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Motion control with optimal nonlinear damping and convergent dynamics

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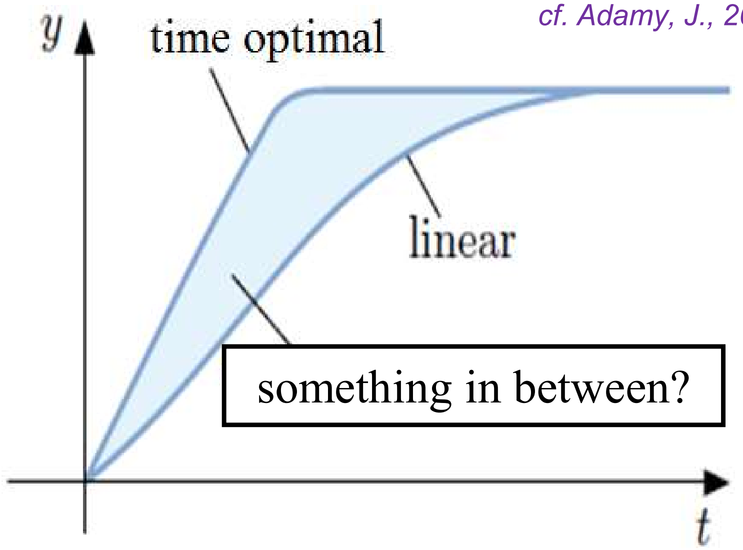
Outline

- Feedback damping in second-order systems

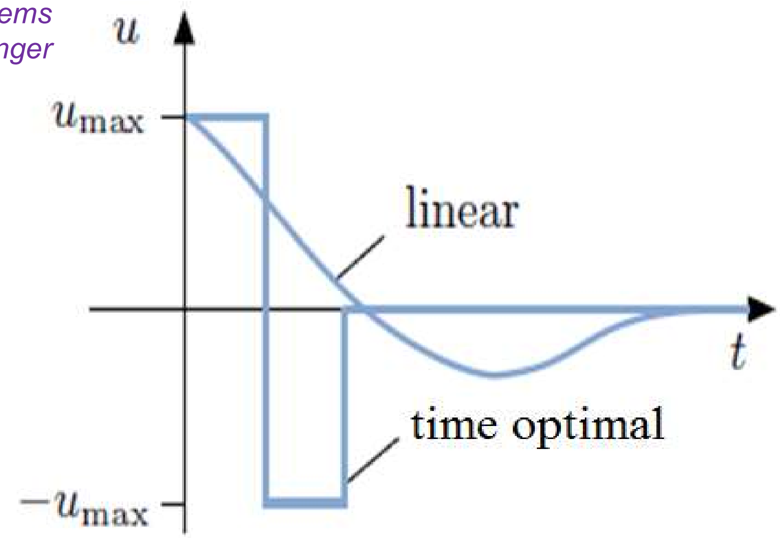
- Motion control with optimal nonlinear damping

- Extension and experimental control example

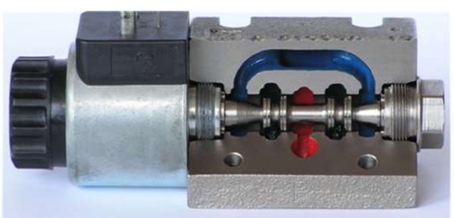
- What is optimal for set-point control of a motion system?

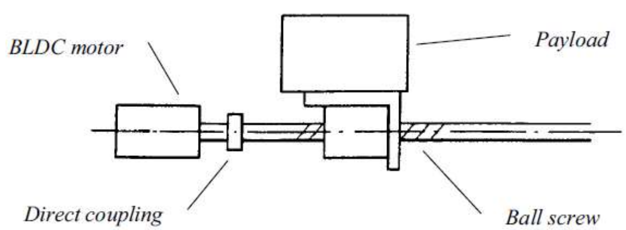


cf. Adamy, J., 2022. Nonlinear Systems and Controls. Springer



In variety of motion systems, including electro-/magneto-/hydro-/mechanical actuators





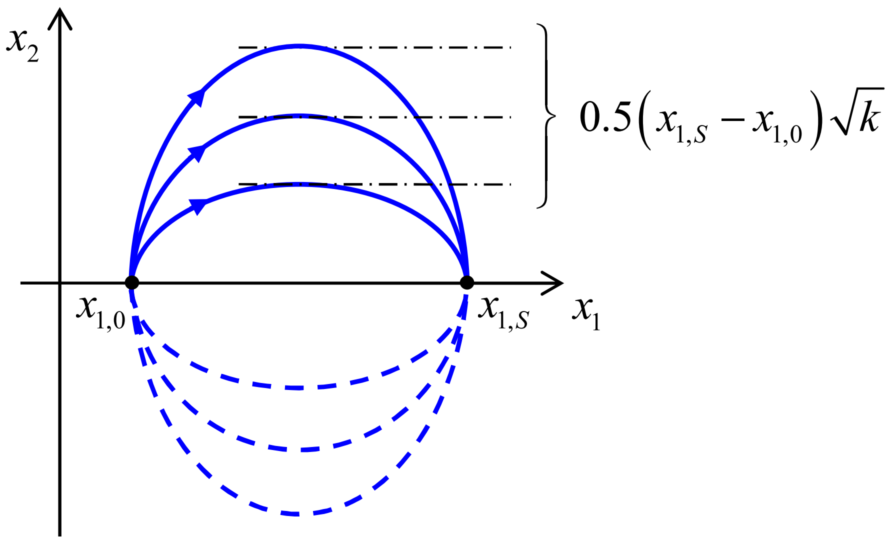
<u>Linear controllers are always suboptimal</u>

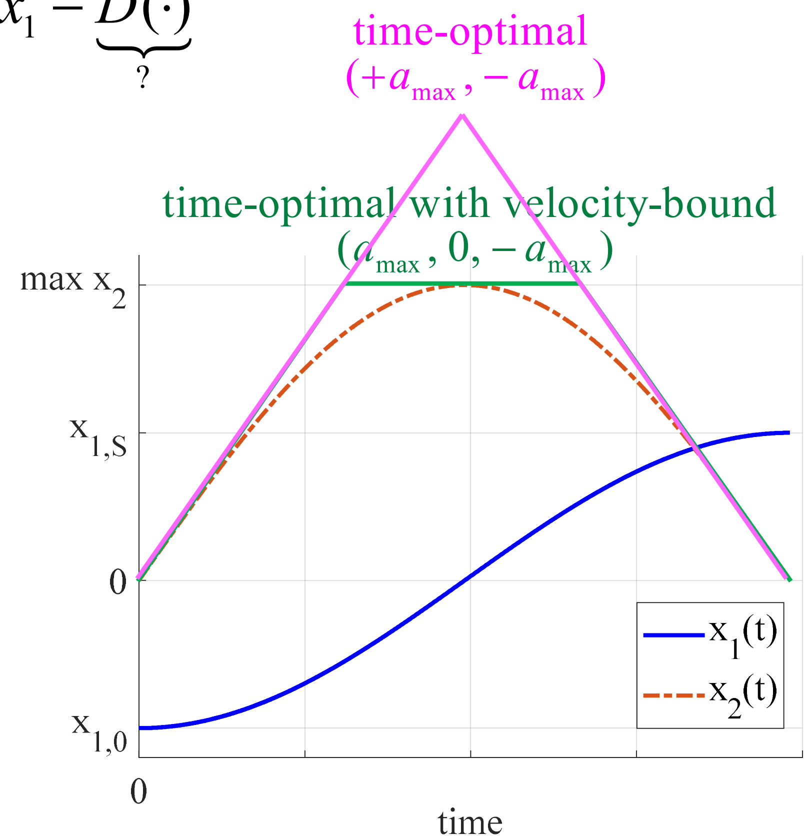
Problem statement (cont.)

