APPROXIMATION
BOUNDEDNESS
SURJECTIVITY

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Introduction

This dissertation deals with three main topics:

1. To explain how the approximation property of a Banach space X forces the space of finite rank operators into X from a fixed, but arbitrary Banach space Y, to be an ideal of its superspace of weakly compact operators from Y into X.

2. To substitute the classical term "second category" with the term "thick" in the Uniform Boundedness Principle and observe that this term is the weakest condition such that the conclusion of the theorem is still true and then to explain how "thickness" is related to surjectivity of operators.

3. To show that unit balls in uniform algebras are as non-dentable as possible by proving, by rather elementary means, that, when such a unit ball is divided into two parts by a hyperplane such that both parts contain at least two distinct points, then both parts have diameter 2.

The first topic is explained through the paper *Factorization of weakly compact operators and the approximation property* [1]. It is a joint work with my supervisor, professor Åsvald Lima at Agder University College, and professor Eve Oja from Tartu University, Estonia. I want to express my warmest thanks for letting me take part in this joint project. This paper is Chapter 2 in the dissertation.

The second topic is discussed through the paper *Boundedness and surjectivity in normed spaces* [3]. This paper is a considerable extention and generalization of the article *A strong uniform boundedness principle in Banach spaces* [2]. The extended paper is Chapter 3 in the dissertation.

The third topic is covered by the paper *Slices in the unit ball of a uniform algebra* [4]. This is a joint work with professor Dirk Werner at Freie Universität, Berlin. I highly appreciated that cooperation and must admit that I probably would not have found such a general result without his help. This paper is Chapter 4 in the dissertation.

This dissertation is the result of more than four years of teaching and doing researches in a 60/40 combination at Agder University College. I want to thank the Institute of Mathematics for giving me the opportunity to work in a nice atmosphere during this "qualifying process". Especially, the goodwill from Arne Holme, will never be forgotten.

During this period I have received approximately \$ 1500 from Agder College University yearly to cover travel expences. This has given me the possibility to take part in four international conferences and three spring schools in Paseky, Czech Republic. Also, the Norwegian Research Council has kindly supported two ten-days stays in Tartu, Estonia. My warmest thanks for financial support.

The formal supervisor of this doctoral work was supposed to be professor Arne Stray at the University of Bergen. The reason why I put it this way is that he has been much more than a formal supervisor. It was him who pointed out the connections to the theory of bounded analytic functions in chapter 3 and he has all the way been very helpful.

Åsvald *must* have his morning coffee some minutes after eight. I now have the same custom, but I want to say that this is only partly because of the coffee. Thank you so much for nice mornings and for all you have done to help and encourage me from I was an undergraduate student and till now. My hope is that although your formal role as a supervisor is over, you will still supervise me for many years and hectoliters of coffee.

My wife and three children have had to get used to a person who is not always listening to what they are saying, because of mathematics. I will probably never change, but I do hope that when you really need me I will always be listening. After all, there is more in life than mathematics.

Kristiansand, August 2001.

Olav Nygaard

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Chapter 1

The framework

In this introductory chapter we will try to give some overview over the development of the concepts studied later on in this dissertation. Most of what is written here can be found in the books [7], [15], [3], [5] and [13]. Only when speaking about results that are not in one of this books, references to original articles will be given. The introduction is meant for persons rather well-oriented in mathematics, but not necessarily experts in functional analysis. The chapter is written in an informative style and ideas important to put the results in Chapters 2-4 in a perspective are given. When some important progresses in functional analysis are not mentioned, it is because there seems not to be connections to the subjects in this dissertation.

Early in the twenties it became clear that the framework complete, normed, linear space was interesting and useful. This framework gives enough structure for strong theorems and, at the same time, the framework is wide enough for a rich variety of problems and applications.

Very central in the development of this framework was the Polish school, with Stefan Banach as the leading person. He also wrote the first book on the subject, the famous *Théorie des Opérations Linéaires* [1]. Although many other mathematicians also contributed heavily, the framework carries Banach's name:

Definition 1.0.1. *A Banach space is a complete, normed linear space. A normed linear space is a (real or complex) vector space X where a function $\|\cdot\| : X \to \mathbb{R}$ is defined with the following three properties*

(a) $\|x\| = 0$ *if and only if* $x = 0$

(b) $\|\alpha x\| = |\alpha|\,\|x\|$ *for all* $x \in X$ *and all scalars* α

(c) $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in X$

The normed, linear space X is called complete if every sequence which is Cauchy in the metric $d(x, y) = \|x - y\|$ converges in X.

The sets $U(0, r) = \{x \in X : \|x\| < r,\ r \in \mathbb{R}\}$ form a basis for the topology on X which the norm gives rise to. The norm $\|\cdot\|$ of the space can also be viewed as the gauge of $U(0,1)$. By (b) multiplication by scalar in the vector space X is continuous from $X \times \mathbb{R}$ to X. By (c) addition is continuous from $X \times X$ to X. Whenever there is a bounded, convex neighbourhood U at the origin in a topological vector space, then the gauge of U defines a norm on X.

The norm function gives a nice compatibility between linear structure and topological structure. For example, the closure $B(0,r)$ of $U(0,r)$ is the set $\{x \in X \ : \|x\| \leq r,\ r \in \mathbb{R}\}$. The set $B(0,1)$ is called the *unit ball* of the Banach space and is most often written as B_X. It is a compact set in the norm-topology if and only if the underlying vector space has but finitely many dimensions. The boundary $S_X = \{x \in X \ : \|x\| = 1\}$ of B_X is called the *unit sphere.*

Since the Banach space X is a vector space, every linear functional on a subspace Y of X can be extended to all of X. This is a consequence of the Axiom of choice. Hahn and Banach proved independently that among the extensions $\hat{f}$ of a given functional f on Y, it is possible to find at least one such that

$$\sup_{x \in B_X} |\hat{f}(x)| = \sup_{y \in Y, \|y\| \leq 1} |f(y)|.$$

Such an extension is called a *norm-preserving extension.*

Theorem 1.0.2 (Hahn-Banach theorem). *Let X be a normed, linear space and let Y be a linear subspace. Then every continuous, linear functional on Y has a norm-preserving extension to all of X.*

The set of continuous, linear functionals X^* on a Banach space X is again a Banach space, with pointwise addition and scalar multiplication, and $U(0,r) = \{x^* \in X^* \ : \sup_{x \in B_X} |x^*(x)| < r\}$. By the Hahn-Banach theorem, for every $x \in X$, there exists an $x^* \in S_{X^*}$ such that $\|x\| = x^*(x)$. Let us call such an $x^* \in S_{X^*}$ a norm-giver. A subset of X^* which contains a norm-giver for every $x \in X$, is called a *James boundary* for X.

The Hahn-Banach theorem can also be used to prove that X^* separates closed convex sets from compact convex sets. In particular, X^* separates points on X and thus determines a locally convex topology on X called *the weak topology.* Here are some facts which show that the weak topology on Banach spaces has to obey strict laws:

Theorem 1.0.3. *Let (X, w) be a Banach space equipped with its weak topology. The following is true:*

(a) *If X^* is separable (contains a norm-dense countable subset), then $(B(0,r), w)$ is a metric space, with metric $d_w(x,y) = \sum_i 2^{-i}|f_i(x) - f_i(y)|$, where (f_i) is any dense, countable subset of S_{X^*}.*

(b) *Weak convergence of bounded nets is equivalent to pointwise convergence on fundamental sets in X^* (Hahn 1922 [8] (sequence case)).*

(c) *The norm closure and the weak closure of a convex set coincide (Mazur).*
(d) B_X *is weakly compact if and only if* X *is the dual of* X^*.
(e) *A set is weakly relatively compact if and only if it is weakly sequentially relatively compact if and only if it is weakly countably relatively compact (Eberlein-Šmulian-Grothendieck).*
(f) *If* A *is weakly compact, then* $\overline{\text{co}}(A)$ *is weakly compact (Krein-Šmulian).*
(g) *A set* A *is weakly compact if and only if every* $x^* \in X^*$ *attains its supremum on* A *(James).*
(h) *A bounded sequence converges weakly if it converges pointwise on a James boundary* J *(Rainwater-Simons).*
(i) *The weak topology of a Banach space is never complete.*

In case (d) we call the space *reflexive.* The above results clearly demonstrate the possibility of finding strong theorems within the framework of a Banach space. X also separates points on X^* and so defines a weak topology on X^* called *the weak-star topology* on X^*. The norm-dual of X^* is denoted X^{**} and the embedding $\tau : X \to X^{**}$, given by $\tau(x)(x^*) = x^*(x)$ is an isometry of X onto a closed subspace of X^{**} which is weak-to-weak-star continuous. Among the general results concerning the w*-topology are:

Theorem 1.0.4. *The following statements are valid:*

(a) *If* X *is separable, then* $(B_{X^*}(0,r), w^*)$ *is a metric space, with metric* $d_{w^*}(f,g) = \sum_i 2^{-i}|f(x_i) - g(x_i)|$, *where* (x_i) *is any dense, countable subset of* S_X.
(b) *Weak-star convergence of bounded nets is equivalent to pointwise convergence on fundamental sets in* X.
(c) S_X *is weak-star dense in* $S_{X^{**}}$ *(Goldstine).*
(d) *Weak-star closed, norm-bounded sets in* X^* *are weak-star compact (Alaoglu 1940).*
(e) *A convex set* $A \subset X^*$ *is weak-star closed if and only if* $A \cap tB_{X^*}$ *is weak-star closed for every* $t > 0$ *(Krein-Šmulian).*
(f) *There is always a weak-star null sequence in* S_{X^*} *(Josefsson-Nissenzweig).*

Concerning (d), it is important to observe that the weakly bounded and the weak-star bounded sets are exactly the norm bounded sets. Thus, the dual of a Banach space with its weak-star topology has the Heine-Borel property.

Many of the above results are true without assuming completeness. The main reason why Banach chose completeness as an assumption, is probably the Baire category theorem, which is valid for complete metric spaces. By using this category theorem, two more corner stones of Banach space theory can be proved (the first cornerstone is the Hahn-Banach theorem(s)). First it is the

Theorem 1.0.5 (Open mapping theorem). *A continuous linear mapping from a Banach space X onto a Banach space Y maps open sets onto open sets. Consequently, a bijective, continuous linear mapping form a Banach space X to a Banach space Y has automatically a continuous inverse.*

From the Open mapping theorem the road is not long to the Closed graph theorem stating that if $T : X \to Y$ is a linear map with closed graph in $X \times Y$, then T is continuous. The third corner stone is the

Theorem 1.0.6 (Uniform boundedness principle). *If (T_α) is a pointwise bounded family of linear continuous operators between the Banach spaces X and Y, then (T_α) is bounded in the norm-topology of the space $\mathcal{L}(X, Y)$ of linear bounded operators from X into Y.*

The norm topology in $\mathcal{L}(X, Y)$ is defined in the same manner as in X^*, that is,

$$B_{\mathcal{L}(X,Y)} = \{T : \sup_{x \in B_X} \|Tx\|_Y \leq 1\}.$$

The Uniform boundedness principle is often called the Banach-Steinhaus theorem, but this is a little misleading, since Hahn proved the theorem first and since the theorem could be proved exactly in the same way as many of it's special cases. However, Banach and Steinhaus were the first to prove it by using Baire's category theorem.

Functional analysis is (as mathematics in general) about giving and taking; the more structure, the more and stronger theorems, the less structure, the fewer and weaker theorems. A typical way of reasoning is to prove a certain theorem within the framework of Banach spaces and then to take away the assumptions that are not needed to reach the conclusion.

To sum up, a Banach space is a certain combination of three variables; linear space, topology given by a norm and completeness. In Chapter 3 we will play with the last of these three variables to see how this affects the Uniform boundedness principle.

In addition to the three above mentioned variables one can add more structure on the Banach space. Assuming a continuous multiplication from $X \times X$ to X, such that X is an algebra, gives a *Banach algebra.* Assuming an order structure, compatibel with the norm, results in a *Banach lattice.* An important Banach algebra is the space $\mathcal{L}(X)$ of continuous linear operators from a Banach space X into itself.

The algebraic assumptions and the topological assumptions of a Banach space are closely connected. The algebraic properties of the norm give possibilities to uncover geometric phenomena in the intersection of the the algebraic and topological properties of the Banach space. Chapter 2 rests heavily on geometrical arguments and, in Chapter 4, we will see how the assumption of a commutative multiplicative structure on the Banach space X forces the unit ball of the (infinite-dimensional) Banach space to have certain geometric properties far from being intuitive from our three-dimensional world.

We will now give some more background and perspectives for each of the three chapters.

1.1 Separability, bases and the approximation property

We have remarked that separability of $X^*[X]$ allows us to work with the weak [w*] -topology as a metric on bounded sets. Some results in Banach space theory are true only when separability assumptions are included, and many theorems are much simpler to prove when invoking some separability assumption.

Arguments in linear algebra often uses the existence of a basis. A generalization of the finite-dimensional basis concept is to define a basis as a countable, bounded set $(x_n) \subset X$, such that every $x \in X$ has a unique representation as an infinite linear combination $\sum_{i=1}^{\infty} a_i x_i$. This definition is due to J. Schauder, who also constructed a basis for the Banach space $C[0,1]$. If a Banach space has a Schauder basis, then the set of finite rational combinations of the x_i's is a countable, dense subset of X, so X is separable. The linear functionals (f_n) given by $f_n(x) = a_n$ are easily shown to be continuous and the Uniform boundedness principle shows that (f_n) is a bounded subset of X^*.

