of solution, 1960

\[ \dot{x} = f(x), \]

- A sliding motion exists if the projections of the vectors \( f^+ = f(x)^+ \) and \( f^- = f(x)^- \) on \( \text{grad}(s) \) are of opposite signs.
- The motion on the surface is \( \dot{x} = f^0 := \mu f^+ + (1 - \mu) f^- \) with \( \mu \) computed to satisfy
  \[ \langle \text{grad}(s), f^0 \rangle = 0 \]
- The surface characterizes the equivalent dynamics \( f_0 \).
Section 2

Stage 1: First Order Sliding Modes
Two Main Concepts of First Order Sliding Mode Control

Figure: Prof. Utkin and Prof. Emel’yanov. IFAC Sensitivity Conference, Dubronovik 1964
Stage 1: First Order Sliding Modes

**Equivalent control**

\[ \dot{x} = f(x, t) + B(x, t)u \]

with \( u \) discontinuous as previously defined.

To find the value of control \( u \) allowing to slide on the given the surface \( s(x) = 0 \) and given dynamics on \( s \):

\[ \dot{s} = Gf + GBu = 0, \quad G = \nabla s \]

If \( GB \) is not singular \( \forall (x, t) \) than an "equivalent control"

\[ u_{eq}(x, t) := -[G(x)B(x, t)]^{-1}G(x)f(x, t) \]

The sliding mode dynamics

\[ \dot{x} = f - B(GB)^{-1}Gf \]
Stage 1: First Order Sliding Modes

Sliding surface

Desired error dynamics

\[ \sigma := x_2 + cx_1 = 0 \implies x_1(t) = x_{10}e^{-ct}, \quad x_2(t) = cx_{20}e^{-ct} \]

then:

- The manifold \( \sigma = 0 \) is known as the sliding surface
- The surface characterizes the desired dynamics
- The control objective of sliding mode control is to reach \( \sigma = 0 \) in finite time
- Once on the surface, the control must keep the trajectories “sliding” on the surface: sliding mode
Stage 1: First Order Sliding Modes

Problem Formulation

\[ \begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u + f(t, x)
\end{align*} \]

Design \( u(t) \) such that \( \lim_{t \to \infty} x_1(t) = \lim_{t \to \infty} x_2(t) = 0 \) and \( \exists T > 0 \) such that

\[ \sigma(t) = 0, \forall t > T, \]

considering bounded uncertainty, i.e.

\[ |f(x, t)| \leq L \]

that represents \textit{modeling imperfections and external perturbations}. 

\[ f(x, t) \]

\[ u \]

\[ x \]

\[ 0 \]
Stage 1: First Order Sliding Modes

Invariance of sliding-modes [B. Drazenovic]


\[ \dot{x} = f(x, t) + B(x, t)u + h(t, x) \]

with \( h(t, x) \) as uncertainty. A sliding-mode is insensitive against uncertainty satisfying

\[ h(t, x) \in \text{span}\{B(x, t)\} \]

(matched perturbations). Under this condition \( \exists \lambda \in \mathbb{R}^m | h = B\lambda \) and then

\[ \dot{x} = f(x, t) + B(x, t)[u + \lambda] \]
Stage 1: First Order Sliding Modes

Design in the regular form [Louk’yanov, 1981]


Figure: Prof. Louk’yanov
Stage 1: First Order Sliding Modes

Design in the regular form \([\text{Louk’yanov, 1981}]\)

\[
\begin{align*}
\dot{x}_1 &= \bar{f}_1(x_1, x_2) \\
\dot{x}_2 &= \bar{f}_2(x_1, x_2) + \bar{B}(x_1, x_2)[u + \lambda]
\end{align*}
\]

- **Fictitious control**: $\bar{x}_2 = -s_0(x_1)$.
- **Sliding surface**: $\sigma(x_1, \bar{x}_2) = \bar{x}_2 + s_0(x_1) = 0$
- **Equations on sliding**
  \[
  \dot{x}_1 = \bar{f}_1(x_1, -s_0(x_1))
  \]
  that does not depend on $f_2(\cdot)$ nor $B_2(\cdot)$. 
Stage 1: First Order Sliding Modes

Example: First Order Sliding Mode

\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = u + f \]