- Consider the problem of optimal damping in 2nd order control systems, with a potential field created by x_1-output feedback

assumed class of the systems
$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -kx_1 - \underbrace{D(\cdot)}_{?} \end{cases}$$

When assuming first a zero damping, i.e. D=0, the system becomes harmonic oscillator





D=0 could be a candidate but (!) requires: (i) 'on-fly' calculation of the reference bias for keeping $x_{1,0} + 0.5\,(x_{1,S} - x_{1,0})$, (ii) exact stop at $x_{1,S}$ and no 'post-regulation'

- Standard state-feedback control (equivalent to PD feedback control)

Closed-loop control system in the state-space form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \Rightarrow \quad \alpha(s) = s^2 + ds + k : \quad \text{characteristic polynomial}$$

Rewriting the loop transfer function $L(s)$ for the Root-locus with d-gain

$$1 + d\,L(s) = 1 + d\,\frac{s}{s^2 + k}$$

Analyzing the closed-loop dynamics for
$d \in [0, \ldots, \infty)$ (while k-gain is fixed)

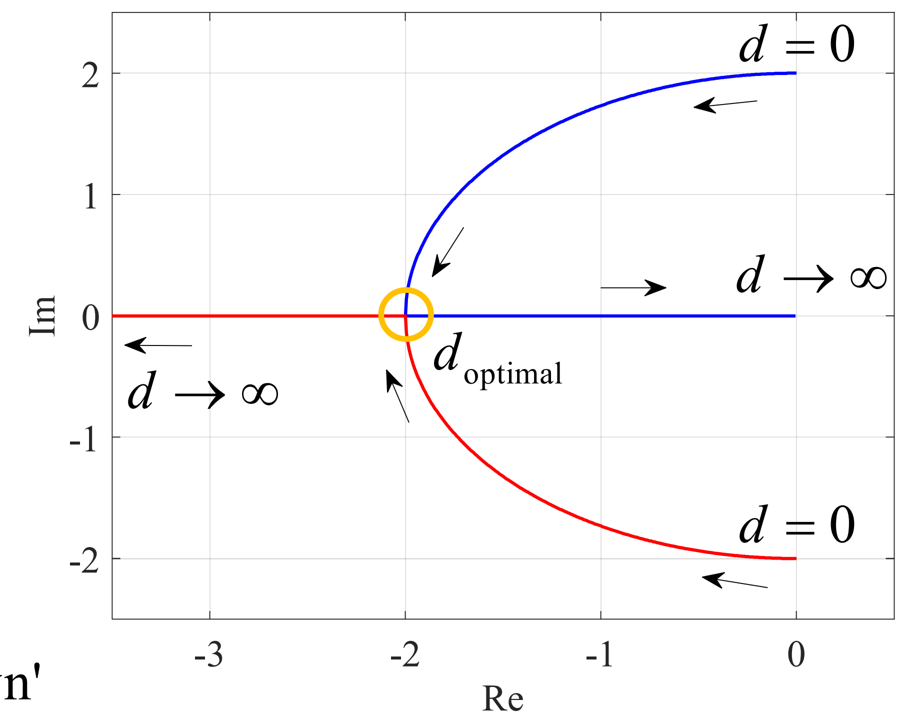
Optimal (i.e. critical) damping:

$$s^2 + ds + k = (s + \lambda)^2 \quad : \text{ double real pole}$$

$$\Rightarrow \quad k = \lambda^2, \; d = 2\sqrt{k}$$

$$d < 2\sqrt{k} \; : \text{transient oscillations}$$

$$d > 2\sqrt{k} \; : \text{dominant pole is 'slowing down'}$$



Feedback with linear damping (cont.)

- Closed-loop dynamics as an initial value problem

$$\ddot{x} + d\dot{x} + k = 0, \quad x(0) = x_{1,0}, \quad \dot{x}(0) = 0; \quad \text{with} \;\; d = 2\sqrt{k} \;\; \text{and} \;\; x_1 \equiv x,\, x_2 \equiv \dot{x}$$

General homogeneous solution (for double real poles)

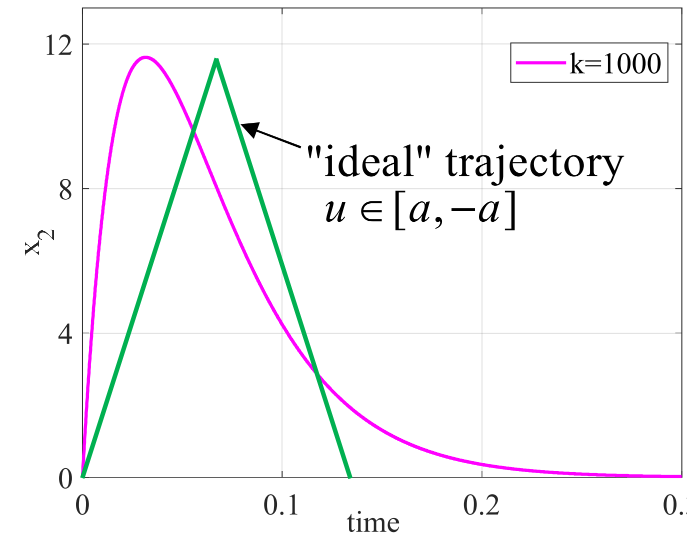
$$x(t) = C_1\, e^{-\lambda t} + C_2\, t\, e^{-\lambda t} \quad \text{with} \quad \lambda = \sqrt{k}$$

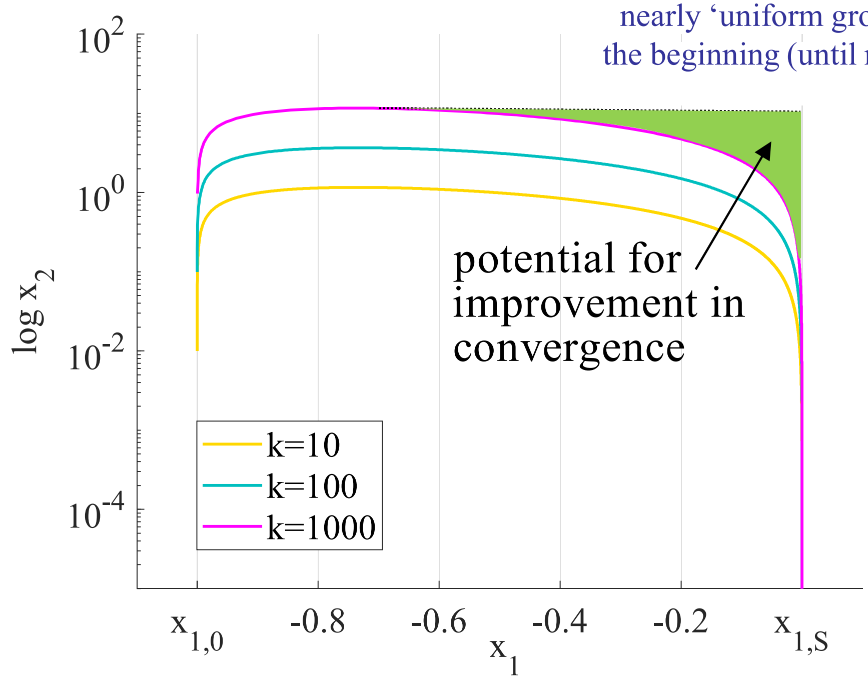
Solving for initial values (i.e. finding C_1 and C_2) results in

$$x(t) = x_{1,0}\, e^{-\sqrt{k}\,t}\left(1 + \sqrt{k}\,t\right), \qquad \dot{x}(t) = -x_{1,0}\, \underbrace{k\,t}\, \underbrace{e^{-\sqrt{k}\,t}}$$

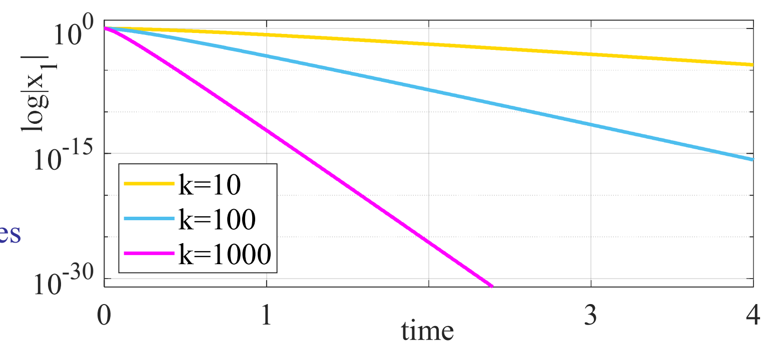
nearly 'uniform growth' at the beginning (until max x_2)

exponential drop towards endpoint (when → zero)