When X has a basis, we can think of X as a linearly ordered sequence of finite-dimensional subspaces X_n. The natural projections P_n from X onto X_n then form a bounded sequence of finite rank operators in $\mathcal{L}(X)$ which converges pointwise on X to the identity operator I_X. It is natural to ask whether all separable Banach spaces are built up this way, that is, whether all separable Banach spaces have a Schauder basis. This question is called the "goose-problem" since a living goose was promised as a reward to the person who could solve it. The question was unsolved until the early seventies when is was solved in the negative by Per Enflo. Enflo constructed an obscure subspace of c_0 which he showed could not have a basis.

The main research on the basis problem, however, was done by A. Grothendieck in the fifties. Let us say a few words about his work. It is not difficult to prove that if X has a basis, then for every Banach space Y, every compact operator $T : Y \to X$ is the limit in norm of a sequence of finite rank operators from Y into X. In different terms: The *approximable operators* from Y into X are exactly the closed subspace of compact operators. The possibility of approximating compact operators doesn't rest on separability. Grothendieck now showed that the approximable operators from Y into X coincides with the compact operators if and only if the finite rank operators on X is dense in $\mathcal{L}(X)$ in the topology of uniform convergence on compact sets. This last result doesn't rest upon separability either. Grothendieck had now proved that if he could find a Banach space such that I_X is not approximable in the topology of uniform convergence on compact sets, then this would imply the existence of a separable space without a basis.

Definition 1.1.1. *A Banach space has the approximation property (AP) if, for every compact set $K \subset X$ and every $\epsilon > 0$, there exists a finite rank operator S on X such that* $\sup_{x \in K} \|Sx - x\| < \epsilon$.

The topology τ of uniform convergence on compact sets is a locally convex vector topology on $L(X, Y)$, generated by the seminorms

$$\|T\|_K = \{\sup_{x \in K} \|Tx\|_Y \ : K \text{ compact}\}.$$

Grothendieck was able to describe $(\mathcal{L}(X, Y), \tau)^*$ and to put this description into the framework of tensor products of topological vector spaces. This observation led him to numerous equivalent formulations of the approximation property. The sad thing was that he couldn't find a counter example to, nor prove, any of them. Among the most interesting equivalent formulations is the following

Theorem 1.1.2. *The following statements are equivalent.*

(a) *There exists a Banach space lacking the AP.*

(b) *There exists a continuous function $K(s, t)$ on $[0, 1] \times [0, 1]$ such that $\int_0^1 K(v, t)K(t, u)\, dt = 0$ for every v and u, but $\int_0^1 K(t, t)\, dt \neq 0$.*

(c) *There exists a matrix $A = (a_{ij})_{i,j=1}^{\infty}$ such that $\lim_j a_{ij} = 0$ for every $i = 1, 2, ...,$ such that $\sum_{i=1}^{\infty} \max_j |a_{ij}| < \infty$ but such that trace$A = \sum_{n=1}^{\infty} a_{nn} \neq 0$.*

A. Davie has given a proof that a matrix satisfying (c) exists. This proof is still the simplest proof (although far from simple) that there exists a Banach space without the AP. Finding new, equivalent formulations of the AP is still interesting, since simpler examples of spaces lacking the AP then could be found. In Chapter 2 at least one new formulation is given. It goes like this: X has the AP if and only if, for every separable, reflexive space Y, $B_{\mathcal{F}(Y,X)}$ is dense in $B_{\mathcal{W}(Y,X)}$ in the strong operator topology. Here $\mathcal{F}(Y, X)$ denotes the (not closed) space of finite rank operators from Y into X and $\mathcal{W}(Y, X)$ means the closed subspace in $\mathcal{L}(Y, X)$ of weakly compact operators from Y into X.

Enflo found his counter example inside c_0. It was not by accident he looked just there. Suppose X is a space lacking the AP. Then there must be a space Y and a compact operator $T : Y \to X$ which is not approximable. Grothendieck had proven that compact operators can always be compactly factorized through a subspace Z of c_0, i.e. there are compact operators $T_1 : Y \to Z$ and $T_2 : Z \to X$ such that $T = T_2 \circ T_1$. Any compact operator defined on c_0 is approximable. Since T is not approximable, neither T_1 nor T_2

can be approximable. Thus Z does not have the AP and Z is a subspace of c_0. In fact, Z^* doesn't have the AP either because Grothendieck had proved that the dual X^* of a Banach space X has the AP if and only if all compact operators defined on X are approximable. So Grothendieck knew there had to be a separable dual space not having the AP, if any.

Just before Enflo's counter example, Figiel proved that compact operators can be compactly factorized through reflexive spaces. More precisely, to every compact operator T, there is a reflexive space Z_T such that T can be compactly factorized through Z_T. Thus, he had shown that there had to be a reflexive space lacking the AP, if any. The matrix in Davie's proof can be used to construct subspaces of c_0 and $l_p, p \neq 2$ lacking the AP. l_2 can't have subspaces without the AP because they are all separable Hilbert spaces and thus isometric to l_2 itself.

Again, let τ be the topology on $\mathcal{L}(X)$ defined by uniform convergence on compact sets. Examinating Davie's proof reveals that, in the non-AP-subspaces of c_0 and $l_p, p \neq 2$ it produces, it isn't just impossible to τ-approximate the identity operator by finite rank operators, the identity operator can't be τ-approximated by compact operators neither. This led to a new concept; the counter examples don't have the *compact approximation property* (CAP). For some time it was open whether the CAP would imply the AP. Willis showed in 1991, given any space without the AP, how to construct a space still not having the AP, but having the CAP. He even constructed a separable reflexive space with the CAP lacking the AP [18].

If a net converges τ to I_X, it need not be norm-bounded in $\mathcal{L}(X)$. If it can be chosen to be norm-bounded by some number λ, as is the case when X has a basis, Grothendieck used the term *bounded* AP (BAP) or, more precisely, λ-AP. If we can take $\lambda = 1$, X has the *metric* AP (MAP). Grothendieck proved that separable duals with the AP must also have the MAP. Figiel and Johnson showed in the early seventies that there exists a separable space (with a separable dual) having the AP but not the BAP. They also found a space having the BAP but not the MAP.

Knowing all this, it is easier to simplify Grothendieck's proofs and to guess new theorems. But still a lot of questions concerning the AP's are open. Let us mention some of them:

Question 1.1.3. (a) *Suppose every compact $T : X \to X$ is approximable. Does X have the AP?*

(b) *Does H^∞ (the bounded analytic functions on the disc, with sup-norm) have the AP?*

(c) *Suppose X has the BAP. Is there an equivalent norm $|||\cdot|||$ on X such that $(X, |||\cdot|||)$ has the MAP?*

(d) *Suppose X^* has the CAP. Does X have the CAP?*

In connection to (b), let us mention that the only classical space which is known not to have the AP is $\mathcal{L}(l_2)$ (see [17]). Concerning (d) it is known that the answer is negative for the MCAP (see [2]).

In 1979, J. Johnson [10] showed how to construct a norm-one projection $P : \mathcal{L}(X)^* \to \mathcal{L}(X)^*$ such that $\ker P = \mathcal{F}(X)^\perp$, when X has the MAP. In more modern terms: If X has the MAP, then $\mathcal{F}(X)$ is an *ideal* in $\mathcal{L}(X)$. In 1993 it was shown by Å. Lima [11] that the converse is true with some additional assumptions on X (RNP). In particular, the converse is true for reflexive spaces.

Question 1.1.4. *Suppose $\mathcal{F}(X)$ is an ideal in $\mathcal{L}(X)$. Does X have the MAP?*

Later, various authors have uncovered new theorems all showing that there are close connections between AP's in X and local complementability of operator subspaces in $\mathcal{L}(Y, X)$. Chapter 2 in this dissertation follows up this research uncovering the following theorem:

Theorem 1.1.5. *X has the AP if and only if for every Banach space Y, $\mathcal{F}(Y, X)$ is an ideal in $\mathcal{W}(Y, X)$.*

Returning to Schauder bases, Mazur proved that every Banach space contains a closed subspace with a Schauder basis. A Schauder basis is called *unconditional* if the unique series expansions are unconditionally convergent. Every $L_p[0, 1]$, $1 < p < \infty$, has an unconditional basis (the Franklin system). Around 1960 $L_1[0, 1]$ was shown, by Pełczynski, not to have an unconditional basis. Banach and Mazur asked whether every Banach space contains a closed subspace with an unconditional Schauder basis. This problem was open for a long time, but was answered in the negative by Gowers and Maurey in the beginning of the 1990's (see [9]).

1.2 The completeness assumption

We have already remarked that one important reason to assume completeness of the normed linear space, is to be able to use the Baire category theorem for the proof of the Open mapping theorem and the Uniform boundedness principle. In a standard proof of the Open mapping theorem we use completeness of the domain space and the possibility of using a category argument in the range space. Rudin formulates the Open mapping theorem this way:

Theorem 1.2.1 (General Open mapping theorem). *Let T be a linear, bounded operator from a topological vector space X into a topological vector space Y. Assume*

(a) *X is an F-space (the topology is induced by a complete invariant metric)*

(b) *TX is of the second category in Y*

Then

(a) *T is onto*
(b) *T maps open sets onto open sets*
(c) *Y is an F-space*

The possibility of proving the Open mapping theorem when X and Y are locally convex spaces has been studied by V. Pták in the fifties (see e.g. [14]). His conclusion is that the domain space X has to be so-called B-complete (see Definition 1.2.3), a property which coincides with completeness in the metric case.

Definition 1.2.2. *Let X and Y be topological vector spaces and let $T : X \to Y$.*

(a) *T is called nearly open if for every neighbourhood U of 0 in X, the closure of TU is a neighbourhood of 0 in Y.*
(b) *T is called nearly continuous if for every neighbourhood V of 0 in Y, $\overline{T^{-1}(V)}$ is a neigbourhood of 0 in X.*

The critical step in the proof of the classical Open mapping theorem is to show that if X is complete, then a nearly open $T : X \to Y$ is already open.

Definition 1.2.3. *A locally convex space X is called B-complete if, for every locally convex space Y, every linear, continuous, nearly open $T : X \to Y$ is open.*

A special case of the Krein-Šmulian theorem (see Theorem 1.0.4(e)) is the

Theorem 1.2.4 (Banach-Dieudonné). *Let X Be a Banach space and suppose Y is a linear subspace of X^*. If $Y \cap B_{X^*}$ is weak-star closed, then Y is weak-star closed.*

One important observation Pták made was the following.

Theorem 1.2.5 (Pták). *A locally convex space is B-complete if and only if the Banach-Dieudonné theorem holds.*

As already mentioned, for metric spaces, B-completeness coincides with completeness. Today locally convex B-complete spaces are usually called Pták spaces.

Using the general Open mapping theorem one can show the

Theorem 1.2.6 (Closed graph theorem, F-spaces). *Let T be a linear, bounded operator from an F-space X into an F-space Y. Assume T has a closed graph in $X \times Y$. Then T is continuous.*

In this theorem, however, the completeness of X is not needed. In fact, if X is a locally convex topological vector space, the above theorem is true exactly when X is a *barrelled space.*

Definition 1.2.7. *Let X be a locally convex topological vector space. A barrell is a subset A of X such that*

(a) *A is closed*

(b) *A is absolutely convex (balanced and convex)*

(c) *A is absorbing (A contains a line segment in every direction)*

X is called barrelled if every barrell in X is a neighbourhood of the origin.

Note that every locally convex topological vector space has a base for the topology at the origin, which is formed by barrells. The point is that there might be more barrells. Barrelled spaces are well suited also for uniform boundedness principles. Let us first note that the bounded subsets of $\mathcal{L}(X,Y)$, X,Y topological vector spaces, are exactly the equicontinuous families of operators.

Theorem 1.2.8 (Corner stones, Barrelled spaces). *Let X be a locally convex topological vector space. The following statements are equivalent.*

(a) *X is barrelled*

(b) *For every locally convex topological vector space Y, every pointwise bounded family $\mathcal{H}$ in $\mathcal{L}(X,Y)$ is equicontinuous*

(c) *For every locally convex topological vector space Y, every linear $T : X \to Y$ is nearly continuous.*

(d) *For every Frechet space (locally convex F-space) Y, every linear $T : X \to Y$ with closed graph in $X \times Y$ is continuous*

(e) *For every Banach space Y, every linear $T : X \to Y$ with closed graph in $X \times Y$ is continuous*

We end this section by summing up in a theorem the corner stones of Banach spaces in the setting of locally convex spaces:

Theorem 1.2.9. *Let X and Y be locally convex topological vector spaces. The following statements are valid.*

(a) *The dual separates convex, compact sets from convex, closed sets*

(b) *The Uniform boundedness principle is true exactly when X is barrelled*

(c) *If X is B-complete and Y is barrelled, any linear bounded T onto Y is open*

(d) *If Y is B-complete and X is barrelled, the Closed graph theorem holds.*

The completeness assumption assures that a Banach space X is both B-complete and barrelled.

1.3 Theory of closed, convex sets

Think of a cube in space. It has vertices, edges and faces. A common property for these three subsets of the cube is that if a point is situated on a vertex, an edge or a face, and lies on a line between two points in the cube, then the endpoints of that line must be inside the vertex, edge or face. This simple geometric idea is generalized to any vector space; we call all non-empty subsets of a given set A with such a property as the vertex, the edge and the face simply *faces* of A. In this terminology, an edge is a one-dimensional face and a vertex is a face containing only one point. Instead of vertex, we use the term *extreme point.* Let us state the formal definition.