- \( x_1, x_2 \) are the states
- \( u \) is the control
- \( f = 2 + 4\sin(t/2) + 0.6\sin(10t) \).
- \( \sigma = x_1 + x_2 \)
Stage 1: First Order Sliding Modes

First Order Sliding Mode: Precision

\[ \tau = 10^{-3} \]
Stage 1: First Order Sliding Modes

**Sliding Mode Differentiator**

- **Signal to differentiate:** \( f(t) \)
- **Assume** \(|\dot{f}(t)| \leq M\)
- **Find differentiator**
  \[
  y = f(t), \quad \dot{y} = \dot{f},
  \]
- **Sliding Mode Differentiator**
  \[
  \dot{x} = k \text{sign}(e), \quad e = y - x
  \]
- **in finite time** \( \dot{e} = 0 \quad \Rightarrow \quad \dot{f} = \text{filtered} \ k \text{sign}(e) \)
SUMMARY: First order sliding modes

Advantages

- **Theoretically exact compensation** of matched uncertainties if the states are available
- Reduces SMC design to control selection for two reduced order systems
- *Saturated* control law
- Ensures finite-time convergence to the sliding surface

Disadvantages

- **Chattering**
- For SISO systems the dimension of sliding dynamics is reduced just for 1 state variable
- State variables converge asymptotically
- High order derivatives are needed to design sliding surfaces. The theory was not complete: theoretically exact compensation needs theoretically exact differentiation
Section 3

Stage 2: Second Order Sliding Modes
Stage 2: Second Order Sliding Modes

Chattering as the relative degree problem

Figure: Prof. Levant and Prof. Fridman
Stage 2: Second Order Sliding Modes

Second Order Sliding Modes

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u + f(x, t) \\
\sigma &= x_1
\end{align*}
\]

- \( f(x, t) \) unknown uncertainties/perturbations.
- All the partial derivatives of \( f(x, t) \) are bounded on compacts

**Main Objective**

To design a control \( u \) such that the origin of system is finite-time stable, in spite of the uncertainties/perturbations \( f(x, t) \), with \( |f(x, t)| < f^+ \) for all \( t, x \)

*Figure: Prof. Emelyanov, Prof. Korovin and Prof. Levantovsky*
Stage 2: Second Order Sliding Modes

Twisting algorithm

\[ u = -a \text{sign}(x_2) - b \text{sign}(x_1), \quad b > a + f^+, \quad a > f^+. \]

- Known bounds $f^+$
- $a$ and $b$ chosen appropriately (Emelyanov et al. 86),
- Ensures finite-time exact convergence for both $x_1$ and $x_2$
Stage 2: Second Order Sliding Modes

### Twisting Algorithm

\[
\dot{x}_1 = x_2 \\
\dot{x}_2 = u + f
\]

- \(x_1, x_2\) are the states
- \(u\) is the control
- \(f = 2 + 4\sin(t/2) + 0.6\sin(10t)\).
Stage 2: Second Order Sliding Modes

Comparison First Stage vs Second Stage: Precision

\[ \tau = 1e - 3 \]
Stage 2: Second Order Sliding Modes

Anti-chattering Strategy

\[ \dot{X} = F(t, X) + G(t, X)u, \quad X \in \mathbb{R}^n, \quad u \in \mathbb{R}, \quad |F| < F^+, \]

The switching variable \( \sigma(X) : \dot{\sigma} = f(\sigma, t) + g(\sigma, t)u. \)

Add an Integrator in control input:

If \( \dot{u} = v = -a\text{sign}(\dot{\sigma}(t)) - b\text{sign}(\sigma(t)) \), so \( u \) is a Lipschitz continuous control signal ensuring finite-time convergence to \( \sigma = 0 \)

Criticism(1987) If it is possible to measure \( \dot{\sigma} = f(t, \sigma) + g(t, \sigma)u \), then the uncertainty \( f(t, \sigma) = \dot{\sigma} - g(t, \sigma)u \) is also known and can be compensated without any discontinuous control!