Only exponential convergence rate: convergence of $x(t) \to 0$ slows down when the state trajectory approaches zero equilibrium



Outline

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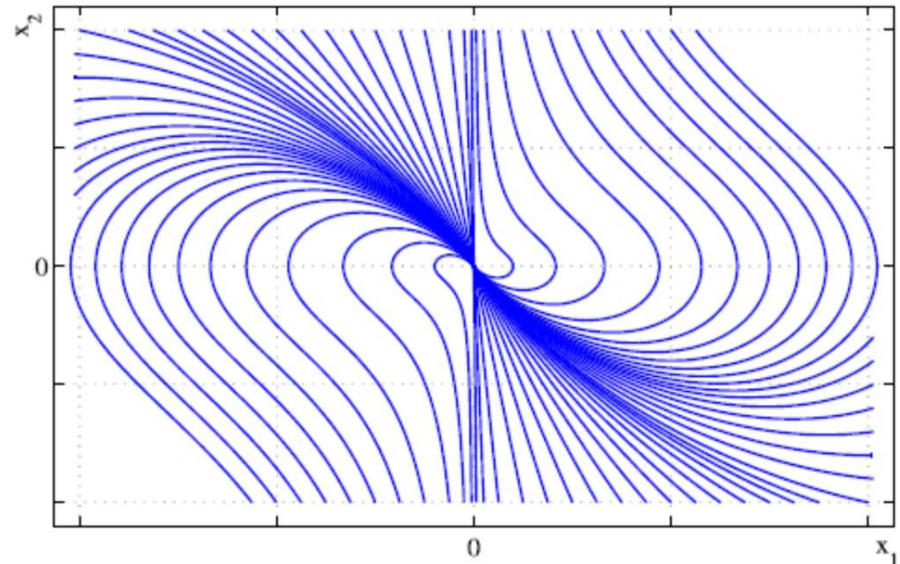
Feedback control with optimal nonlinear damping

- Introduced optimal nonlinear damping (OND) control

$$\dot{x}_1 = x_2, \tag{4}$$

$$\dot{x}_2 = \underbrace{-kx_1 - x_2^2|x_1|^{-1}\operatorname{sign}(x_2)}_{v}, \tag{5}$$

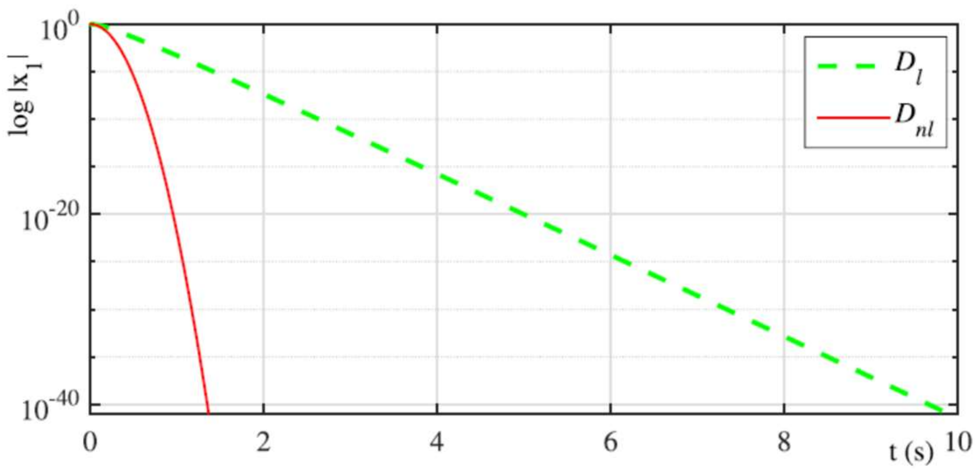
$$(x_1, x_2) \in \mathbb{R}^2 \setminus \{x_1 = 0 \,|\, x_2 \neq 0\}$$

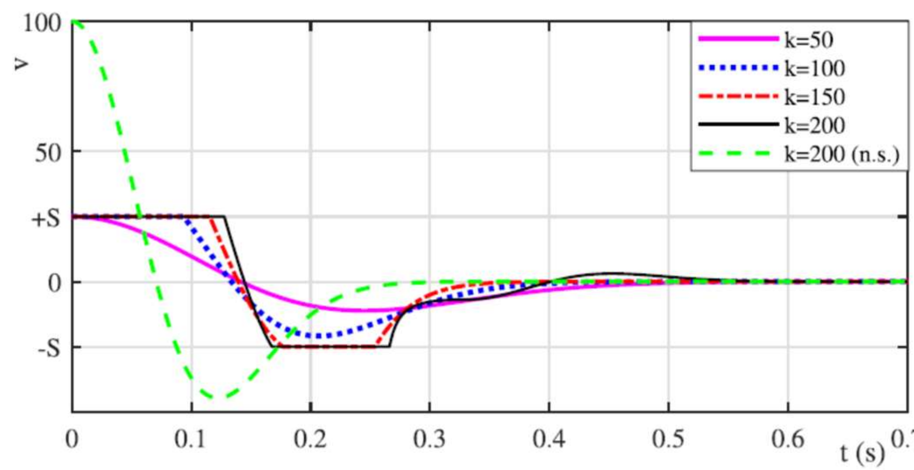


Ruderman, 2021, Journal of Franklin Institute

Main difference, comparing to linear (PD) feedback control, is the damping map D

$$\dot{x}_2 = -kx_1 - D$$



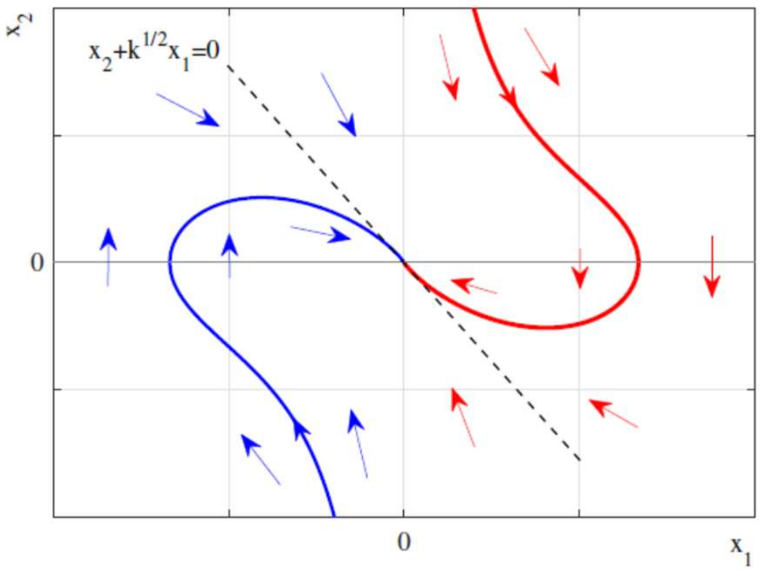


OND-control in (4), (5) allows also for input signal $v(t)$ to be saturated, $v \in [-S, \ldots +S]$