Definition 1.3.1. *Let A be a subset of a vector space. A subset F of A is called a face of A if, whenever $x, y \in A$ and $\lambda > 0, \mu > 0, \lambda + \mu = 1$ are such that $\lambda x + \mu y \in F$, then $x, y \in F$. A point $e \in A$ is called extreme if whenever $x, y \in A$ and $\lambda > 0, \mu > 0, \lambda + \mu = 1$ are such that $\lambda x + \mu y = e$, then $x = y = e$.*

Fundamental in functional analysis is the following discovery of Krein and Milman:

Theorem 1.3.2 (The Krein-Milman theorem). *Suppose A is a non-empty, compact subset of a locally convex topological vector space X. Then A contains an extreme point. If A in addition is convex, and E is the set of extreme points of A, then*

$$A = \overline{co}(E).$$

Later Milman proved the following "converse":

Theorem 1.3.3 (Milman's converse to the Krein-Milman theorem). *Suppose A is a compact, convex subset of a locally convex topological vector space X and E is the set of extreme points of A. If F is any subset of A such that*

$$A = \overline{co}(F),$$

then every $e \in E$ lies in the closure of F.

The cube in $\mathbb{R}^3$ obviously fulfils both the assumption and the conclusion of the Krein-Milman theorem. As we have remarked earlier, the unit ball is no longer compact if the Banach space is infinite-dimensional. Also, the Krein-Milman theorem does not in general hold for the unit ball of a Banach space. As examples, the unit ball of $L_1[0,1]$ has no extreme points. The unit ball of $C_{\mathbb{R}}[0,1]$ has two extreme points, $\{\pm 1\}$. The unit ball of l_1 is the closed convex hull of its extreme points, the same is true in l_∞. The unit ball of the space of regular Borel measures equipped vith variational norm, $M[0,1]$ has every $\pm\delta_x : x \in [0,1]$ as an extreme point, but is not the closed, convex hull of its extreme points. However, by invoking Alaoglu's theorem, it has to be the weak-star closed, convex hull of its extreme points. Since the dual ball in any reflexive space is weakly compact, and the norm closure and the weak closure of convex sets coincide (see Theorem 1.0.3), unit balls of reflexive spaces are the closed, convex hull of their extreme points.

l_1 is not reflexive, but it is a separable dual space. Bessaga and Pełczynski proved that in separable dual spaces closed, convex sets are always the closed convex hull of the extreme points. Later on, spaces with such a property are said to have the Krein-Milman property (KMP). The $M[0,1]$-example shows that not every dual space has the KMP.

The extreme points of the dual unit ball of a Banach space X is clearly a James boundary for X. The Rainwater-Simons theorem (see Theorem 1.0.3) was first proved (and published by Rainwater, a group of analysts in Seattle) for this particular boundary, but Simons' later research showed that in that theorem extreme point structure had nothing to do with it.

The KMP gives a dichotomy of Banach spaces, but unfortunately no equivalent formulation of that dichotomy is known. We will return to this dichotomy in a moment, but first we have to explain the Radon-Nikodým property. To do this we need some more geometrical concepts.

Let us again return to finite dimensions: Give $\mathbb{R}^2$ the norm defined by a unit ball which is put together the following way: Draw a straight line from $(-1,1)$ to $(1,1)$ and from $(-1,-1)$ to $(1,-1)$. Then close the open ends with half-circles. The resulting unit ball looks like a skating rink. Every point of the half-circles are now extreme. A point e in a set A is called an *exposed point of* A if there is a functional taking its maximum over A at e and nowhere else in A. Geometrically, there is a unique tangent plane at e. Every exposed point is clearly extreme, but not every extreme point need to be exposed. Just look at the four points where the straight lines meet the half-circles. Here, the tangent planes are not unique.

The extreme points of the unit ball of l_∞ are all of the form $(\pm1, \pm1, ...)$. To any such extreme point, it is easy to find a unit vector in l_1 which exposes. This exposer is even weak-star continuous, so the extreme points of l_∞ are all *weak-star exposed.* The set of weak-star exposed points of the dual unit ball is of some interest, since it has to be contained in any James boundary of X.

An exposed point e in a subset A of a normed space X is called *strongly exposed* if, whenever f is an exposer of e and (x_n) is a sequence in A such that $f(x_n) \to f(e)$, then $\|x_n - e\| \to 0$. If the strongly exposed point is exposed by a weak-star continuous functional, it is called *weak-star strongly exposed.*

It was shown early that in a Banach space every norm-compact convex set is the closed convex hull of its exposed points. However, this generalization of the Krein-Milman theorem theorem can not be done in arbitrary locally convex spaces. Returning to compact convex subsets of Banach spaces, it can be seen that all exposed points are strongly exposed. The same is not true for weakly compact convex sets. It is therefore remarkable that Lindenstrauss in the sixties proved

Theorem 1.3.4. *A weakly compact, convex subset of a Banach space is the closed convex hull of its strongly exposed points.*

Even in a Hilbert space a weakly compact body can be found, having only countably many strongly exposed points. Fonf [6] has shown that in reflexive spaces, the set of exposed points of a closed convex body is always uncountable. More on this will be commented in Chapter 3.

It turns out that Lindenstrauss' theorem above is a special case of a deep theorem of Bourgain:

Theorem 1.3.5. *Let C be a closed bounded and convex subset of a Banach space X. Suppose C has the Radon-Nikodým property. Then C is the closed convex hull of its strongly exposed points.*

We now explain the Radon-Nikodým property.

Definition 1.3.6. *Let C be a closed, bounded and convex subset of a Banach space X. C is said to have the Radon-Nikodým property (RNP) if the following Radon-Nikodým theorem holds:*

Let (Ω, Σ) be a measurable space, let F be an X-valued measure and μ a probability measure on (Ω, Σ). Assume $F(E)/\mu(E) \in C$ for all μ-positive $A \in \Sigma$. Then there is an $f \in L_1(\mu, X)$ such that

$$F(E) = \int_A f(\omega) d\mu(\omega)$$

for all $A \in \Sigma$.

The space X is said to have the RNP if its unit ball has the RNP.

The condition $F(E)/\mu(E) \in C$ for all μ-positive $A \in \Sigma$ implies that F is of bounded variation, that $F \ll \mu$ and that $f \in L_\infty(\mu, X)$ whenever it exists.

An important step towards a proof of Theorem 1.3.5 is to establish the concept of dentability. First we need the simple idea of a *slice*. A slice of a bounded, closed, convex set C is a subset $S(C, x^*, \alpha)$ defined by

$$S(C, x^*, \alpha) = \left\{ y \in C \ : x^*(y) > \sup_{x \in C} x^*(x) - \alpha \right\}.$$

Here $x^* \in X^*$ and $\alpha > 0$. Geometrically, a slice is "what is left of C above the hyperplane". An interesting thing is now to study what happens to the diameter of the slices when α decreases to 0. As an example, look at the cube in $\mathbb{R}^3$. If the dividing hyperplane is parallel to an edge or a face (now using geometrical terms), the diameter of the slices will remain 2, no matter how small α is. If the dividing hyperplane is not parallel an edge or a face, the diameter of the slices decreases to 0 as $\alpha \to 0^+$, and the hyperplane "leaves" the cube at an extreme point. In Chapter 4 we prove that any slice of the unit ball of a uniform algebra (e.g. every $C(K)$-space) has diameter 2!

Now, a bounded, closed, convex set C is called *dentable* if it has slices of arbitrary small diameter. If a sequence of slices S_n is such that $\mathrm{diam}(S_n) \to 0$ and a point $y \in C$ lies in S_n for every n, then y is called a *denting point* of C. Note that every denting point is extreme and also a point of continuity for C (the identity is weak-norm continuous in the relative weak and norm topologies on C). Lin, Lin and Trojanski [12] proved in 1988 that an extreme point of a closed, bounded and convex set C in a Banach space is a denting point of C excatly when it is a point of continuity for C.

If X has the RNP, then every closed, bounded, convex subset of X will have to have the RNP. Thus, if X has the RNP it also has the KMP. Whether the converse is true is a longstanding problem which goes back to the early seventies. It has been proven, by Huff and Morris, that the KMP and the RNP are equivalent in dual Banach spaces. Also the equivalence has been established for large classes of Banach spaces not necessarily dual.

There is a stronger version of the Krein-Milman theorem valid for metrizable compact convex sets. In this case the set of extreme points is a $\mathcal{G}_\delta$, and hence Borel-measurable.

Theorem 1.3.7 (Choquet's theorem). *Let C be a metrizable, compact, convex subset of a locally convex topological vector space X. If $x \in K$, there exists a regular probability Borel measure μ defined on K which is concentrated on the extreme points of K and with the property that given any affine, continuous (in particular, linear continuous) f, then*

$$f(x) = \int_K f(k)\, d\mu(k).$$

We say that x is the *barycenter* of μ and that μ *represents* x if the above identity is valid for all $f \in X^*$. It is not very difficult to show the following

Theorem 1.3.8. *Let X be a locally convex topological vector space and C be one of its compact, convex subsets. The following statements are equivalent:*

(a) *C is the closed convex hull of its extreme points (which is true by the Krein-Milman theorem)*

(b) *Every point x of C is the barycenter of a regular probability Borel measure supported on the closure of the extreme points of C*

The set of extreme points is not a closed set in general, not even in the metrizable case.

Throughout this dissertation we will use heavily concrete representation of the described subsets of the extreme points of various sets.

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Chapter 2

Factorization of weakly compact operators and the approximation property

2.1 Introduction

Let us recall that a linear subspace F of a Banach space E is an **ideal** in E if $F^\perp$ is the kernel of a norm one projection in E^*. The notion of an ideal was introduced and studied by Godefroy, Kalton, and Saphar in [14].

J. Johnson [20] proved that if X is a Banach space with the metric approximation property, then, for every Banach space Y, $\mathcal{F}(Y,X)$, the space of finite rank operators from Y to X, is an ideal in $\mathcal{L}(Y,X)$, the space of bounded operators from Y to X. Lima [23] has shown that the converse is true if X has the Radon-Nikodým property. It is not known whether the converse is true in general.

In [25], Lima and Oja proved that X has the approximation property if and only if $\mathcal{F}(Y,X)$ is an ideal in $\mathcal{K}(Y,X)$, the space of compact operators from Y to X, for all Banach spaces Y. In fact, they showed something stronger: X has the approximation property if (and only if) $\mathcal{F}(Y,X)$ is an ideal in $\mathcal{K}(Y,X)$ for all separable reflexive spaces Y, or, equivalently, for all closed subspaces Y of c_0.

It is natural to ask what happens if we look at $\mathcal{F}(Y,X)$ as a subspace of $\mathcal{W}(Y,X)$, the space of weakly compact operators from Y to X, instead of looking at $\mathcal{F}(Y,X)$ as a subspace of $\mathcal{K}(Y,X)$. The answer to this question is the main result of this paper: *X has the approximation property if and only if $\mathcal{F}(Y,X)$ is an ideal in $\mathcal{W}(Y,X)$ for all Banach spaces Y, which in turn, is equivalent to the condition that, for every Banach space Y and every $T \in \mathcal{W}(Y,X)$, there is a net (T_α) in $\mathcal{F}(Y,X)$ with $\sup_\alpha \|T_\alpha\| \leq \|T\|$ such that $T_\alpha y \to Ty$ for all $y \in Y$.*

We depart from the remarkable factorization theorem due to Davis, Figiel, Johnson, and Pełczyński [5] asserting that any weakly compact operator factors through a reflexive Banach space. In Section 2 (cf. Lemma 2.2.1), we make a quantitative change in the Davis-Figiel-Johnson-Pełczyński construction which enables us to show, in Section 3, that one can factorize weakly compact operators through reflexive Banach spaces isometrically and even uniformly. In Theorem 2.2.2, we give a new characterization of the approximation property in terms of the Davis-Figiel-Johnson-Pełczyński factorization. We apply these results in Corollary 2.2.5 where we prove that X has the approximation property if and only if every weakly compact operator into X can be approximated in the strong operator topology by finite rank operators whose norms are at most equal to the norm of the weakly compact operator.

In Section 3 (cf. Lemma 2.3.1), we show that on the absolutely convex weakly compact set that is used in the factorization theorem of Davis, Figiel, Johnson, and Pełczyński to construct the reflexive Banach space, the two norm topologies coincide. (It was a part of the original construction that the two weak topologies coincide on the unit ball of the reflexive Banach space.) This, together with the quantitative modification of the Davis-Figiel-Johnson-Pełczyński construction made in Section 2, leads us to an isometric version of the Davis-Figiel-Johnson-Pełczyński factorization theorem (cf. Theorem 2.3.2). This also applies to show that the isometric factorization can even be uniform with respect to finite dimensional subspaces in the space of weakly compact operators (cf. Theorem 2.3.3 and Corollaries 2.3.4 and 2.3.5).

We apply the uniform isometric factorization from Section 3 in Sections 4 and 5. Our main results in Section 4 are Theorem 2.4.3 and Theorem 2.4.4. They characterize the approximation property of X and X^* in terms of ideals of finite rank operators. In particular, Theorem 2.4.3 shows that X has the approximation property if and only if $\mathcal{F}(Y, X)$ is an ideal in $\mathcal{W}(Y, X)$ for all Banach spaces Y, and Theorem 2.4.4 shows that X^* has the approximation property if and only if $\mathcal{F}(X, Y)$ is an ideal in $\mathcal{W}(X, Y)$ for all Banach spaces Y.