Counter-argument

If \( g \) is uncertain so \( \ddot{\sigma} \) depends on \( u \) through uncertainty! The anti-chattering strategy is reasonable for the case of uncertain control gains.
Stage 2: Second Order Sliding Modes

Discussion about SOSM

Advantages of SOSM

1. Allows to compensate **bounded** matched uncertainties for the systems with relative degree two with discontinuous control signal

2. Allows to compensate **Lipschitz** matched uncertainties with continuous control signal using the first derivative of sliding inputs

3. Ensures quadratic precision of convergence with respect to the sliding output

4. For one degree of freedom mechanical systems: the sliding surface design is no longer needed.

5. For systems with relative degree $r$: the order of the sliding dynamics is reduced up to $(r - 2)$. The design of the sliding surface of order $(r - 2)$ is still necessary!
OPEN PROBLEMS: EARLY 90th

- To reduce the chattering substituting **discontinuous control signal with continuous one** the derivative of the sliding input still needed!

- The problem of exact finite-time stabilization and exact disturbance compensation for SISO systems with arbitrary relative degree remains open. More deep decomposition is still needed

- Theoretically exact differentiators are needed to realize theoretically exact compensation of the Lipschitz matched uncertainties
First Stage vs Second Stage

Figure: First Stage

Figure: Second Stage
Stage 2: Second Order Sliding Modes

Terminal Algorithm

\[
\dot{x}_1 = x_2, \quad \dot{x}_2 = u(x),
\]

\[
u(x) = -\alpha \text{sign}(s(x)),
\]

\[
s(x) = x_2 + \beta \sqrt{|x_1|} \text{sign}(x_1).
\]

Figure: Prof. Z. Man
Stage 2: Second Order Sliding Modes

Relative Degree of Terminal Sliding Variable

Time derivative of the switching surface

\[
\dot{s}(x) = \dot{x}_2 + \beta \frac{x_2}{2\sqrt{|x_1|}} = -\alpha \text{sign}(s(x)) + \beta \frac{x_2}{2\sqrt{|x_1|}}.
\]

- \( s(x) \) is singular for \( x_1 = 0 \), and the relative degree of the switching surface does not exist.
- On \( x_2 = -\beta \sqrt{|x_1|} \text{sign}(x_1) \)

\[
\dot{s} = -\alpha \text{sign}(s(x)) - \frac{\beta^2}{2} \text{sign}(x_1).
\]

Two types of behavior for the solution of the system are possible.
Stage 2: Second Order Sliding Modes

Terminal mode:

- $\beta^2 < 2\alpha$,

  Trajectories of the system reach the surface $s(x) = 0$ and remain there.

Twisting mode

- $\beta^2 > 2\alpha$

  Trajectories do not slide on the surface $s(x) = 0$
Section 4

Stage 3: Super-Twisting Algorithm
Stage 3: Super-Twisting Algorithm

The Super–Twisting Algorithm (STA)

Emalyanov, Korovin, Levantovsky, 1990, Levantovsky 1993

\[
\dot{x} = f(t) + g(t)u, \\
u = -k_1|x|^{\frac{1}{2}} \text{sign}(x) + \nu,
\]

Integral extension

\[
\dot{\nu} = -k_2 \text{sign}(x),
\]

\(f(x(t), t)\) is Lipschitz disturbance

- Continuous control signal
- Exact finite time convergence to \(x(t) = \dot{x}(t) = 0, \ \forall t \geq T\)
- The derivative of \(x\) is not used!!!
- If \(x\) is measured, the STA is an output-feedback controller
Stage 3: Super-Twisting Algorithm

Example Super-Twisting Algorithm

\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = u + f \]

- \( x_1, x_2 \) are the states
- \( u \) is the control
- \( f = 2 + 4 \sin(t/2) + 0.6 \sin(10t) \).
- \( \sigma = x_1 + x_2 \)
Stage 3: Super-Twisting Algorithm

Second Stage vs Third Stage

Figure: First Stage

Figure: Second Stage

Figure: Third Stage
Stage 3: Super-Twisting Algorithm

Comparison Second Stage to Third Stage: Precision

\[ \tau = 1 \epsilon - 3 \]
Stage 3: Super-Twisting Algorithm

Robust Exact Differentiator, Levant (1998)