Stability properties of OND-control

- OND-control is globally asymptotically stable, converging to origin

alongside attractor $x_2 + \sqrt{k} x_1 = 0$

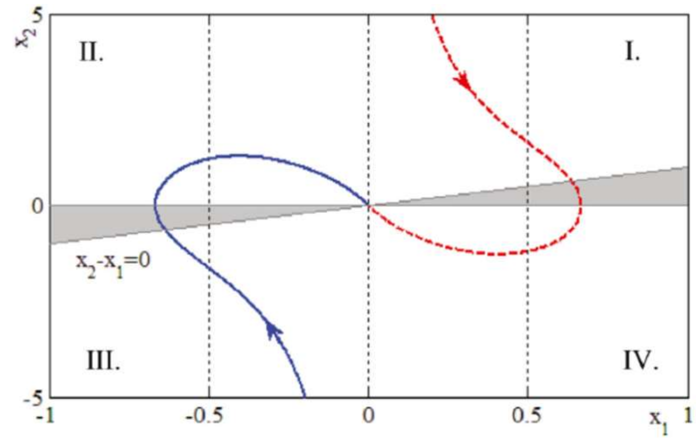


For Lyapunov function candidate

$$V = \frac{1}{2} x_2^2 + k \frac{1}{2} x_1^2 \quad \Rightarrow \quad \dot{V} = -x_2^3 |x_1|^{-1} \operatorname{sign}(x_2) \leq 0$$

Asymptotic stability proof is by invariance (LaSalle) principle, i.e. for $\{x_1 \neq 0 \mid x_2 = 0\}$

Closed-loop passive if $\dfrac{|x_2|}{|x_1|} \geq \operatorname{sign}(x_2) \operatorname{sign}(x_1)$



For attractor, consider steady-state of (4), (5):

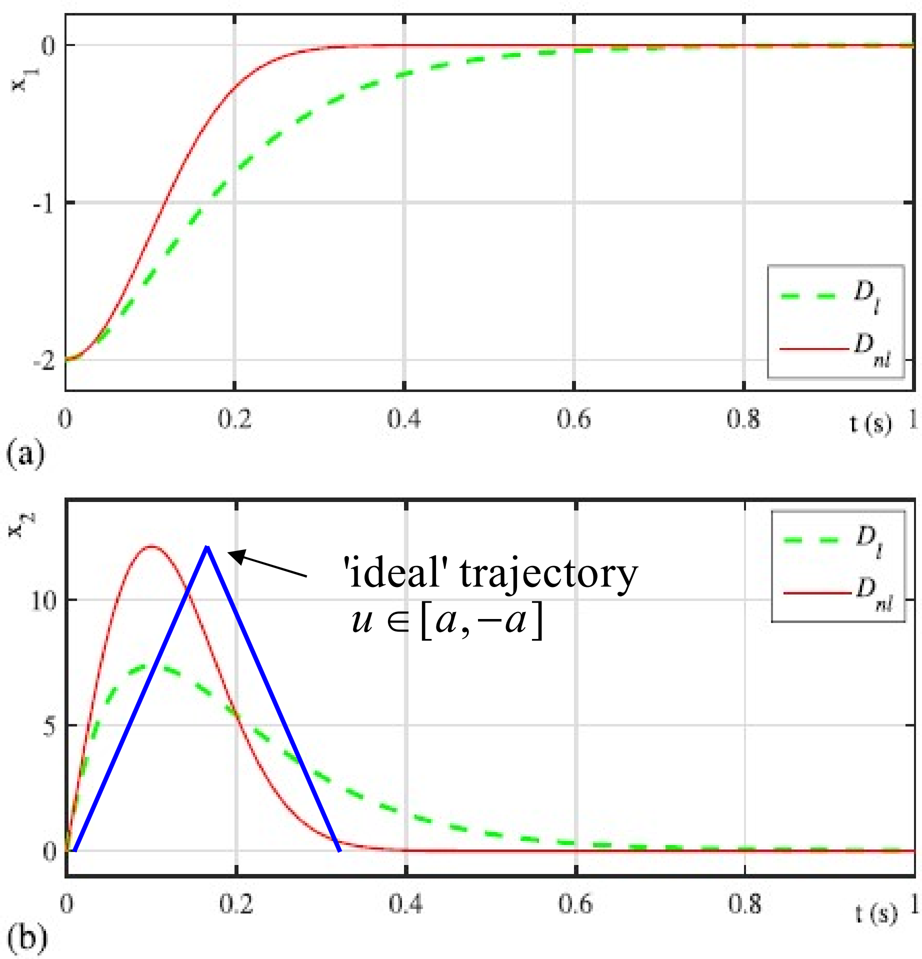
$$\mathbf{0} = \begin{bmatrix} 0 & 1 \\ -k & -|x_2||x_1|^{-1} \end{bmatrix} \cdot [x_1, x_2]^T$$

$$\Rightarrow \quad k|x_1|x_1 = -|x_2|x_2 \quad \Rightarrow \quad k x_1^2 \operatorname{sign}(x_1) = -x_2^2 \operatorname{sign}(x_2)$$

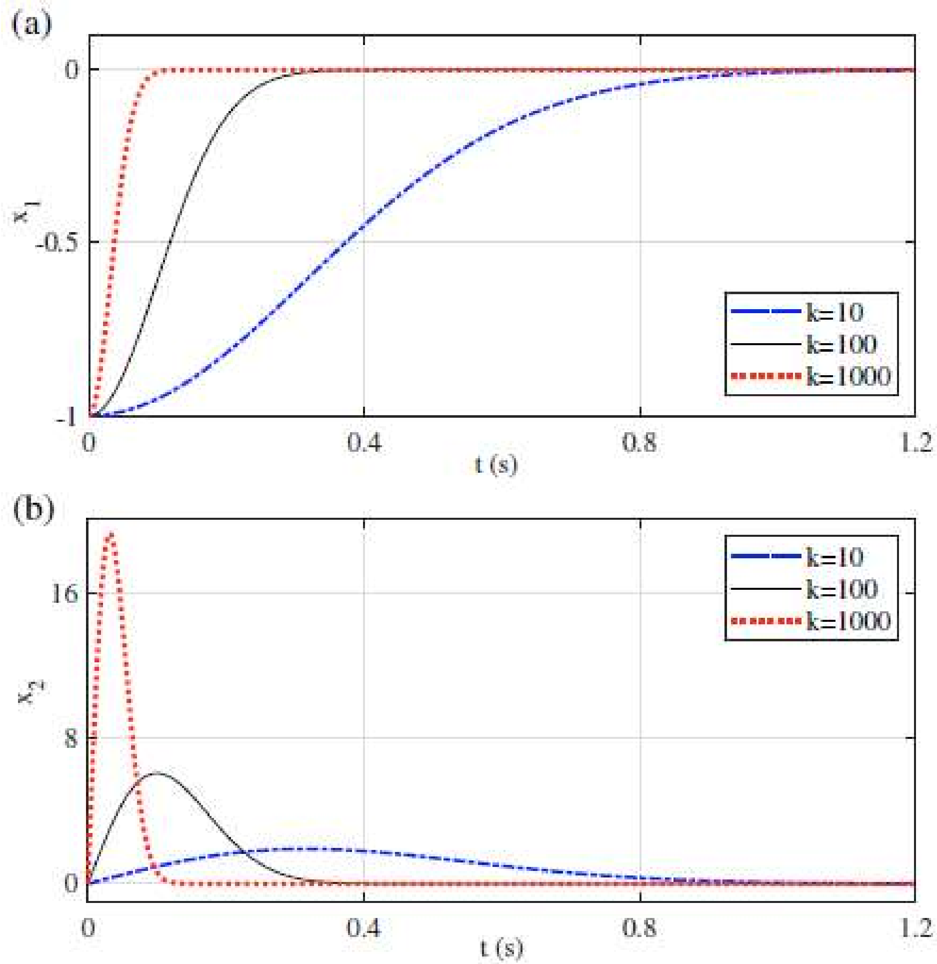
allowing only for real solutions of above, results in $x_2 + \sqrt{k} x_1 = 0$

Linearly- & nonlinearly-damped response

- Nonlinear damping brings x_2-trajectory closer to an 'ideal' bang-bang-type response



- Nonlinearly-damped shape is independent of k, which only scales/stretches x_2-trajectory



Energy analysis of feedback-damped system

- For total energy function (equivalent to Lyapunov function)

$$E = \frac{1}{2} x_2^2 + \frac{1}{2} k x_1^2 \quad \Rightarrow \quad \dot{E} = x_2 \dot{x}_2 + k x_2 x_1 = x_2 (\underbrace{\dot{x}_2}_{\text{dynamics}: \; \dot{x}_2 = -k x_1 - D(\cdot)} + k x_1)$$