In Section 5, an easy example shows that it is not possible to characterize the compact approximation property of X by $\mathcal{K}(Y, X)$ being an ideal in $\mathcal{W}(Y, X)$ for all Y (although this property characterizes the compact approximation property for reflexive X). In Theorem 2.5.1, we give some conditions equivalent to $\mathcal{K}(Y, X)$ being an ideal in $\mathcal{W}(Y, X)$ for all Y. We also show, by using the description of duals of spaces of compact operators due to Feder and Saphar [12], that these conditions are implied by the compact approximation property of X (cf. also Theorem 2.5.1).

In Theorems 2.6.1 and 2.6.2 of the final Section 6, we demonstrate how the method of proof of Theorem 2.2.2 can be further developed to give alternative proofs (through ideals of finite rank or compact operators) for known results about cases when the (compact) approximation property implies the metric (compact) approximation property. In particular, as an immediate corollary, we obtain the result due to Godefroy and Saphar [15] that X^* has the metric compact approximation property with conjugate operators whenever X^* has the compact approximation property with conjugate operators and X^* or X^{**} has the Radon-Nikodým property.

Let us fix some more notation. In a linear normed space X, we denote the closed unit ball by B_X and the closed ball with center x and radius r by $B_X(x,r)$. For a set $A \subset X$, its norm closure is denoted by $\overline{A}$, its linear span by $\operatorname{span} A$, its convex hull by $\operatorname{conv} A$, and the set of its strongly exposed points by $\operatorname{sexp} A$.

We shall write $\mathcal{K}_X$ (resp. $\mathcal{W}_X$) for the family of all compact (resp. weakly compact) absolutely convex subsets of B_X.

2.2 Criteria of the approximation property in terms of the Davis-Figiel-Johnson-Pełczyński factorization

In this section, we depart from the famous Davis, Figiel, Johnson, and Pełczyński factorization construction (cf. Lemma 1 on p. 313 in [5], [6, pp. 160-161], [7, p. 227], [33, p. 51] or Lemma 2.2.1 below) and apply the Grothendieck-Feder-Saphar description of duals of spaces of compact operators (cf. [16] or [8] and [12]) to obtain several conditions equivalent to the approximation property of Banach spaces, all of them expressed in terms of the Davis-Figiel-Johnson-Pełczyński construction (cf. Theorem 2.2.2 below). This leads us to an interesting "metric" characterization of the approximation property (cf. Corollary 2.2.5) similar to the well-known characterization of the **metric** approximation property as the denseness of $B_{\mathcal{F}(Y,X)}$ in $B_{\mathcal{L}(Y,X)}$ in the topology of uniform convergence on compact sets, for all Banach spaces Y.

We shall need a quantitative version of the classical Davis, Figiel, Johnson, Pełczyński factorization construction, which in fact consists in replacing the number 2 in the original construction by $\sqrt{a}$ for any $a > 1$. We now fix the notation to describe the Davis-Figiel-Johnson-Pełczyński construction, and we shall also use this notation in the following sections.

Let $a > 1$. Let X be a Banach space and let K be a closed absolutely convex subset of its unit ball B_X. For each $n \in \mathbb{N} = \{1, 2, \dots\}$, put $B_n = a^{n/2}K + a^{-n/2}B_X$ and denote by $\|\ \|_n$ the equivalent norm on X defined by the gauge of B_n. Let $\|x\|_K = (\sum_{n=1}^{\infty} \|x\|_n^2)^{1/2}$, $X_K = \{x \in X \colon \|x\|_K < \infty\}$ and $C_K = \{x \in X \colon \|x\|_K \leq 1\}$. Further, let J_K denote the identity embedding of X_K into X. Finally, we put

$$f(a) = \left(\sum_{n=1}^{\infty} \frac{a^n}{(a^n+1)^2} \right)^{1/2}$$

and note that $f\colon (1,\infty) \to \mathbb{R}$ is a continuous, strictly decreasing function with $\lim_{a\to 1^+} f(a) = \infty$ and $\lim_{a\to\infty} f(a) = 0$. Hence, there is a unique point $\tilde{a} \in (1,\infty)$ such that $f(\tilde{a}) = 1$. (A "good" estimate of this $\tilde{a}$ is $\exp(4/9) = 1.55962349761....$) For this $\tilde{a}$, one has $K \subset C_K \subset B_X$ (this is clear from Lemma 2.2.1 below).

The following is the classical Davis-Figiel-Johnson-Pełczyński factorization lemma with some "cosmetic" changes.

Lemma 2.2.1 (cf. p. 313 in [5]).

(i) *$K \subset f(a)C_K$.*

(ii) *X_K is a Banach space with the closed unit ball C_K, and $J_K \in \mathcal{L}(X_K, X)$, and $\|J_K\| \le 1/f(a)$.*

(iii) *J_K^{**} is injective.*

(iv) *X_K is reflexive if and only if K is weakly compact.*

Proof. Only (i) and $\|J_K\| \le 1/f(a)$ in (ii) need to be verified.

Suppose $x \in K$. Since $x \in B_X$, we get

$$a^{n/2}x + a^{-n/2}x \in B_n,$$

so that

$$\|x\|_n \le \frac{1}{a^{n/2} + a^{-n/2}} = \frac{a^{n/2}}{a^n + 1}$$

for all n. Hence $\|x\|_K \le f(a)$. This proves (i).

Since B_X is convex and $K \subset B_X$, we have

$$\frac{1}{a^{n/2} + a^{-n/2}}(a^{n/2}K + a^{-n/2}B_X) \subset B_X,$$

that is

$$\frac{a^{n/2}}{a^n + 1}B_n \subset B_X.$$

Hence

$$\|x\|_n \ge \frac{a^{n/2}}{a^n + 1}\|x\|$$

and therefore $\|x\|_K \ge f(a)\|x\|$ for all $x \in X_K$, meaning that $\|J_K\| \le 1/f(a)$. □

Theorem 2.2.2. *For a Banach space X, the following assertions are equivalent.*

(i) *X has the approximation property.*

(ii) *$\mathcal{F}(X_K, X)$ is an ideal in $\mathcal{L}(X_K, X)$ for every $K \in \mathcal{W}_X$.*

(iii) *For every $K \in \mathcal{W}_X$, there exists a net (A_α) in $\mathcal{F}(X_K, X)$ with* $\sup_\alpha \|A_\alpha\| \le \|J_K\|$ *such that $A_\alpha x \underset{\alpha}{\to} J_K x$ for all $x \in X_K$.*

(iv) *For every $K \in \mathcal{W}_X$, there exists a bounded net (A_α) in $\mathcal{F}(X_K, X)$ such that $A_\alpha x \underset{\alpha}{\to} J_K x$ for all $x \in X_K$.*

(v) *For every $K \in \mathcal{K}_X$, there exists a net (A_α) in $\mathcal{F}(X_K, X)$ such that $\|A_\alpha - J_K\| \underset{\alpha}{\to} 0$.*

Remark 2.2.1. Condition (v) means that J_K belongs to the norm closure of $\mathcal{F}(X_K, X)$ in $\mathcal{L}(X_K, X)$ and (iii) can be viewed as its “metric” version: J_K belongs to the closure of the ball $\mathcal{F}(X_K, X) \cap B(0, \|J_K\|)$ in the strong operator topology of $\mathcal{L}(X_K, X)$.

For the proofs of Theorem 2.2.2 and Theorem 2.5.1, we shall need the following well-known description of duals of spaces of compact operators due to Feder and Saphar [12]. Let us recall that if X and Y are Banach spaces, then for any $v \in X^* \hat{\otimes}_\pi Y^{**}$, $v = \sum_{n=1}^\infty x_n^* \otimes y_n^{**}$ with $\sum_{n=1}^\infty \|x_n^*\| \|y_n^{**}\| < \infty$, and for any $T \in \mathcal{L}(Y, X)$, the element $T^{**}v \in X^* \hat{\otimes}_\pi X^{**}$ is defined by $T^{**}v = \sum_{n=1}^\infty x_n^* \otimes T^{**} y_n^{**}$.

Lemma 2.2.3 (cf. [12, Theorem 1]). *Let X and Y be Banach spaces such that X^* or Y^{**} has the Radon-Nikodým property. Let $\Phi\colon X^* \hat{\otimes}_\pi Y^{**} \to \mathcal{L}(Y, X)^*$ be defined by*

$$(\Phi v)(T) = \operatorname{trace}(T^{**}v), \quad T \in \mathcal{L}(Y, X), \quad v \in X^* \hat{\otimes}_\pi Y^{**}.$$

Then, for all $g \in \mathcal{K}(Y, X)^*$, there exists $v \in X^* \hat{\otimes}_\pi Y^{**}$ such that $g = (\Phi v)|_{\mathcal{K}(Y,X)}$ and $\|g\| = \|\Phi v\|$.

The proof of Theorem 2.2.2, as well as some other proofs of this paper, will also use the following result.

Lemma 2.2.4. *Let X and Y be Banach spaces. Let $\mathcal{A}$ be a subspace of $\mathcal{L}(Y, X)$ containing $\mathcal{F}(Y, X)$ and let $T \in \mathcal{L}(Y, X)$. If $\mathcal{A}$ is an ideal in $\mathcal{L} :=$* $\operatorname{span}(\mathcal{A} \cup \{T\})$ *and P is an ideal projection, then there exists a net $(A_\alpha) \subset \mathcal{A}$ with* $\sup_\alpha \|A_\alpha\| \le \|T\|$ *such that*

$$y^{**}(A_\alpha^* x^*) \underset{\alpha}{\to} (P(y^{**} \otimes x^*))(T) \quad \textit{for all } x^* \in X^* \textit{ and } y^{**} \in Y^{**}.$$

Moreover, if Y has the Radon-Nikodým property (in particular, if Y is reflexive), then (A_α) can be chosen to satisfy

$$A_\alpha y \to Ty \quad \textit{for all} \quad y \in Y.$$

Proof. Let P be a norm one projection on $\mathcal{L}^*$ with $\ker P = \mathcal{A}^\perp$. Since $P^*(T) \in \mathcal{A}^{\perp\perp} \subset \mathcal{L}^{**}$ and $\|P^*(T)\| \le \|T\|$, there exists a net $(A_\alpha) \subset \mathcal{A}$

with $\sup_\alpha \|A_\alpha\| \le \|T\|$ such that $A_\alpha \to P^*(T)$ weak* in $\mathcal{L}^{**}$. In particular, for $x^* \in X^*$ and $y^{**} \in Y^{**}$, we have

$$y^{**}(A_\alpha^* x^*) = (y^{**} \otimes x^*)(A_\alpha) \underset{\alpha}{\longrightarrow} (y^{**} \otimes x^*)(P^*(T)) = (P(y^{**} \otimes x^*))(T).$$

It is straightforward to verify that, for any $f \in \mathcal{L}^*$, Pf is a norm-preserving extension of $f|_{\mathcal{A}} \in \mathcal{A}^*$. On the other hand, it is proved in [24, Lemma 3.4, (b)] that $y \otimes x^* \in \mathcal{F}(Y, X)^*$ has a unique norm-preserving extension to the whole $\mathcal{L}(Y, X)$ whenever $x^* \in X^*$ and $y \in \operatorname{sexp} B_Y$. Therefore $P(y \otimes x^*) = y \otimes x^* \in \mathcal{L}^*$ and

$$(A_\alpha^* x^*)(y) \underset{\alpha}{\longrightarrow} (y \otimes x^*)(T) = (T^* x^*)(y) \text{ for all } x^* \in X^* \text{ and } y \in \operatorname{sexp} B_Y.$$

If Y has the Radon-Nikodým property, then $Y = \overline{\operatorname{span}}\,(\operatorname{sexp} B_Y)$, and we get that

$$(A_\alpha^* x^*)(y) \underset{\alpha}{\longrightarrow} (T^* x^*)(y) \text{ for all } x^* \in X^* \text{ and } y \in Y.$$

This means that $A_\alpha \to T$ in the weak operator topology of $\mathcal{L}(Y, X)$. Since the weak and strong operator topologies yield the same dual space (cf. e.g. [9, Theorem VI.1.4]), after passing to convex combinations, we may assume that $A_\alpha \to T$ strongly. $\square$

Proof of Theorem 2.2.2. (i) $\Rightarrow$ (ii). Let us assume that X has the approximation property. We shall show that $\mathcal{F}(Y, X)$ is an ideal in $\mathcal{L}(Y, X)$ for any reflexive Banach space Y.

Consider $f \in \mathcal{L}(Y, X)^*$. For $g = f|_{\mathcal{F}(Y,X)}$, let $v = \sum_{n=1}^\infty x_n^* \otimes y_n \in X^* \hat{\otimes}_\pi Y$ with $\sum_{n=1}^\infty \|x_n^*\| < \infty$ and $\|y_n\| \to 0$ be given by Lemma 2.2.3. We assume that $(K_\alpha) \subset \mathcal{F}(X, X)$ converges to I_X uniformly on the compact subsets of X. Then, for $T \in \mathcal{L}(Y, X)$,

$$\begin{aligned} |(\Phi v)(T) - f(K_\alpha T)| &= |(\Phi v)(T - K_\alpha T)| \\ &= |\sum_{n=1}^\infty x_n^*((T - K_\alpha T) y_n)| \\ &\le \sup_n \|(I_X - K_\alpha)(T y_n)\| \sum_{n=1}^\infty \|x_n^*\| \underset{\alpha}{\longrightarrow} 0 \end{aligned}$$

because $\{0, Ty_1, Ty_2, \dots\}$ is a compact subset of X. Since $\Phi v \in \mathcal{L}(Y, X)^*$ is a norm-preserving extension of $f|_{\mathcal{F}(Y,X)}$, the mapping $P \colon \mathcal{L}(Y, X)^* \to \mathcal{L}(Y, X)^*$ defined by

$$(Pf)(T) = \lim_\alpha f(K_\alpha T) = (\Phi v)(T), \quad f \in \mathcal{L}(Y, X)^*, \quad T \in \mathcal{L}(Y, X),$$

is a norm one projection with $\ker P = \mathcal{F}(Y, X)^\perp$.