- Signal to differentiate: $f(t)$
- Assume $|\ddot{f}(t)| \leq L$
- Find an observer for $\dot{x}_1 = x_2$, $\dot{x}_2 = \ddot{f}$, $y = x_1$,
- $\ddot{f}(t)$ bounded perturbation.
- STA observer
  $$\dot{\hat{x}}_1 = k_1 |y - \hat{x}_1|^{\frac{1}{2}} \text{sign}(y - \hat{x}_1) + \hat{x}_2,$$
  $$\dot{\hat{x}}_2 = k_2 \text{sign}(y - \hat{x}_1), \quad k_2 > L$$
- Convergence of STA assures: $(f - \hat{x}_1) = (\dot{f} - \hat{x}_2) = 0$ after finite time without filtration!
Stage 3: Super-Twisting Algorithm

SUMMARY

Advantages

1. Continuous control signal compensating Lipschitz uncertainties
2. Chattering attenuation but not its complete removal! (Boiko, Fridman 2005)
3. Differentiator obtained using the STA:
   - Finite-time exact estimation of derivatives in the absence of both noise and sampling;
   - Best possible asymptotic approximation in the sense of Kolmogorov 62.
Stage 3: Super-Twisting Algorithm

Open problems: End of 20th century

1. Relative degree $r \geq 2$: Need sliding surface. Consequently the states converge to the origin asymptotically. Deeper decomposition is needed!

2. STA based differentiator for the sliding surface design is not enough: he can not provide the best possible precision for highest derivatives

3. STA signal can grow together with perturbation! Saturation is needed

4. Direct application of STA together with the first order differentiator for control of mechanical system can not be done because it is necessary to form sliding surface with Lipschitz derivatives
Section 5

Stage 4: Arbitrary Order Sliding Mode Controllers
\[
\dot{X} = F(t, X) + G(t, X)u, \quad X \in R^n, \quad u \in R \\
\sigma = \sigma(X, t), \in R.
\]

- \(\sigma\) has a fixed and known relative degree \(r\).
- Control problem is transformed into the finite-time stabilization of an uncertain differential equation
  \[
  \sigma^{(r)} = f(t, X) + g(t, X)u,
  \]
  and corresponding differential inclusion
  \[
  \sigma^{(r)} \in [-C, C] + [K_m, K_M]u,
  \]
  where \(C, K_m\) and \(K_M\) are known constants.
Stage 4: Nested arbitrary order sliding-mode controllers

- 2001: Nested arbitrary order SM controller
- Solve the finite-time exact stabilization problem for an output with an arbitrary relative degree.
- Bounded Lebesgue measurable uncertainties.
- "Nested" higher order sliding-mode (HOSM) controllers are constructed using a recursion

Figure: Prof. Levantovsky
Nested Third Order Singular Terminal Algorithm

- **Third Order**

\[ u = -\alpha \text{sign} \left( \ddot{\sigma} + 2(\dot{\sigma}|^3 + |\sigma|^2)^{\frac{1}{6}} \times \text{sign}(\dot{\sigma} + |\sigma|^{\frac{2}{3}} \text{sign}(\sigma)) \right) \]

- **Fourth Order**

\[ u = -\alpha \text{sign} \left( \dddot{\sigma} + 3(\ddot{\sigma}|^6 + \dot{\sigma}|^4 + \sigma|^3)^{\frac{1}{12}} \times \text{sign} \left( \ddot{\sigma} + (\dot{\sigma}|^4 + |\sigma|^3)^{\frac{1}{6}} \text{sign}(\dot{\sigma} + 0.5|\sigma|^{\frac{3}{4}} \text{sign}(\sigma)) \right) \)

Finite-time stabilization of \( \sigma = 0 \) and its successive derivatives up to \( r - 1 \).
Stage 4: Arbitrary Order Sliding Mode Controllers

HOSM Differentiator

- The Nested Controller needs the output and its successive derivatives
- Instrument: HOSM arbitrary order differentiator
- Let $\sigma(t)$ signal to be differentiated $k - 1$ times
- Assume that $|\sigma^{(k)}| \leq L$
- 3-th order HOSM differentiator

$$
\begin{align*}
\dot{z}_0 &= v_0 = -3L^{1/4}|z_0 - \sigma|^{3/4} \text{sign}(z_0 - \sigma) + z_1, \\
\dot{z}_1 &= v_1 = -2L^{1/3}|z_1 - v_0|^{2/3} \text{sign}(z_1 - v_0) + z_2, \\
\dot{z}_2 &= v_2 = -1.5L^{1/2}|z_2 - v_1|^{1/2} \text{sign}(z_2 - v_1) + z_3, \\
\dot{z}_3 &= -1.1L \text{sign}(z_3 - v_2)
\end{align*}
$$