- linear damping

$$D(\cdot) \equiv d x_2 \quad \Rightarrow \dot{E} = -d x_2^2$$

- nonlinear constant damping (Coulomb-type)

$$D(\cdot) \equiv d \, \mathrm{sign}(x_2) \quad \Rightarrow \dot{E} = -d \, | x_2 |$$

- proposed optimal nonlinear damping

$$D(\cdot) \equiv x_2^2 \, \mathrm{sign}(x_2) \, | x_1 |^{-1} \quad \Rightarrow \dot{E} = - | x_2 |^3 | x_1 |^{-1}$$

- Considering the energy- and, correspondingly, power-balance yields

$$\dot{E}_{\text{supply}} + \dot{E}_{\text{dissipated}} + \underbrace{\dot{E}_{\text{conservative}}}_{\equiv x_2 \dot{x}_2} = 0$$

supply power of control through potential field $E_{\text{supply}} = \frac{1}{2} k x_1^2$

- linear damping control:

$$\underbrace{k x_1 x_2}_{\dot{E}_{\text{supply}}} - \underbrace{d x_2^2}_{} = -\dot{E}_{\text{conservative}}$$

independent of the control errors (i.e. set-point distance)

- nonlinear damping control:

$$\underbrace{k x_1 x_2}_{\dot{E}_{\text{supply}}} - \frac{| x_2 |^3}{\underbrace{| x_1 |}_{}} = -\dot{E}_{\text{conservative}}$$

decreases for larger control errors (i.e. set-point distance)

Energetic (corresp. Lyapunov-function) aspects of regularized OND-control

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Remark 5. When assuming a quadratic Lyapunov function candidate

$$V(x) = x^T P x = \frac{1}{2} k e_1^2 + \frac{1}{2} e_2^2, \qquad (12)$$

which represents the total energy level (i.e. potential energy plus kinetic energy) of the system (7), (8), its time derivative results in

$$\frac{d}{dt} V(x) = - \frac{|e_2| \, e_2^2}{|e_1| + \mu}. \qquad (13)$$

Thus, the rate at which the control system (7), (8) reduces its energy is cubic in the error rate, i.e. $\sim |e_2|^3$, and hyperbolic in the error size, i.e. $\sim |e_1|^{-1}$, cf. Figure 3.

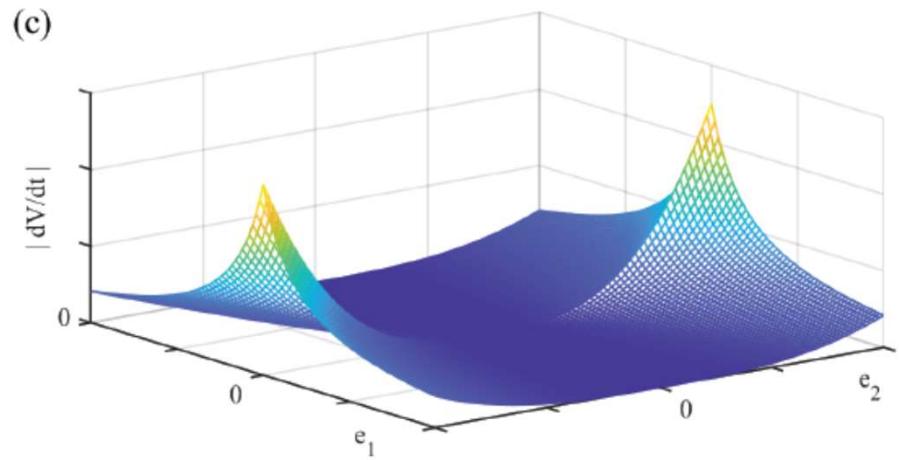
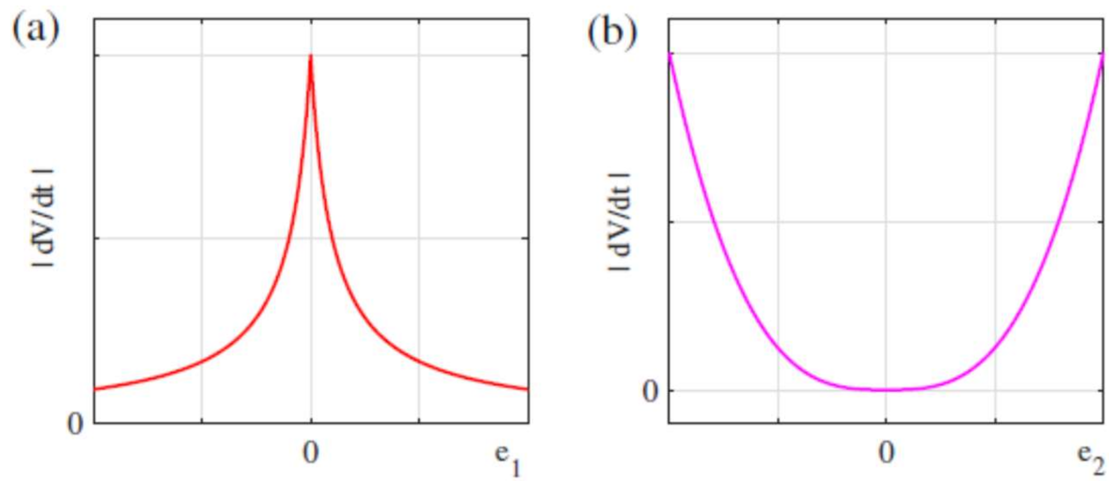


Fig. 3. Energy reduction rate $|\dot{V}|$ of the system (7), (8): depending on e_1 in (a), depending on e_2 in (b), and as overall error-states function according to (13) in (c).



- Regularization factor $0<\mu<<k$ prevents an infinite energy-rate and, thus, ensures a finite control action when $|e_1| \rightarrow 0$
- Cubic dependency of energy-rate from the error-rate enables the control to react faster to the error dynamics, like in case of non-steady trajectory phases or sudden external perturbations

- With introduced regularization factor

$0 < \mu \ll k \qquad e_1 = x_1 - r \qquad \dot{e}_1 \equiv e_2 = x_2 - \dot{r}$

the error dynamics of OND-controlled system becomes

$$\dot{e}_1 = e_2, \qquad \dot{e}_2 = -ke_1 - \frac{|e_2|\, e_2}{|e_1| + \mu}$$

- All previously shown OND-properties are preserved, while preventing singularities when $(e_1, e_2) \in \{e_1 = 0 \,|\, e_2 \neq 0\}$

❑ One is interested in stability of not a particular solution or invariant set, but stability properties of all solutions and some limit solution $\bar{x}(t)$, to which all other solutions will converge

❑ We focus on convergent dynamics by Demidovich*, while a lot of research was done in association with incremental stability and contraction analysis (contraction theory)

Key differences:

- Incremental stability does not imply the boundedness of solutions in forward time and the existence of a well-defined bounded steady-state solution
- Convergence does not imply decay of the 'distance' between any two solutions uniform in the initial distance

On compact sets: Convergence ⇔ Incremental Stability

** Pavlov et al., 2004, Systems & Control Letters*

cf. e.g. Rüffer, Van De Wouw, Mueller, 2013, Systems & Control Letters

Preliminaries of convergent dynamics

Notation of convergent system

$$\dot{x} = f(x, t) \quad (1)$$

Definition 1. The system (1) is said to be convergent if for all initial conditions $t_0 \in \mathbb{R}$, $\bar{x}_0 \in \mathbb{R}^n$ there exists a solution $\bar{x}(t) = x(t, t_0, \bar{x}_0)$ which satisfies:

(i) $\bar{x}(t)$ is well-defined and bounded for all $t \in (-\infty, \infty)$;
(ii) $\bar{x}(t)$ is globally asymptotically stable.