(ii) $\Rightarrow$ (iii). This is immediate from Lemma 2.2.4 because X_K is reflexive.

(iii) $\Rightarrow$ (iv). This is obvious.

(iv) $\Rightarrow$ (v). Let $K \in \mathcal{K}_X$. Then $J_K \in \mathcal{K}(X_K, X)$ because

$$J_K(C_K) = C_K \subset a^{n/2}K + a^{-n/2}B_X, \quad \text{for all} \quad n \in \mathbb{N},$$

implies that $J_K(C_K)$ has, for any $\epsilon > 0$, a finite ϵ-net and therefore it is relatively compact in X. By the description of the weak convergence in spaces of compact operators due to Feder and Saphar [12, Corollary 1.2] (the reflexivity of X_K and the boundedness of (A_α) are used here), we get that $(A_\alpha - J_K) \to 0$ weakly in $\mathcal{K}(X_K, X)$. After passing to convex combinations, we may assume that $\|A_\alpha - J_K\| \to 0$.

(v) $\Rightarrow$ (i). Let K be a compact subset of X and let $\epsilon > 0$. We have to show that there is an operator $T \in \mathcal{F}(X, X)$ such that $\|Tx - x\| < \epsilon$ for all $x \in K$. We may assume that $K \in \mathcal{K}_X$ (note that, by a theorem of Mazur, the absolutely convex hull of a compact set in a Banach space is compact). By (v), there is an operator $A = \sum_{i=1}^n y_i^* \otimes x_i \in \mathcal{F}(X_K, X)$ (with $y_i^* \in X_K^*$, $x_i \in X$) such that $\|A - J_K\| < \epsilon/2f(a)$. Since J_K^{**} is injective (cf. Lemma 2.2.1), $J_K^*(X^*)$ is norm dense in X_K^*. Let $x_i^* \in X^*$ satisfy $\|y_i^* - J_K^* x_i^*\| < \epsilon/2f(a)\sum_{i=1}^n \|x_i\|$ and let $T = \sum_{i=1}^n x_i^* \otimes x_i \in \mathcal{F}(X, X)$. Then, for every $x \in K$ (recall from Lemma 2.2.1 that $K \subset f(a)C_K$), we have

$$\begin{aligned}
\|Tx - x\| &= \|TJ_Kx - J_Kx\| \\
&\leq \|A - J_K\|\|x\|_K + \|TJ_K - A\|\|x\|_K \\
&< \frac{\epsilon}{2} + f(a)\|\sum_{i=1}^n (J_K^* x_i^* - y_i^*) \otimes x_i\| \\
&\leq \frac{\epsilon}{2} + f(a)\sum_{i=1}^n \|J_K^* x_i^* - y_i^*\|\|x_i\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

□

Remark 2.2.2. A famous theorem due to Grothendieck [16] (cf. e.g. [26, p. 32]) asserts that X has the approximation property if and only if $\overline{\mathcal{F}(Y, X)} = \mathcal{K}(Y, X)$ for all Banach spaces Y. Here the "only if" part is easy and straightforward (cf. e.g. [26, p. 32]). The "classical" proof of the "if" part relies on Grothendieck's characterization of a compact set as a subset of the closed convex hull of a norm-null sequence (cf. e.g. [26, pp. 32-33]) which is used to construct a Banach space Y - a linear subspace of X - such that the formal identity map from Y into X is compact. The proof of the implication (v) $\Rightarrow$ (i) above provides an alternative easier proof to the "if" part (where X_K together with the identity map J_K plays the role of Y). And combined together with Theorem (AP) in [25], this also gives an easy short proof for the classical fact (due to Grothendieck) that X^* has the approximation property if and only if $\overline{\mathcal{F}(X, Y)} = \mathcal{K}(X, Y)$ for all Banach spaces Y.

Remark 2.2.3. The idea to define a norm one projection with $\ker P = \mathcal{F}(Y,X)^\perp$ on $\mathcal{L}(Y,X)^*$ by $(Pf)(T) = \lim_\alpha f(K_\alpha T)$, $f \in \mathcal{L}(Y,X)^*$, $T \in \mathcal{L}(Y,X)$, whenever $K_\alpha \in B_{\mathcal{F}(X,X)}$ and $K_\alpha \to I_X$, is due to J. Johnson [20]. In the proof of the implication (i) $\Rightarrow$ (ii), the set of operators K_α is not necessarily bounded.

A Banach space X has the approximation property if and only if, for every Banach space Y, the finite rank operators are dense in $\mathcal{L}(Y,X)$ in the topology τ of uniform convergence on compact sets, and X has the **metric** approximation property if and only if the "metric" version of this condition holds: for every Banach space Y, the finite rank operators of norm ≤ 1 are dense in the unit ball of $\mathcal{L}(Y,X)$ in the topology τ (cf. e.g. [26, pp. 32, 39]). The next result provides a similar "metric" criterion for the approximation property.

Corollary 2.2.5. *For a Banach space X, the following assertions are equivalent.*

(i) *X has the approximation property.*

(ii) *For every Banach space Y, $B_{\mathcal{F}(Y,X)}$ is dense in $B_{\mathcal{W}(Y,X)}$ in the strong operator topology.*

(iii) *For every Banach space Y and every $T \in \mathcal{W}(Y,X)$, there is a net (T_α) in $\mathcal{F}(Y,X)$ with $\sup_\alpha \|T_\alpha\| \leq \|T\|$ such that $T_\alpha y \to Ty$ for all $y \in Y$.*

(iv) *For every separable reflexive Banach space Y and every $T \in \mathcal{W}(Y,X)$ there is a sequence (T_n) in $\mathcal{F}(Y,X)$ with $\sup_n \|T_n\| \leq \|T\|$ such that $T_n y \to Ty$ for all $y \in Y$.*

Proof. (i) $\Rightarrow$ (iii). We may assume that $\|T\| = 1$. Then $K := \overline{T(B_Y)} \in \mathcal{W}_X$. Let the number a be fixed so that $f(a) = 1$. Then (cf. Lemma 2.2.1) $T(B_Y) \subset C_K = B_{X_K}$ and $\|J_K\| \leq 1$. By Theorem 2.2.2 ((i) $\Rightarrow$ (iii)), there exists a net (A_α) in $\mathcal{F}(X_K,X)$ with $\sup_\alpha \|A_\alpha\| \leq 1$ such that $A_\alpha x \to J_K x$ for all $x \in X_K$. Define $T_\alpha \colon Y \to X$ by $T_\alpha y = A_\alpha T y$, $y \in Y$. Then $T_\alpha \colon Y \to X$ is linear and of finite rank, $T_\alpha y \to Ty$ for all $y \in Y$, and $\|T_\alpha\| \leq \sup\{\|Ty\|_K \colon y \in B_Y\} \leq 1$ for all α.

(iii) $\Rightarrow$ (ii). This is obvious.

(ii) $\Rightarrow$ (i). By Lemma 2.2.1, $J_K \in \mathcal{W}(X_K,X)$ whenever $K \in \mathcal{W}_X$ (because X_K is reflexive). Therefore, (ii) implies assertion (iv) of Theorem 2.2.2, which is equivalent to (i).

(iii)$\Rightarrow$(iv). Let Y be a separable Banach space and let $T \in \mathcal{W}(Y,X)$. Let (y_n) be a dense sequence in B_Y. By a standard argument, picking from the given net (T_α), for each $n = 1, 2, \dots$, operators T_{α_n} so that $\|T_{\alpha_n} y_1 - Ty_1\| < 1/n, \dots, \|T_{\alpha_n} y_n - Ty_n\| < 1/n$, one obtains the desired sequence $(T_n) = (T_{\alpha_n})$.

(iv)$\Rightarrow$(i). Let Z be any reflexive Banach space and let $T \in \mathcal{W}(Z,X)$. Recall that every separable subspace of Z is contained in a separable 1-complemented subspace Y of Z, meaning that there exists a norm one projection P_Y from Z onto Y (this so-called "separable 1-complementation property" is shared by all weakly compactly generated spaces (cf. [1] or e.g.

[6, p. 149])). Therefore the set of all triples $\alpha = (F, Y, \varepsilon)$, where F is a finite dimensional subspace of Z, Y is a separable 1-complemented subspace of Z containing F, and $\epsilon > 0$, is a directed set in the natural way. For any $\alpha = (F, Y, \varepsilon)$, considering $T|_Y \in \mathcal{W}(Y, X)$, we choose an operator $T_\varepsilon \in \mathcal{F}(Y, X)$ with $\|T_\varepsilon\| \leq \|T|_Y\|$ such that $\|T_\varepsilon y - Ty\| < \varepsilon$ for all $y \in B_F$ and let $T_\alpha = T_\varepsilon \circ P_Y$. Then $(T_\alpha) \subset \mathcal{F}(Z, X)$ satisfies $\sup_\alpha \|T_\alpha\| \leq \|T\|$ and $T_\alpha z \to Tz$ for all $z \in Z$. In particular, this gives assertion (iii) of Theorem 2.2.2 which is equivalent to (i). $\square$

Remark 2.2.4. Concerning the implication (i) $\Rightarrow$ (ii) of Corollary 2.2.5, we note that, by a result due to Grothendieck [16, Corollary 2, p. 141], the approximation property of the dual space X^* implies condition (ii) of Corollary 2.2.5. We are grateful to the Referee for pointing out this for us. Grothendieck's proof relies on his theorem stating that if A and B are, respectively, integral and weakly compact operators, then $A \circ B$ is a nuclear operator with the nuclear norm not greater than $\|B\|$ multiplied by the integral norm of A.

2.3 Uniform isometric factorization

The remarkable factorization theorem due to Davies, Figiel, Johnson, and Pełczyński [5] asserts that any weakly compact operator T factors through a reflexive space. In this case, if we write $T = A \circ B$, it is clear that the operators A and B are weakly compact. By a theorem of Figiel and Johnson ([13] and [21]), if T is a compact operator, then it admits a factorization $T = A \circ B$ where A and B are compact. (This fact can also be deduced from the Davis-Figiel-Johnson-Pełczyński theorem (cf. e.g. [19, p. 374]).)

In Theorem 2.3.2 below, we shall see that the quantitative modification in the Davis-Figiel-Johnson-Pełczyński construction made in Section 2, together with the following Lemma 2.3.1, leads to an isometric factorization in the Davis-Figiel-Johnson-Pełczyński and the Figiel-Johnson theorems. (In particular, if $\|T\| = 1$, then $\|A\| = \|B\| = 1$; the estimates from [33, p. 51] would give $\|A\|, \|B\| \leq 4$.)

Lemma 2.3.1 (Lemma 2.2.1 continued).

(i) *For $x \in K$, one has*

$$\|x\|_K^2 \leq (\frac{1}{4} + \frac{1}{\ln a})\|x\|.$$

(ii) *The X-norm and X_K-norm topologies coincide on K.*

(iii) *The weak topologies defined by X^* and X_K^* coincide on C_K.*

(iv) *C_K as a subset of X is compact, weakly compact, or separable if and only if K has the same property.*

Proof. (i) Let $x \in K$, $x \neq 0$. Then we have

$$a^{n/2}x + a^{-n/2}\frac{x}{\|x\|} \in B_n,$$

so that

$$\|x\|_K^2 \leq \sum_{n=1}^{\infty} \frac{1}{(a^{n/2} + a^{-n/2}\|x\|^{-1})^2} = \|x\| \sum_{n=1}^{\infty} \frac{a^n\|x\|}{(a^n\|x\| + 1)^2}.$$

Let $h(t) = a^t\|x\|/(a^t\|x\| + 1)^2$, $1 \leq t < \infty$. The graph of h has a bell-shaped form and $\max h(t) = 1/4$. Let $k \in \mathbb{N}$ be such that

$$h(1) \leq h(2) \leq \cdots \leq h(k-1) \leq h(k) \geq h(k+1) \geq \cdots .$$

Then

$$\begin{aligned}
\frac{\|x\|_K^2}{\|x\|} &\leq \sum_{n=1}^{\infty} h(n) \leq h(k) + \int_1^{\infty} h(t)\,dt \\
&\leq \frac{1}{4} + \frac{1}{\ln a}\int_{1+a\|x\|}^{\infty} \frac{du}{u^2} \\
&= \frac{1}{4} + \frac{1}{\ln a}\left(\frac{1}{1+a\|x\|}\right) \leq \frac{1}{4} + \frac{1}{\ln a}.
\end{aligned}$$

(ii) For $x, y \in K$, we have $\frac{x-y}{2} \in K$. By (i),

$$\|x-y\|_K^2 \leq (\frac{1}{2} + \frac{2}{\ln a})\|x-y\|.$$

This together with (ii) in Lemma 2.2.1 gives (ii).

(iii) This is proved in [5].