- $z_i$ true derivative $\sigma^{(i)}(t)$. 
Stage 4: Arbitrary Order Sliding Mode Controllers

Black Box Control Concept for HOSM

Figure: Black Box Control Concept for HOSM
Advantages of nested HOSM for SISO systems with relative degree $r$

- Theoretically exact disturbance compensation basing on output information only
- Full dynamical collapse: ensures $\sigma = \dot{\sigma} = \ddot{\sigma} = \cdots = \sigma^{(r-1)} = 0$ in finite-time
- Ensures the $r$-th order precision for the sliding output with respect to the discretization step and fast parasitic dynamics
- The sliding surface design is no longer needed
Open problems: After 2005

- For SISO systems with relative degree $r$ still produces a discontinuous control signal
- Anti-chattering strategy: the information about $\sigma^{(r)}$ containing perturbations is needed
Stage 4: Arbitrary Order Sliding Mode Controllers

Example Nested 3rd Order Singular Terminal Controller with anti-chattering strategy

\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = u + f \]
\[ \dot{u} = u_2 \]

- \( x_1, x_2 \) are the states
- \( u_2 \) is the control
- \( f = 2 + 4\sin(t/2) + 0.6\sin(10t) \)
- \( \sigma = x_1 \)

Convergence of states

Phase portrait of \( x_1, x_2 \) and \( \dot{x}_2 \)

Control Input and Negative of the Perturbation

5 Stages of SM
Stage 4: Arbitrary Order Sliding Mode Controllers

First Stage to Fourth Stage

**First Stage**

![First Stage Image](image1)

**Second Stage**

![Second Stage Image](image2)

**Third Stage**

![Third Stage Image](image3)

**Fourth Stage**

![Fourth Stage Image](image4)
Comparison of Third Stage to Fourth Stage: Precision

\[ \tau = 1e^{-3} \]
Section 6

Stage 5: Continuous Arbitrary Order Sliding-Mode Controllers
Stage 5: Continuous Arbitrary Order Sliding-Mode Controllers

Figure: UNAM: J. Moreno IIT: B. Bandyopadhyay, S. Kamal, A. Chalanga, Shtessel Edwards
Properties of CHOSM

For the systems with relative degree $r$

- Continuous control signal
- Finite-time convergence to the $(r + 1)$-th order sliding-mode set
- Derivatives of the output up to the $(r - 1)$ order
Continuous arbitrary order sliding-mode controllers

- Discontinuous–Integral Algorithm (D-I), (Zamora, Moreno, 2013)
- Two versions of the Continuous Terminal Sliding Mode Algorithm (CTSMA) (Mexico-India 2014-16)
  - (a) Continuous Singular Terminal Sliding Mode Algorithm (CSTSMA);
  - (b) Continuous Nonsingular Terminal Sliding Mode Algorithm (CNTSMA);
- Continuous Twisting Algorithm (CTA) (Moreno, Fridman et al 2015-18)
Stage 5: Continuous Arbitrary Order Sliding-Mode Controllers

Discontinuous - Integral (D-I) Algorithm

\[
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u + f(t) \\
\sigma &= x_1
\end{aligned}
\]

\[u = -k_1 [x_1]^{1/3} - k_2 [x_2]^{1/2} - \int_0^t (k_3 [x_1(\tau)]^0) d\tau,
\]

where \(k_1, k_2, k_3\) are appropriate positive gains.

New Notation: \(\lfloor z \rceil^p = |z|^p \text{sgn}(z)\)

NONLINEAR PID!
Stage 5: Continuous Arbitrary Order Sliding-Mode Controllers

Continuous Twisting Algorithm (CTA)

\[
\begin{cases}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u + f(t) \\
\sigma &= x_1
\end{cases}
\]

\[
u = - k_1 \left\lfloor x_1 \right\rfloor^{1/3} - k_2 \left\lfloor x_2 \right\rfloor^{1/2} - \int_0^t (k_3 \left\lfloor x_1(\tau) \right\rfloor^0 + k_4 \left\lfloor x_2(\tau) \right\rfloor^0) d\tau,
\]

where \( k_1, k_2, k_3, k_4 \) are appropriate positive gains.