Convergent systems according to Demidovich [1967]

Such solution $\bar{x}(t)$ is called a *limit solution*, to which all other solutions of the system (1) converge as $t \to \infty$. In other words, all solutions of a convergent system 'forget' their initial conditions after some transient time, which depends on exogenous values like the reference or disturbance, and thus converge asymptotically to $\bar{x}(t)$.

Sufficient condition for system to be convergent

Theorem 1. Consider the system (1). Suppose, for some positive definite matrix $P = P^T > 0$ the matrix

$$J(x,t) := \frac{1}{2}\left(P \frac{\partial f}{\partial x}(x,t) + \left[\frac{\partial f}{\partial x}(x,t)\right]^T P\right) \qquad (2)$$

is negative definite uniformly in $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ and $|f(0,t)| \leq \text{const} < +\infty$ for all $t \in \mathbb{R}$. Then the system (1) is convergent.

More details on Demidovich's definitions and proof of the Theorem 1 are in Pavlov, Pogromsky, et al. [2004]

Convergent dynamics of OND-control

For output tracking of the reference trajectory $r(t) \in \mathcal{C}^1$, we introduce the error state $e_1 = x_1 - r$. Its time derivative is $e_2 = x_2 - \dot{r}$, respectively. Note that for an output tracking of $\mathcal{C}^1$-trajectories, one can assume $\ddot{r}(t) = 0$ for $t > \tau$, while $t \leq \tau$ characterizes certain transient phase where $\dot{r} \neq$ const. In the sense of a motion control, for instance, the time $t \leq \tau$ will correspond to the transient phases of a system acceleration or deceleration when moving. If a reference trajectory $r(t)$ contains multiple, but finite in time, transient phases with $\ddot{r}(t) \neq 0$, they will appear as temporary perturbations upon which the convergent dynamics of the control error, i.e. $\|e_1, e_2\| \to 0$, must be guaranteed for $t > \tau$.

Note that the introduced here regularization term $0 < \mu \ll k$ does not act as an additional design parameter, yet it prevents singularity in solutions of the system (4), (5), cf. Section 2.2. Evaluating the Jacobian of $f(x,t)$ with $x = [e_1, e_2]^T$, cf. (7), (8) and (1), one obtains

$$\frac{\partial f}{\partial x} = \tag{9}$$

$$= \begin{bmatrix} 0 & 1 \\ -k + |e_2|\, e_2 \,\mathrm{sign}(e_1)/(|e_1| + \mu)^2 & -2|e_2|/(|e_1| + \mu) \end{bmatrix}.$$

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OND-control error dynamics becomes

$$\dot{e}_1 = e_2, \tag{7}$$

$$\dot{e}_2 = -ke_1 - \frac{|e_2|\, e_2}{|e_1| + \mu}. \tag{8}$$

Then, suggesting the positive definite matrix

$$P = \frac{1}{2}\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}, \tag{10}$$

one can show that the matrix $J(x,t)$, which is the solution of (2), is negative definite and, correspondingly, the Theorem 1 holds. For proving it, we substitute (9) and (10) into (2) and evaluate the matrix definiteness as

$$x^T J(x,t)\, x = -\frac{3}{4}\, \frac{|e_2|\, e_2^2 \left(|e_1| + 2\mu\right)}{\left(e_1 + \mu \,\mathrm{sign}(e_1)\right)^2} \leq 0 \quad \forall\, x \neq 0. \tag{11}$$

Definiteness (not only semi-) of (11) implies; since substituting $e_2 = 0$ into (8) $\Rightarrow$ $\dot{e}_2 = -ke_1$

$\Rightarrow$ *the system (7),(8) is uniformly convergent, and* $[e_1, e_2](t) = 0 \equiv \bar{x}$ *is unique limit solution*

Numerical example of convergent OND-control system

Output trajectories for different initial values

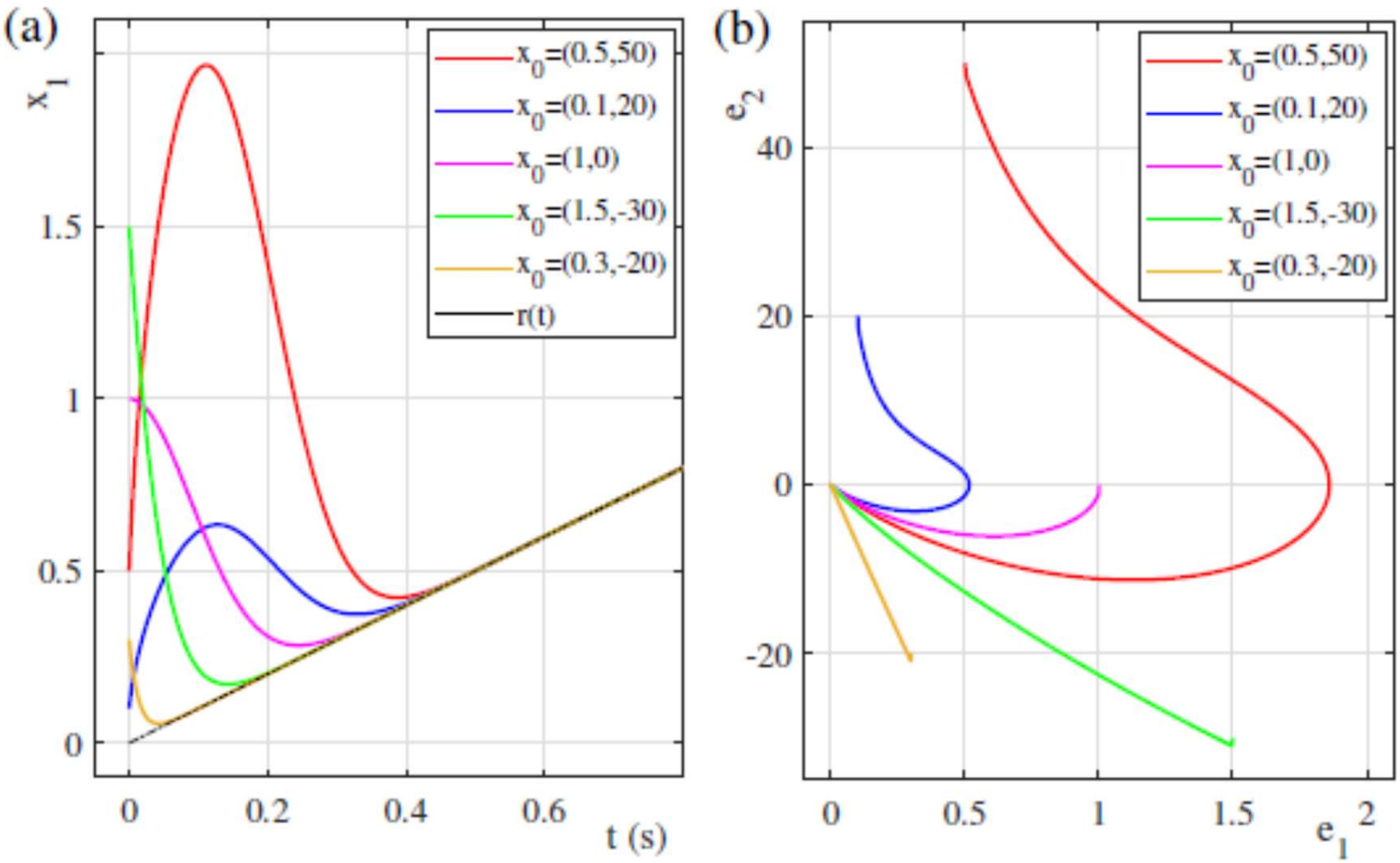

Fig. 4. Trajectories of the system (7), (8), with $k = 100$, $\mu = 0.0001$, for different initial values $x_0 \equiv [x_1, x_2](t_0)$: the output $x_1(t)$ versus reference $r(t)$ in (a), phase portrait of the error states in (b).