(iv) This is essentially known (cf. [5] or [7, p. 228]) and follows from the inclusions $(f(a))^{-1}K \subset C_K \subset a^{n/2}K + a^{-n/2}B_X$, for all n, and from the fact that $C_K = \bigcap_{n=1}^{\infty}\{x \in X\colon \sum_{k=1}^{n} \|x\|_k^2 \leq 1\}$ is closed and weakly closed. □

Theorem 2.3.2. *Suppose $T \in \mathcal{L}(Y, X)$. Let $K = \overline{T(B_Y(0, 1/\|T\|))}$ and let $T_K\colon Y \to X_K$ be defined by $T_K y = Ty$, $y \in Y$. Then $T = J_K \circ T_K$ and*

(i) *T is separably valued, weakly compact, compact, or of finite rank if and only if T_K has the same property if and only if J_K has the same property.*

(ii) *$\|T\| = \|T_K\|$ and $\|J_K\| = 1$ whenever $f(a) = 1$.*

Proof. (i) We only need to prove that the above-mentioned properties of T imply the same properties for T_K and J_K. Since T_K is algebraically the same operator as T, they have the same rank and, by Lemma 2.3.1, (ii) and (iii),

T_K is separably valued, compact, or weakly compact whenever T is. If T is of finite rank, then J_K has finite rank since

$$J_K(B_{X_K}) = C_K \subset \bigcap_{n=1}^{\infty}(T(Y) + a^{-n/2}B_X) = \overline{T(Y)} = T(Y).$$

That the other properties of T imply the same properties for J_K, it is clear from Lemma 2.3.1, (iv).

(ii) If $f(a) = 1$, then $\|J_K\| \leq 1$ by Lemma 2.2.1, (ii). Without loss of generality, we may assume that $\|T\| = 1$. Since $K \subset C_K$ (cf. Lemma 2.2.1), (i), we get $\|T_K\| = \sup_{y \in B_Y} \|Ty\|_K \leq \sup_{z \in K} \|z\|_K \leq \sup_{z \in C_K} \|z\|_K = 1$. But then

$$1 = \|T\| = \|J_K \circ T_K\| \leq \|J_K\|\|T_K\| \leq \min\{\|T_K\|, \|J_K\|\}.$$

Therefore $\|T_K\| = \|J_K\| = 1$. □

By developing the method of proof of Theorem 2.3.2, we shall show (cf. Theorem 2.3.3 and Corollaries 2.3.4 and 2.3.5) that the isometric factorization can even be uniform with respect to finite dimensional subspaces in the space of weakly compact operators.

Theorem 2.3.3. *Let F be a finite dimensional subspace of $\mathcal{W}(Y, X)$. Then there exist a reflexive space Z, a norm one operator $J\colon Z \to X$, and a linear isometry $\Phi\colon F \to \mathcal{W}(Y, Z)$ such that $T = J \circ \Phi(T)$ for all $T \in F$. Moreover,*

(i) *$Z = X_K$ and $J = J_K$ for some $K \in \mathcal{W}_X$ whenever the number a is fixed so that $f(a) = 1$,*

(ii) *T is compact if and only if $\Phi(T)$ is compact,*

(iii) *T has finite rank if and only if $\Phi(T)$ has finite rank.*

Proof. Let $K = \overline{\mathrm{conv}}\{Ty\colon T \in B_F \text{ and } y \in B_Y\}$. Then K is a weakly closed absolutely convex subset of B_X. We shall use Grothendieck's lemma (cf. e.g. [7, p. 227]) to show that K is weakly compact. For given $\varepsilon > 0$, let $\{T_1, \ldots, T_n\}$ be an $\varepsilon/2$-net of B_F. Let K_ε be the closed convex hull of the weakly compact set $\overline{T_1(B_Y)} \cup \ldots \cup \overline{T_n(B_Y)}$. By the Krein-Šmulian theorem, K_ε is weakly compact. Since $K \subset K_\varepsilon + \varepsilon B_X$, the weak compactness of K follows from Grothendieck's lemma.

Choose a such that $f(a) = 1$. Put $Z = X_K$, $J = J_K$, and define $\Phi\colon F \to \mathcal{W}(Y, Z)$ by $\Phi(T)y = Ty$, $y \in Y$. Then Z is reflexive (since K is weakly compact), Φ is linear, and $T = J \circ \Phi(T)$ for all $T \in F$. As in the proof of Theorem 2.3.2, we show (i) and (ii), and we also obtain that $\|\Phi(T)\| = 1$, whenever $\|T\| = 1$, and that $\|J\| = 1$. □

Remark 2.3.1. The proof of Theorem 2.3.3 shows how norm compact sets in the space of weakly compact operators can be uniformly and isometrically factorized.

Corollary 2.3.4. *Let F be a finite dimensional subspace of $\mathcal{W}(X,Y)$. Then there exist a reflexive space Z, a norm one operator $J\colon X \to Z$, and a linear isometry $\Phi\colon F \to \mathcal{W}(Z,Y)$ such that $T = \Phi(T) \circ J$ for all $T \in F$. Moreover,*

(i) *T is compact if and only if $\Phi(T)$ is compact,*

(ii) *T has finite rank if and only if $\Phi(T)$ has finite rank.*

Proof. Let us consider the finite dimensional subspace $G = \{T^*\colon T \in F\}$ of $\mathcal{W}(Y^*, X^*)$. By Theorem 2.3.3, there exist a reflexive space Z, a norm one operator $I\colon Z^* \to X^*$, and a linear isometry $\Psi\colon G \to \mathcal{W}(Y^*, Z^*)$ so that $T^* = I \circ \Psi(T^*)$ for all $T \in F$. Put $J = I^*|_X$ and define $\Phi(T) = (\Psi(T^*))^*$ for $T \in F$. Since $T^{**}|_X = T$ whenever $T \in F$, we have $T = \Phi(T) \circ J$ and $\Phi(T) \in \mathcal{W}(Z,Y)$ for all $T \in F$. Moreover, $\|\Phi(T)\| = \|(\Psi(T^*))^*\| = \|\Psi(T^*)\| = \|T^*\| = \|T\|$ for $T \in F$. The linearity of Φ and properties (i) and (ii) are also clear from the definition of Φ. Finally, it is easily seen that $\|J\| = 1$. $\square$

Corollary 2.3.4 will be applied in the next section to prove that $\mathcal{F}(Y,X)$ is an ideal in $\mathcal{W}(Y,X)$ for all Banach spaces Y whenever X has the approximation property. We conclude this section with an immediate corollary from Theorem 2.3.3 and Corollary 2.3.4.

Corollary 2.3.5. *For every finite dimensional subspace F of $\mathcal{W}(X,Y)$, there exist reflexive spaces Z and W, norm one operators $J\colon X \to Z$ and $I\colon W \to Y$, and a linear isometry $\Phi\colon F \to \mathcal{W}(Z,W)$ such that $T = I \circ \Phi(T) \circ J$ for all $T \in F$.*

2.4 The approximation property and ideals of finite rank operators

In this section, our main objective is to prove that a Banach space X has the approximation property if and only if $\mathcal{F}(Y,X)$ is an ideal in $\mathcal{W}(Y,X)$ for all Banach spaces Y (see Theorem 2.4.3 below which also lists other criteria of the approximation property in terms of ideals of finite rank operators). In fact, we have already proved (see Theorem 2.2.2 and the proof of its implication (i)$\Rightarrow$(ii)) that X has the approximation property if and only if $\mathcal{F}(Y,X)$ is an ideal in $\mathcal{W}(Y,X)$ for all reflexive Banach spaces Y. The next result extends this assertion from reflexive spaces to all Banach spaces.

Theorem 2.4.1. *Let X be a Banach space. Then $\mathcal{F}(Y,X)$ (resp. $\mathcal{K}(Y,X)$) is an ideal in $\mathcal{W}(Y,X)$ for all Banach spaces Y if and only if $\mathcal{F}(Z,X)$ (resp. $\mathcal{K}(Z,X)$) is an ideal in $\mathcal{W}(Z,X)$ for all reflexive spaces Z.*

The proof of Theorem 2.4.1 will use the uniform isometric factorization of weakly compact operators from Section 3 and the following alternative characterization of ideals (proved e.g. in Lima [23], Fakhoury [11], and Kalton [22]).

Theorem 2.4.2. *Let F be a closed subspace of a Banach space E. The following statements are equivalent.*

(i) *F is an ideal in E.*

(ii) *F is locally 1-complemented in E, i.e. for every finite dimensional subspace G of E and for all $\varepsilon > 0$, there is an operator $A\colon G \to F$ such that $\|A\| < 1 + \varepsilon$ and $Ax = x$ for all $x \in G \cap F$.*

Remark 2.4.1. It is straightforward to verify that the condition $Ax = x$ for all $x \in G \cap F$ in Theorem 2.4.2 can be replaced by $\|Ax - x\| \leq \varepsilon$ for all $x \in B_{G\cap F}$.

Let us recall that, for a linear subspace F of a Banach space E (as it is clear from the definition of the ideal), F is an ideal in E if and only if $\overline{F}$ is an ideal in E.

Proof of Theorem 2.4.1. We shall first consider the case of ideals of compact operators. Let $\mathcal{K}(Z, X)$ be an ideal in $\mathcal{W}(Z, X)$ for all reflexive Banach spaces Z. For a Banach space Y, let G be a finite dimensional subspace of $\mathcal{W}(Y, X)$ and let $\varepsilon > 0$. By Corollary 2.3.4, we can find a reflexive space Z, a norm one operator $J\colon Y \to Z$, and an isometry Φ taking G into $\mathcal{W}(Z, X)$ and preserving compact operators such that $T = \Phi(T) \circ J$ for $T \in G$. By Theorem 2.4.2, there is an operator $A\colon \Phi(G) \to \mathcal{K}(Z, X)$ which "locally 1-complements" $\mathcal{K}(Z, X)$ in $\mathcal{W}(Z, X)$. Then $B\colon G \to \mathcal{K}(Y, X)$ defined by $B(T) = A(\Phi(T)) \circ J$, $T \in G$, "locally 1-complements" $\mathcal{K}(Y, X)$ in $\mathcal{W}(Y, X)$. This proves the claim about compact operators.

Now, if $\mathcal{F}(Z, X)$ is an ideal in $\mathcal{W}(Z, X)$ for all reflexive spaces Z, then, as we mentioned above, X has the approximation property. Consequently, $\overline{\mathcal{F}(Y, X)} = \mathcal{K}(Y, X)$ for all Banach spaces Y (cf. e.g. Remark 2.2.2). Therefore, by the first part of the proof, $\mathcal{F}(Y, X)$ is an ideal in $\mathcal{W}(Y, X)$ for all Banach spaces Y. □

Remark 2.4.2. The assertion of Theorem 2.4.1 concerning ideals of finite rank operators can also be proved similarly to the case of ideals of compact operators in Theorem 2.4.1, using that the isometry from Corollary 2.3.4 preserves finite rank operators. However, in this case, one should apply Remark 2.4.1 and notice that the condition from Remark 2.4.1 works also for subspaces F which are not necessarily closed.

In the next result, we summarize criteria of the approximation property expressed in termes of ideals of finite rank operators obtained in this paper and in the paper [25] by Lima and Oja.

Theorem 2.4.3. *Let X be a Banach space. The following statements are equivalent.*

(i) *X has the approximation property.*

(ii) *$\mathcal{F}(Y, X)$ is an ideal in $\mathcal{W}(Y, X)$ for all Banach spaces Y.*

(iii) *$\mathcal{F}(Y,X)$ is an ideal in $\mathcal{W}(Y,X)$ for all separable reflexive Banach spaces Y.*
(iv) *$\mathcal{F}(Y,X)$ is an ideal in $\mathcal{W}(Y,X)$ for all closed subspaces $Y \subset c_0$.*
(v) *$\mathcal{F}(Y,X)$ is an ideal in $\mathcal{K}(Y,X)$ for all Banach spaces Y.*
(vi) *$\mathcal{F}(Y,X)$ is an ideal in $\mathcal{K}(Y,X)$ for all separable reflexive Banach spaces Y.*
(vii) *$\mathcal{F}(Y,X)$ is an ideal in $\mathcal{K}(Y,X)$ for all closed subspaces $Y \subset c_0$.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) has just been proved above. The implications (vi) $\Rightarrow$ (i) and (vii) $\Rightarrow$ (i) are proved in [25, Theorem 2.6.1]. The other required implications (e.g. (ii) $\Rightarrow$ (v) $\Rightarrow$ (vi) & (vii)) are obvious. $\square$

In the paper [25] by Lima and Oja, it was proved that interchanging the roles of X and Y in statements (v), (vi), and (vii) of Theorem 2.4.3 gives conditions equivalent to the approximation property of X^*. This result will be used and extended in the following symmetric version of Theorem 2.4.3.

Theorem 2.4.4. *The following statements are equivalent:*
(i) *X^* has the approximation property.*
(ii) *$\mathcal{F}(X,Y)$ is an ideal in $\mathcal{W}(X,Y)$ for all Banach spaces Y.*
(iii) *$\mathcal{F}(X,Y)$ is an ideal in $\mathcal{W}(X,Y)$ for all separable reflexive Banach spaces Y.*
(iv) *$\mathcal{F}(X,Y)$ is an ideal in $\mathcal{W}(X,Y)$ for all closed subspaces $Y \subset c_0$.*
(v) *$\mathcal{F}(X,Y)$ is an ideal in $\mathcal{K}(X,Y)$ for all Banach spaces Y.*
(vi) *$\mathcal{F}(X,Y)$ is an ideal in $\mathcal{K}(X,Y)$ for all separable reflexive Banach spaces Y.*
(vii) *$\mathcal{F}(X,Y)$ is an ideal in $\mathcal{K}(X,Y)$ for all closed subspaces $Y \subset c_0$.*

Let us recall that, by a fundamental result due to Grothendieck [16] (cf. e.g. [26, p. 33]), X^* has the approximation property if and only if $\overline{\mathcal{F}(X,Y)} = \mathcal{K}(X,Y)$ for all Banach spaces Y.