New Notation: \( \lfloor z \rfloor^p = |z|^p \text{sgn}(z) \)
Stage 5: Continuous Arbitrary Order Sliding-Mode Controllers

Continuous Twisting Algorithm (CTA)

- Closed Loop System

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -k_1 [x_1]^{1/3} - k_2 [x_2]^{1/2} + x_3 \\
\dot{x}_3 &= -k_3 [x_1]^0 - k_3 [x_2]^0 + \rho,
\end{align*}
\]

\[\rho = \frac{\partial f}{\partial x} \dot{x} + \frac{\partial f}{\partial t}, \text{ and } |\rho| \leq \Delta.\]

- Twisting structure to reject perturbations.
Stage 5: Continuous Arbitrary Order Sliding-Mode Controllers

Simulation: CTA

Figure: Numerical results for a double integrator with perturbation
Stage 5: Continuous Arbitrary Order Sliding-Mode Controllers

Discussion about the CTA

Advantages

- Algorithms homogeneous of degree $\delta_f = -1$, with weights $\rho = 3, 2, 1$.
- The only information needed to maintain finite time convergence of all three variables $x_1, x_2$ and $x_3$ is the output ($x_1$) and its derivative ($x_2$).
- Precision corresponds to a 3rd order sliding mode.
Continuous Singular Terminal Sliding Mode Algorithm (CSTSMA)

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u + f(t) \\
\sigma &= x_1 \\
u &= -k_1 [\phi]^{1/2} - k_3 \int_0^t [\phi]^0 d\tau,
\end{align*}
\tag{7}
\]

where \( \phi = (x_2 + k_2 [x_1]^{2/3}) \), and \( k_1, k_2, k_3 \) are appropriate positive gains.
Stage 5: Continuous Arbitrary Order Sliding-Mode Controllers

Continuous Singular Terminal Sliding Mode Algorithm (CSTSMA)

Closed Loop System

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -k_1 \lfloor \phi \rfloor^{1/2} + x_3 \\
\dot{x}_3 &= -k_3 \lfloor \phi \rfloor^0 + \rho,
\end{align*}
\]

\(\rho = \frac{\partial f}{\partial x} \dot{x} + \frac{\partial f}{\partial t}, \text{ and } |\rho| \leq \Delta.\)

Combination of the Super-Twisting algorithm with the Singular Terminal Sliding mode.
Stage 5: Continuous Arbitrary Order Sliding-Mode Controllers

Continuous Nonsingular Terminal Sliding Mode Algorithm (CNTSMA)

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u + f(t) \\
\sigma &= x_1
\end{align*}
\]

\[ u = -k_1 [\phi_N]^{1/3} - k_3 \int_0^t [\phi_N]^{0} d\tau, \quad (9) \]

where \( \phi_N = (x_1 + k_2 [x_2]^{3/2}) \), and \( k_1, k_2, k_3 \) are appropriate positive gains.
Stage 5: Continuous Arbitrary Order Sliding-Mode Controllers

Continuous Nonsingular Terminal Sliding Mode Algorithm (CNTSMA)

- Closed Loop System

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -k_1 [\phi_N]^{1/3} + x_3 \\
\dot{x}_3 &= -k_3 [\phi_N]^0 + \rho,
\end{align*}
\]

(10)

- Combination of the Super-Twisting algorithm with the Nonsingular Terminal Sliding Mode algorithm.

\[
\rho = \frac{\partial f}{\partial x} \dot{x} + \frac{\partial f}{\partial t} \quad \text{and} \quad |\rho| \leq \Delta.
\]
Stage 5: Continuous Arbitrary Order Sliding-Mode Controllers

Simulation: CSTSMA

(a)

(b)

Figure: Convergence and precision of states with $\tau = 0.001$ for 3-CSTSMA
Continuous Singular Terminal Sliding Mode Control (CSTSMC)

Twisting controller-like Behavior.