Numerical example of motion control trajectory

Control performance for piecewise smooth trajectory (e.g. motion control)

Compared with a standard (critically damped) PD linear feedback controller with

$$\dot{e}_2 = -100e_1 - 20e_2$$

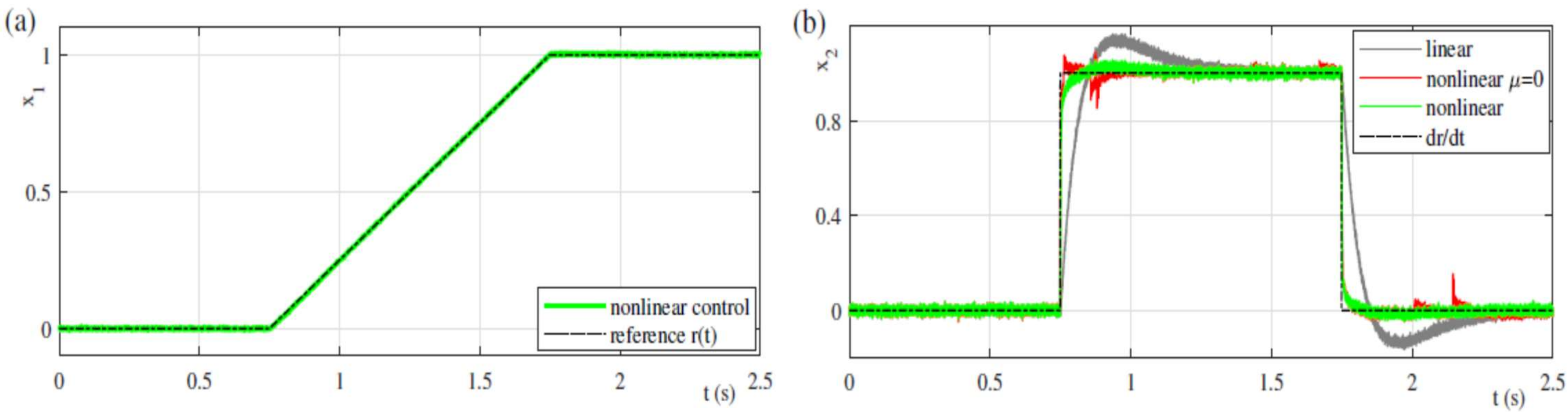

Fig. 5. Trajectories of the system (7), (8), with $k = 100$, $\mu = 0.0001$: the output $x_1(t)$ versus reference $r(t)$ in (a), the $x_2(t)$ state in (b) – compared with a case without regularization (i.e. $\mu = 0$) and with a critically damped proportional-derivative linear controller.

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Scaled OND-control for motion systems

- Extension for common motion systems

$$\ddot{x}_1(t) + \frac{1}{a}\dot{x}_1(t) = \frac{b}{a}u(t) \qquad \tau\dot{x}_2(t) + x_2(t) = Ku(t)$$

Here, x_1 is the output motion state (i.e. relative displacement in the generalized coordinates) of interest and u is the control input (i.e. generalized driving force). The parameters $a, b > 0$ are identifiable, either from the frequency response (FR) measurements or from the technical data sheets of the motion system under consideration.

- Scaled OND-control has the same properties as before

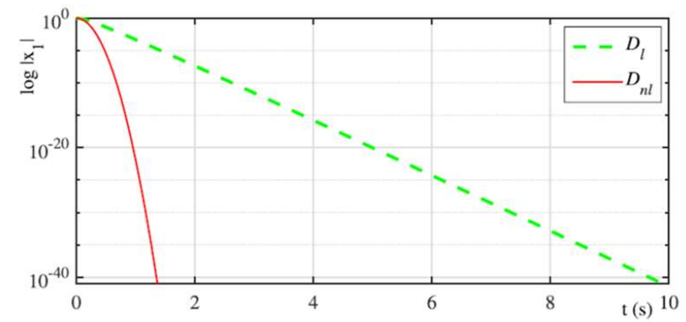
$$u(t) = ke + \frac{a}{b}\frac{|\dot{e}|\,\dot{e}}{|e| + \mu} + \frac{1}{b}\dot{x}_1(t)$$

$e = r - x_1$, where $r \in \mathcal{C}^1$ is the reference value

- If motion is perturbed by matched input ξ

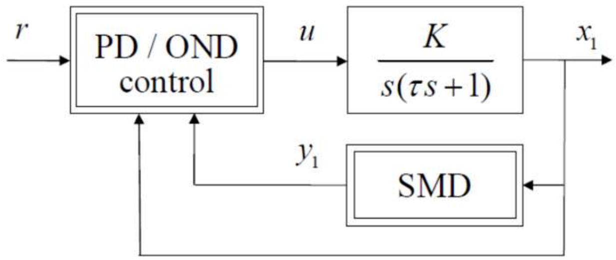
$$\tau\ddot{x}_1 + \tau\frac{|\dot{x}_1|\dot{x}_1}{|x_1| + \mu} + kx_1 = \xi$$

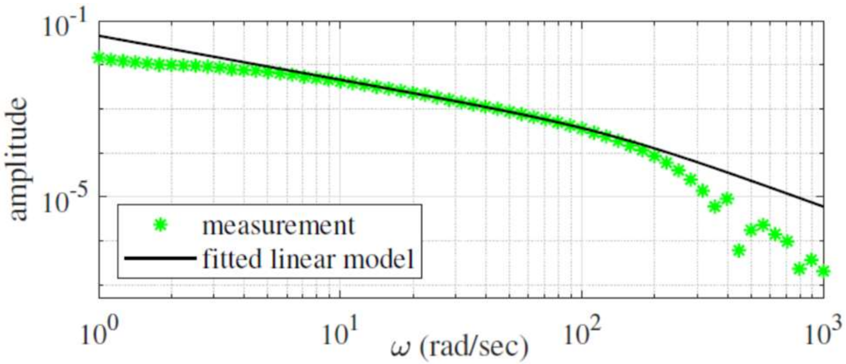
when assuming $r = 0$, for the sake of simplicity, and non-zero initial conditions $(x_1, x_2)(t) \neq 0$