In the proof of Theorem 2.4.4, we shall need the following symmetric version of Theorem 2.4.1.

Theorem 2.4.5. *Let X be a Banach space. Then $\mathcal{F}(X,Y)$ (resp. $\mathcal{K}(X,Y)$) is an ideal in $\mathcal{W}(X,Y)$ for all Banach spaces Y if and only if $\mathcal{F}(X,Z)$ (resp. $\mathcal{K}(X,Z)$) is an ideal in $\mathcal{W}(X,Z)$ for all reflexive Banach spaces Z.*

Proof. The case of compact operators can be proved as in Theorem 2.4.1 by applying Theorem 2.3.3 instead of Corollary 2.3.4.

Let $\mathcal{F}(X,Z)$ be an ideal in $\mathcal{W}(X,Z)$ for all reflexive spaces Z. Then, by the natural isometry $T \to T^*|_X$ between $\mathcal{W}(Z^*,X^*)$ and $\mathcal{W}(X,Z)$, we have that $\mathcal{F}(Y,X^*)$ is an ideal in $\mathcal{W}(Y,X^*)$ for all reflexive Banach spaces Y, meaning that X^* has the approximation property. Therefore, as we recalled above, $\overline{\mathcal{F}(X,Y)} = \mathcal{K}(X,Y)$ for all Banach spaces Y. And the already proved case of compact operators implies that $\mathcal{F}(X,Y)$ is an ideal in $\mathcal{W}(X,Y)$ for all Banach spaces Y. $\square$

Proof of Theorem 2.4.4. The equivalence (i) $\Leftrightarrow$ (ii) is clear from Theorem 2.4.5 and its proof. The implications (vi) $\Rightarrow$ (i) and (vii) $\Rightarrow$ (i) are proved in [25, Theorem 5.2], and the other required implications are obvious. □

2.5 The compact approximation property and ideals of compact operators

Replacing the finite rank operators by compact operators gives the definition of the compact approximation property: one says that a Banach space X has the **compact approximation property** (resp. the **metric compact approximation property**) if I_X belongs to the closure of $\mathcal{K}(X,X)$ (resp. $B_{\mathcal{K}(X,X)}$) with respect to the topology of uniform convergence on compact subsets in X. It is known that even the metric compact approximation property does not imply the approximation property [32].

By the previous section, X has the approximation property if and only if $\mathcal{F}(Y,X)$ is an ideal in $\mathcal{W}(Y,X)$ for all Banach spaces Y. We shall show that one can replace finite rank operators by compact operators in the "only if" part of this characterization (cf. Theorem 2.5.1), but one cannot do this in the "if" part (cf. the following example).

Example 2.5.1. There is a Banach space X without the compact approximation property such that $\mathcal{K}(Y,X) = \mathcal{W}(Y,X)$ (i.e. $\mathcal{K}(Y,X)$ is trivially an ideal in $\mathcal{W}(Y,X)$) for all Banach spaces Y.

Let X be a closed subspace of ℓ_1 without the compact approximation property (cf. [31] or e.g. [27, p. 107]). If $T \in \mathcal{W}(Y,X)$ for a Banach space Y, then by the Eberlein-Šmulian theorem and the Schur property of ℓ_1, it follows that T is compact.

Theorem 2.5.1. *Let X be a Banach space and let the number a be fixed so that $f(a) = 1$. The following assertions are equivalent and they hold whenever X has the compact approximation property.*

(a) *$\mathcal{K}(Y,X)$ is an ideal in $\mathcal{W}(Y,X)$ for all Banach spaces Y.*
(b) *$\mathcal{K}(Y,X)$ is an ideal in $\mathcal{W}(Y,X)$ for all separable reflexive Banach spaces Y.*
(c) *$\mathcal{K}(X_K,X)$ is an ideal in $\operatorname{span}(\mathcal{K}(X_K,X) \cup \{J_K\})$ for every $K \in \mathcal{W}_X$.*
(d) *For every Banach space Y and every $T \in \mathcal{W}(Y,X)$, there is a net (T_α) in $\mathcal{K}(Y,X)$ with $\sup_\alpha \|T_\alpha\| \le \|T\|$ such that $T_\alpha y \underset{\alpha}{\to} Ty$ for all $y \in Y$.*
(e) *For every separable reflexive Banach space Y and every $T \in \mathcal{W}(Y,X)$, there is a sequence (T_n) in $\mathcal{K}(Y,X)$ with $\sup_n \|T_n\| \le \|T\|$ such that $T_n y \underset{n}{\to} Ty$ for all $y \in Y$.*
(f) *For every $K \in \mathcal{W}_X$, there is a net (A_α) in $B_{\mathcal{K}(X_K,X)}$ such that $A_\alpha x \underset{\alpha}{\to} J_K x$ for all $x \in X_K$.*

Proof. The implications (a) ⇒ (b), (a) ⇒ (c), and (d) ⇒ (f) are obvious. The implications (c) ⇒ (f) and (b) ⇒ (e) are immediate from Lemma 2.2.4 (for (b) ⇒ (e), one should also use the standard argument from the proof of (iii) ⇒ (iv) in Corollary 2.2.5). The proofs of (e) ⇒ (f) and (f) ⇒ (d) are essentially the same as, respectively, the proofs of (iv) ⇒ (i) and (i) ⇒ (iii) in Corollary 2.2.5.

(f) ⇒ (a). We shall apply Theorem 2.4.2 together with Remark 2.4.1 to show that $\mathcal{K}(Y,X)$ is an ideal in $\mathcal{W}(Y,X)$. Let G be a finite dimensional subspace of $\mathcal{W}(Y,X)$ and let $\varepsilon > 0$. By Theorem 2.3.3, there exist $K \in \mathcal{W}_X$ and a linear isometry $\Phi\colon G \to \mathcal{W}(Y,X_K)$ preserving compact operators such that $T = J_K \circ \Phi(T)$ for all $T \in G$. Let a net (A_α) in $B_{\mathcal{K}(X_K,X)}$ satisfy $\|(A_\alpha - J_K)x\| \underset{\alpha}{\to} 0$ for all $x \in X_K$. Since $\{\Phi(T)y\colon T \in B_{G\cap\mathcal{K}(Y,X)}, y \in B_Y\}$ is a relatively compact subset of X_K, there is an α so that $\|(A_\alpha - J_K)\Phi(T)y\| \le \varepsilon$ for all $T \in B_{G\cap\mathcal{K}(Y,X)}$ and $y \in B_Y$. This means that $\|A_\alpha \circ \Phi(T) - T\| \le \varepsilon$ for all $T \in B_{G\cap\mathcal{K}(Y,X)}$. And denoting $A(T) = A_\alpha \circ \Phi(T)$, $T \in G$, we get an operator $A\colon G \to \mathcal{K}(Y,X)$ as desired.

Finally, let us assume that X has the compact approximation property. Replacing $\mathcal{F}(Y,X)$ by $\mathcal{K}(Y,X)$ and $\mathcal{F}(X,X)$ by $\mathcal{K}(X,X)$ in the proof of the implication (i) ⇒ (ii) of Theorem 2.2.2 shows that $\mathcal{K}(Y,X)$ is an ideal in $\mathcal{W}(Y,X)$ for any reflexive Banach space Y. □

Remark 2.5.1. Since X_K is reflexive whenever $K \in \mathcal{W}_X$, Theorem 2.4.1 immediately follows from Lemma 2.2.4 and the implication (f) ⇒ (a) of Theorem 2.5.1. However, the proof of Theorem 2.4.1 we gave in Section 4 is easier and more direct.

Remark 2.5.2. As we saw above, $\mathcal{F}(Y,X)$ is an ideal in $\mathcal{W}(Y,X)$ for all Banach spaces Y whenever there exists a number $a > 1$ so that $\mathcal{F}(X_K,X)$ is an ideal in $\mathcal{W}(X_K,X)$ for all $K \in \mathcal{W}_X$. By Theorem 2.5.1, (c)⇒(a), the similar assertion for compact operators holds for the number a for which $f(a) = 1$.

We say that a Banach space X has the **weakly compact approximation property** if I_X belongs to the closure of $\mathcal{W}(X,X)$ with respect to the topology of uniform convergence on compact subsets in X. This notion was considered by Reinov [30] and by Grønbæk and Willis [17]. Note that Astala and Tylli [2] use this notion when I_X belongs to the closure of $\mathcal{W}(X,X)$ with respect to the topology of uniform convergence on **weakly** compact subsets in X.

Corollary 2.5.2. *The assertions of Theorem 2.5.1 are equivalent to the compact approximation property of X whenever X has the weakly compact approximation property.*

Proof. Let $K \in \mathcal{K}_X$, let $\varepsilon > 0$, and choose $T \in \mathcal{W}(X,X)$ such that $\|Tx - x\| < \varepsilon/2$ for all $x \in K$. By assertion (d) of Theorem 2.5.1, there is a bounded net (T_α) in $\mathcal{K}(X,X)$ such that $T_\alpha x \to Tx$ for all $x \in X$. By compactness of K, $\sup_{x\in K} \|T_\alpha x - Tx\| \to 0$ and therefore $\sup_{x\in K} \|T_\alpha x - x\| < \varepsilon$ for some α. □

Remark 2.5.3. Corollary 2.5.2 applies, in particular, to Banach spaces X which are reflexive. However in this case, the assertions of Theorem 2.5.1 are equivalent to the metric compact approximation property of X and also to the fact that $\mathcal{K}(X,X)$ is an ideal in $\mathcal{W}(X,X)$ (cf. [23, Theorem 14]).

Corollary 2.5.3. *Let X be a Banach space and let the number a be fixed so that $f(a)=1$. The following assertions are equivalent and they hold whenever X^* has the compact approximation property.*

(a) *$\mathcal{K}(X,Y)$ is an ideal in $\mathcal{W}(X,Y)$ for all Banach spaces Y.*

(b) *$\mathcal{K}(X,Y)$ is an ideal in $\mathcal{W}(X,Y)$ for all separable reflexive Banach spaces Y.*

(c) *For every Banach space Y and every $T\in\mathcal{W}(X,Y)$, there is a net (T_α) in $\mathcal{K}(X,Y)$ with $\sup_\alpha\|T_\alpha\|\le\|T\|$ such that $T_\alpha^*y^*\underset{\alpha}{\to}T^*y^*$ for all $y^*\in Y^*$.*

(d) *For every separable reflexive Banach space Y and every $T\in\mathcal{W}(X,Y)$, there is a sequence (T_n) in $\mathcal{K}(X,Y)$ with $\sup_n\|T_n\|\le\|T\|$ such that $T_n^*y^*\underset{n}{\to}T^*y^*$ for all $y^*\in Y^*$.*

Proof. We shall use the natural isometry $T\to T^*|_X$ between $\mathcal{W}(Z^*,X^*)$ and $\mathcal{W}(X,Z)$ for reflexive Banach spaces Z. By this isometry, $\mathcal{K}(X,Y)$ is an ideal in $\mathcal{W}(X,Y)$ for all reflexive Banach spaces Y if and only if $\mathcal{K}(Y,X^*)$ is an ideal in $\mathcal{W}(Y,X^*)$ for all reflexive Banach spaces Y. Applying Theorems 2.4.5 and 2.4.1, this yields the equivalence of (a) to condition (a) of Theorem 2.5.1 for X^*. Furthermore, by the same isometry, (b) and (d) are respectively equivalent to conditions (b) and (e) of Theorem 2.5.1 for X^*, (c) implies condition (d) of Theorem 2.5.1 for X^* which, in its turn, implies the particular case of (c) where Y is assumed to be reflexive. Hence, by Theorem 2.5.1, (c)$\Rightarrow$(a)$\Leftrightarrow$(b)$\Leftrightarrow$(d), the last equivalent conditions hold whenever X^* has the compact approximation property, and they imply the particular case of (c) with reflexive Y. To finish the proof, we have to show that this particular case of (c) actually implies (c). Let Y be a Banach space and let $T\in\mathcal{W}(X,Y)$. Let K, Y_K, T_K, and J_K be as in Theorem 2.3.2. Since K is weakly compact, Y_K is reflexive. Hence, for $T_K\in\mathcal{W}(X,Y_K)$, there is a net (S_α) in $\mathcal{K}(X,Y_K)$ with $\sup_\alpha\|S_\alpha\|\le\|T_K\|=\|T\|$ such that $S_\alpha^*z^*\underset{\alpha}{\to}T_K^*z^*$ for all $z^*\in Y_K^*$. Since $\|J_K\|=1$, the net $T_\alpha=J_K\circ S_\alpha$ clearly satisfies what is needed. □

2.6 From approximation properties to metric approximation properties

We would like to demonstrate how the method of proof of Theorem 2.2.2 can be further developed to give alternative proofs for known results about cases when the (compact) approximation property implies the metric (compact) approximation property. (Note that the following results could have been obtained already in Section 2, but by their nature, they fit more properly to conclude this paper.)

The dual space X^* of a Banach space X is said to have the **compact approximation property with conjugate operators** if I_{X^*} belongs to the closure of $\{K^*\colon K \in \mathcal{K}(X,X)\}$ with respect to the topology of uniform convergence on compact subsets of X^*. By an example due to Grønbæk and Willis [17], the compact approximation property of X^* does not imply the compact approximation property with conjugate operators. Moreover, Casazza and Jarchow [3] have shown that there is a Banach space X failing the metric compact approximation property such that all its duals X^*, X^{**}, ... have the metric compact approximation property. Let us recall that if X^* has the approximation property, then X^* has the approximation property with conjugate operators (this is clear from the local reflexivity principle).