Figure: Numerical example uncertain double integrator
Stage 5: Continuous Arbitrary Order Sliding-Mode Controllers

Continuous Nonsingular Terminal Sliding Mode Control (CNTSMC)

Figure: Numerical example uncertain triple integrator
Sliding-Like Behavior of CNTSMC

Figure: Phase portrait of Plant’s states $x_1$ and $x_2$, and locus of the switching curve $\phi = \phi_N = 0$, showing a Sliding-Like behavior of the CNTSMC
Twisting-Like Behavior of CNTSMC

Figure: Phase portrait of Plant’s states $x_1$ and $x_2$, and locus of the switching curve $\phi = \phi_N = 0$, showing a Twisting-Like behavior of the CNTSM controller.
Stage 5: Continuous Arbitrary Order Sliding-Mode Controllers

All five Stages

Figure: First Stage

Figure: Second Stage

Figure: Third Stage

Figure: Fourth Stage

Figure: Fifth Stage: CSTSMC

Figure: Fifth Stage: CNTSMC

Stage 1
Stage 2
Stage 3
Stage 4
Stage 5
Fourth Stage vs Fifth Stage: Precision

Same precision and smoothness of control without using $\sigma$
Stage 5: Continuous Arbitrary Order Sliding-Mode Controllers

Third Stage vs Fifth Stage: Precision

\[ \tau = 1e^{-3} \]
### Stage 5: Continuous Arbitrary Order Sliding-Mode Controllers

#### Comparison 5 Stages

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Control Signal</th>
<th>Information</th>
<th>Stability</th>
<th>Precision w.r.t. sampling</th>
</tr>
</thead>
<tbody>
<tr>
<td>First SMC</td>
<td>Discontinuous</td>
<td>$\sigma, \dot{\sigma}$</td>
<td>Asymptotic</td>
<td>$O(h)$</td>
</tr>
<tr>
<td>2SMC</td>
<td>Discontinuous</td>
<td>$\sigma, \dot{\sigma}$</td>
<td>Finite time</td>
<td>$0(h^2)$</td>
</tr>
<tr>
<td>Super-twisting</td>
<td>Continuous</td>
<td>$\sigma, \dot{\sigma}$</td>
<td>Asymptotic</td>
<td>$0(h^2)$</td>
</tr>
<tr>
<td>3SMC + anti-chattering</td>
<td>Continuous</td>
<td>$\sigma, \dot{\sigma}, \ddot{\sigma}$</td>
<td>Finite time</td>
<td>$0(h^3)$</td>
</tr>
<tr>
<td>Continuous 2SMC</td>
<td>Continuous</td>
<td>$\sigma, \dot{\sigma}$</td>
<td>Finite time</td>
<td>$0(h^3)$</td>
</tr>
</tbody>
</table>
Stage 5: Continuous Arbitrary Order Sliding-Mode Controllers

Third Order CTA (Mendoza, Fridman, Moreno, 2017)

- 3-CTA

\[
\dot{x}_1 = x_2 \\
\dot{x}_2 = x_3 \\
\dot{x}_3 = -k_1 \lfloor x_1 \rfloor^{\frac{1}{4}} - k_2 \lfloor x_2 \rfloor^{\frac{1}{3}} - k_3 \lfloor x_2 \rfloor^{\frac{1}{2}} + x_4 \\
\dot{x}_4 = -k_4 \lfloor x_1 \rfloor^0 - k_5 \lfloor x_2 \rfloor^0
\]
Stage 5: Continuous Arbitrary Order Sliding-Mode Controllers

3-CTA

Figure: 3-CTA and states precision with $\tau = 0.001$
### C2SM
- Homogeneous of degree $\delta_f = -1$, with weights $\rho = 3, 2, 1$.
- The only information that needed to maintain finite time convergence of all three variables $x_1, x_2$ and $x_3$ is the output ($x_1$) and its derivative ($x_2$).
- Can work for an uncertain system with relative degree 2 with respect to its output.
- Can compensate bounded Lebesgue measurable perturbations.

### r-CHOSM
- Homogeneous of degree $\delta_f = -1$, with weights $\rho = r, r - 1, \cdots, 2, 1$.
- Can be used for uncertain system with relative degree $r - 1$ with respect to output.
- Insensible to perturbations whose time derivative is bounded! (can not grow faster than a linear function of time!)
Section 7

Conclusions
Last three decades new generations of controllers:

- second order sliding mode controllers (1985);
- super-twisting controllers (1993);

We have presented the next generation: two families of continuous nested sliding-mode controllers, that can be used on Lipschitz systems with relative degree $r$, providing a continuous control signal.

New controllers ensure a finite-time convergence of the sliding output to the $(r + 1)$-th order sliding set using information on the sliding output and its derivatives up to the order $(r - 1)$. 