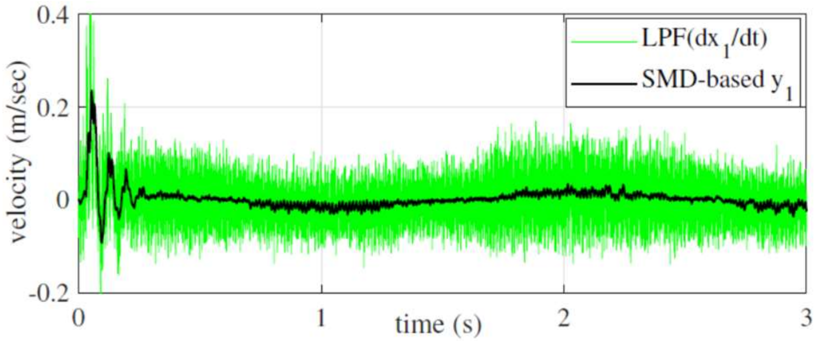


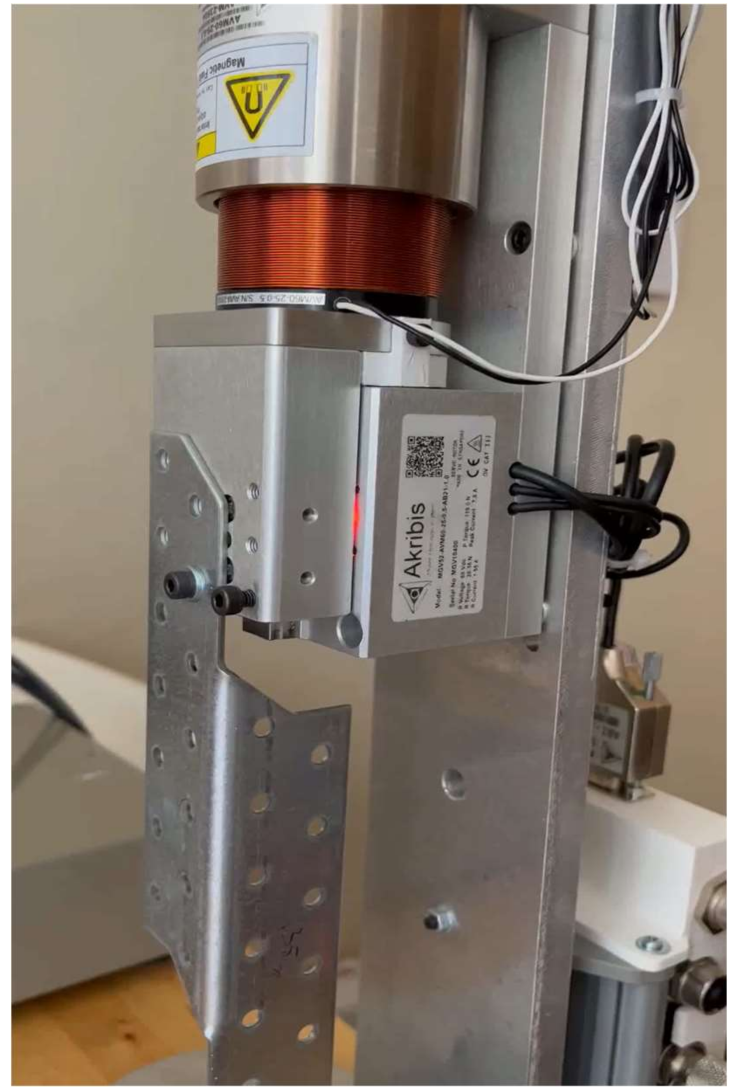
$x_1(t) \to \xi k^{-1}$ at steady-state

- 1DOF laboratory experimental setup (voice-coil actuator)



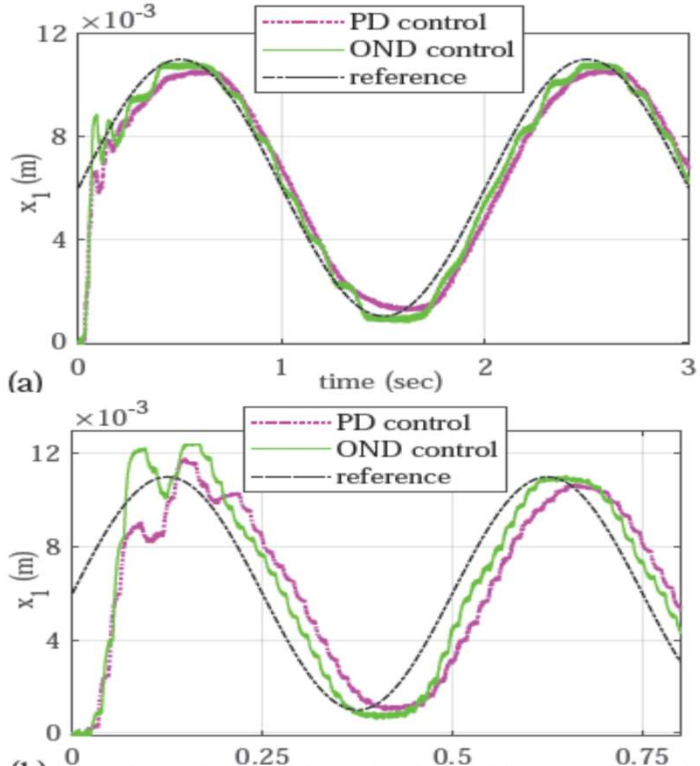




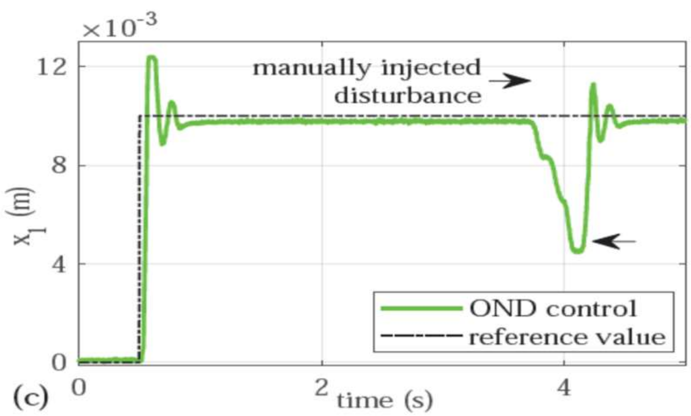


Experimental example (cont.)

- Comparison of OND and PD (critically damped) controllers, $k = 1000$

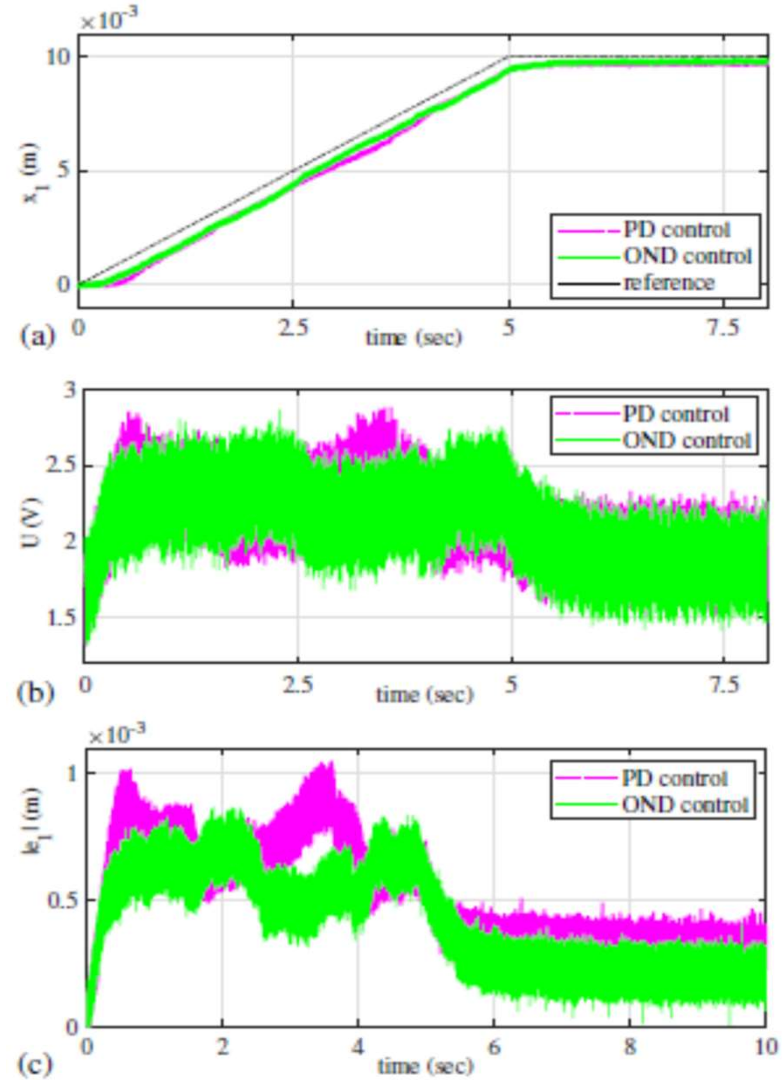


sinusoidal trajectory (of 0.5Hz and 2Hz frequency)



OND with disturbance

Measured position response of OND and PD controls to the slope reference (a), control value (b), absolute control error (c)



References to related works

[1] Ruderman, M. Optimal nonlinear damping control of second-order systems. Journal of The Franklin Institute, Vol. 358(8), 2021, pp. 4292-4302

[2] Ruderman, M. Convergent dynamics of optimal nonlinear damping control. IFAC-PapersOnLine, Vol. 54(17), 2021, pp. 141-144

[3] Ruderman, M. Application of motion control with optimal nonlinear damping and convergent dynamics. 22nd Styrian Workshop on Automatic Control, Graz, Austria, Sep 2022, pp. 7-8

[4] Ruderman, M. Motion control with optimal nonlinear damping: from theory to experiment. Control Engineering Practice, Vol. 127, 2022

Thank you for attention

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https://prosjektbanken.forskningsradet.no/en/project/FORISS/340782