The following two results will explain surprisingly well why, in certain important cases, the (compact) approximation property implies the metric (compact) approximation property.

Theorem 2.6.1. *Let X and Y be Banach spaces such that Y^* or X^{**} has the Radon-Nikodým property. If X^* has the compact approximation property with conjugate operators, then $\mathcal{K}(X,Y)$ is an ideal in $\mathcal{L}(X,Y)$ with an ideal projection P such that*

$$P(x^{**} \otimes y^*) = x^{**} \otimes y^* \quad \textit{for all } y^* \in Y^* \textit{ and } x^{**} \in X^{**}.$$

Proof. We assume that (K_α^*) with $K_\alpha \in \mathcal{K}(X,X)$ converges to I_{X^*} uniformly on compact subsets of X^*. Similarly to the proof of Theorem 2.2.2, we can define an ideal projection P by

$$(Pf)(T) = \lim_\alpha f(TK_\alpha), \quad f \in \mathcal{L}(X,Y)^*, \quad T \in \mathcal{L}(X,Y). \qquad (*)$$

In particular, for $f = x^{**} \otimes y^*$ and $T \in \mathcal{L}(X,Y)$, this implies

$$(P(x^{**} \otimes y^*))(T) = \lim_\alpha x^{**}(K_\alpha^* T^* y^*) = x^{**}(T^* y^*) = (x^{**} \otimes y^*)(T).$$

□

Theorem 2.6.2. *Let X be a Banach space. The following statements are equivalent.*

(a) *X^* has the metric compact approximation property with conjugate operators.*

(b) *For all Banach spaces Y, $\mathcal{K}(X,Y)$ is an ideal in $\mathcal{L}(X,Y)$ with an ideal projection P such that*

$$P(x^{**} \otimes y^*) = x^{**} \otimes y^* \quad \textit{for all } y^* \in Y^* \textit{ and } x^{**} \in X^{**}.$$

(c) $\mathcal{K}(X,X)$ *is an ideal in* span $(\mathcal{K}(X,X) \cup \{I\})$ *with an ideal projection* P *such that*

$$P(x^{**} \otimes x^*) = x^{**} \otimes x^* \quad \textit{for all } x^* \in X^* \textit{ and } x^{**} \in X^{**}.$$

Proof. (a) $\Rightarrow$ (b). Let (K_α) be a net in $B_{\mathcal{K}(X,X)}$ such that $K_\alpha^* x^* \to x^*$ for all $x^* \in X^*$. Applying a well-known result due to J. Johnson [20], by passing to a subnet of (K_α), one can define an ideal projection P by $(*)$. As in the proof of Theorem 2.6.1, we have $P(x^{**} \otimes y^*) = x^{**} \otimes y^*$ for all $y^* \in Y^*$ and $x^{**} \in X^{**}$.

(b) $\Rightarrow$ (c). This is obvious.

(c) $\Rightarrow$ (a). By Lemma 2.2.4, there exists a net (K_α) in $B_{\mathcal{K}(X,X)}$ such that

$$x^{**}(K_\alpha^* x^*) \underset{\alpha}{\to} P(x^{**} \otimes x^*)(I_X) = (x^{**} \otimes x^*)(I_X) = x^{**}(x^*)$$

for all $x^* \in X^*$ and $x^{**} \in X^{**}$. Thus $K_\alpha^* \to I_{X^*}$ in the weak operator topology of $\mathcal{L}(X^*, X^*)$. Since the weak and strong operator topologies yield the same dual space, after passing to convex combinations, we may assume that $K_\alpha^* \to I_{X^*}$ in the strong operator topology. □

As an immediate corollary of Theorems 2.6.1 and 2.6.2, we obtain the following result due to Godefroy and Saphar [15].

Corollary 2.6.3 (cf. [15, Corollary 1.6]). *Let X be a Banach space such that X^* or X^{**} has the Radon-Nikodým property. If X^* has the compact approximation property with conjugate operators, then X^* has the metric compact approximation property with conjugate operators.*

Remark 2.6.1. The original proof of Corollary 2.6.3 due to Godefroy and Saphar [15] was also based, like ours, on Lemma 2.2.3, but by using the local reflexivity principle, it was modeled after Grothendieck's classical proof in [16]. Another proof of Corollary 2.6.3 (under the assumption that X^* has the Radon-Nikodým property) is given by Cho and Johnson [4] by an adaption of the alternative proof due to Lindenstrauss and Tzafriri [26, pp. 39-40].

The similar argument as in Theorem 2.6.2 yields the next result.

Theorem 2.6.4. *Let X be a Banach space. The following statements are equivalent.*

(a) *X has the metric compact approximation property.*

(b) *For all Banach spaces Y, $\mathcal{K}(Y,X)$ is an ideal in $\mathcal{L}(Y,X)$ with an ideal projection P such that*

$$P(y \otimes x^*) = y \otimes x^* \quad \textit{for all } x^* \in X^* \textit{ and } y \in Y.$$

(c) $\mathcal{K}(X,X)$ *is an ideal in* span $(\mathcal{K}(X,X) \cup \{I\})$ *with ideal projection* P *such that*

$$P(x \otimes x^*) = x \otimes x^* \quad \textit{for all } x^* \in X^* \textit{ and } x \in X.$$

The equivalence (a) $\Leftrightarrow$ (c) of Theorem 2.6.4 is contained in [10, Proposition 4].

An immediate corollary of Theorem 2.6.4 and Lemma 2.2.4 is the following result due to Lima [23].

Corollary 2.6.5 (cf. [23, Theorem 14]). *Let X be a Banach space with the Radon-Nikodým property. X has the metric compact approximation property if and only if $\mathcal{K}(X,X)$ is an ideal in* $\operatorname{span}(\mathcal{K}(X,X)\cup\{I\})$.

Theorems 2.6.2 and 2.6.4 remain valid for the metric approximation property if one replaces $\mathcal{K}(Y,X)$ by $\mathcal{F}(Y,X)$ and $\mathcal{K}(X,X)$ by $\mathcal{F}(X,X)$ (this is clear from the proofs). Therefore we have the following modifications of Corollaries 2.6.3 and 2.6.5.

Corollary 2.6.6 (cf. [8, p. 246]). *Let X be a Banach space such that X^* or X^{**} has the Radon-Nikodým property. If X^* has the approximation property, then X^* has the metric approximation property.*

Corollary 2.6.7 (cf. [23, Theorem 13]). *Let X be a Banach space with the Radon-Nikodým property. X has the metric approximation property if and only if $\mathcal{F}(X,X)$ is an ideal in* $\operatorname{span}(\mathcal{F}(X,X)\cup\{I\})$.

There are several important results on the (metric) approximation property for which it is not known whether or not they hold in the case of the (metric) compact approximation property. For instance, it is known, as we already mentioned above, that the (metric) approximation property for X^* implies the same for X. Casazza and Jarchow [3] have shown that this is not true for the metric compact approximation property, but it seems to be an open question whether or not this is true for the compact approximation property. It is not known whether Corollary 2.6.3 remains true if X^* has the compact approximation property (and not necessarily the compact approximation property with conjugate operators) (this question was posed by Godefroy and Saphar in [15]). It is known that the metric approximation property is separably determined: X has the metric approximation property whenever every separable subspace is contained in a separable subspace of X with the metric approximation property. In [28] (see also [29]), similar results were shown for the metric approximation property having some special geometric features (like unconditionality). We do not know whether these results hold for the metric compact approximation property.

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Chapter 3

Boundedness and surjectivity

3.1 Introduction

The following question is fundamental in the theory of linear operators: Given two topological vector spaces U and V. Suppose we are given a linear continuous operator $T : U \to V$ and suppose we can show that the range of T contains a certain set $A \subset V$. Are there properties (S) such that the following is true:

If A has property (S), then T must be onto

In one dimension the following is of course true: *Suppose $A \subset V$ contains one point different from the origin. Then $TU \supset A \Rightarrow TU = V$.* In arbitrary n-dimensional spaces V the theorem goes like this: *Suppose $A \subset V$ contains n independent vectors. Then $TU \supset A \Rightarrow TU = V$.*

Thus, the question is easy in finite dimensional spaces. When V does not have finitely many dimensions, the question is not easy. The problem is, naively spoken, that operators may very well have dense range without being onto. But from classical theorems we know something: For example, when V is a normed space and U is a Banach space, it was shown already by Banach in the twenties that

If $A \subset V$ is of second (Baire-)category in V, then $TU \supset A \Rightarrow TU = V$.

However, there are examples of "smaller" sets than second category sets which allow one to draw the conclusion that the operator is onto. An example of such a situation is provided by taking $V = \ell_\infty$ and A as the set of 0-1 sequences (this follows as a special case of Seever's theorem [3, p.17]).

Let us say that the set $A \subset V$ has the surjectivity property if "onto A implies onto V". Since, by linearity, also the symmetric, convex hull of A must be contained in the range of A, we may assume A to be symmetric and convex. On the other hand, the theorems are most useful when A is "small", as in the ℓ_∞ situation. Also, by passing to quotients, A has the surjectivity property if "onto A implies onto V for every injection".

Again, speaking naively and intuitively: By continuity, T maps every convergent net in U to a convergent net in V. So if TU is dense, but T is not onto, that must be because there are limits in V that can't be reached by the convergent nets which T produces in V, from nets in U.

When U and V are Banach spaces, a necessary and sufficient condition was given for a *bounded* set A to have the surjectivity property, by Kadets and Fonf in 1982 [14, Proposition 1].

Theorem 3.1.1 (Kadets-Fonf). *Let V be a Banach space and $A \subset S_V$. Then the following are equivalent statements:*

(a) *For any Banach space U and any bounded linear operator $T : U \to V$ such that $TU \supset A$, one has $TU = V$.*

(b) *For every representation of A as the union of an increasing sequence of sets, $A = \cup_{i=1}^{\infty} A_i$, $(A_i \uparrow)$, there is an index j such that*

$$\inf_{f \in S_{V^*}} \sup_{v \in A_j} |f(v)| > 0,$$

(i.e. A_j is a norming set for V^* (see Definition 3.2.1.))

We call the property in statement (b) in the Fonf-Kadets theorem "thickness". Here we show that boundedness is not needed in their theorem and we show that if (b) fails, then there is a Tauberian injection onto A, but not onto Y (see Theorem 3.4.2). The latter fact is obtained by using the Davis, Figiel, Johnson, and Pełczyński construction [1].

The main theorem, however, is that for Banach spaces, U and V, the surjectivity property is equivalent to a boundedness property. More precisely:

Main Theorem: *Let U and V be Banach spaces. Suppose $A \subset V$ has the surjectivity property and suppose we have a family of linear continuous operators from V into U which is pointwise bounded on A. Then this family is uniformly bounded (i.e. bounded in $\mathcal{L}(V, U)$). Moreover, if A does not have the surjectivity property, then there is a sequence of operators which is pointwise bounded on A, but not bounded as a subset of $\mathcal{L}(V, U)$).*

Thus, boundedness conclusions and surjectivity conclusions can be drawn from the common property, thickness.

Next we study the situation when U and V are duals of normed spaces, equipped with weak star topologies. Again surjectivity and boundedness are connected and this time to a weaker thickness property. Combined with a theorem of H. Shapiro, this will have as an immediate corollary the following new theorem in complex function theory: *Suppose T is a weak-star continuous linear operator from a dual Banach space U into $H^\infty(D)$ such that TU contains the Blaschke products. Then T is onto $H^\infty(D)$.*

We will return to the surjectivity property towards the end of this section. Let us now concentrate on boundedness. Recall the Banach-Steinhaus theorem for Banach spaces: *A family of linear continuous operators on a Banach space X, which is pointwise bounded on a set of second category, is bounded.*

Let V be a normed linear space. Motivated by the Banach-Steinhaus theorem we say that $A \subset V$ has the *boundedness property* if every family of linear continuous operators on V, which is pointwise bounded on A, is bounded. More generally, if U is a normed space and $\mathcal{A}$ is a subset of $\mathcal{L}(V, U)$, we say that A has the $\mathcal{A}$- *restricted boundedness property* if every family of linear continuous operators in $\mathcal{A}$, which is pointwise bounded on A, is bounded. In the latter definition, if $\mathcal{A}$ is the space of adjoint operators between duals, we use the term *w^*-boundedness property* .

From the Banach-Steinhaus theorem we conclude that every second category set A in a Banach space has the boundedness property. However, the Nikodým-Grothendieck boundedness theorem (see e.g. [3, p. 14] or [2, p. 80]) says, in our terminology, that the set of characteristic functions in the unit sphere of $B(\Sigma)$ has the boundedness property. This set is certainly not of the second category, it is even nowhere dense.

Let us have a look at a more recent theorem of J. Fernandez [5] (see also [23]), which is in the same spirit as the Nikodým-Grothendieck theorem.

Theorem 3.1.2 (Fernandez' theorem). *Suppose (f_n) is a sequence in $L_1(T)$ such that*

$$|\sup_n \int_T f_n \phi \, d\theta| < \infty$$

for every inner function ϕ. Then

$$|\sup_n \int_T f_n g \, d\theta| < \infty$$

for every $g \in H^\infty(D)$ (and hence (f_n) is bounded in the pre-dual of $H^\infty(D)$).

It is well known that the pre-dual of H^∞ is L_1/H_0^1. Thus, by Theorem 3.3.4, in our language, Fernandez' theorem says that the set of inner functions has the w^*-boundedness property in $H^\infty(D)$. In [5] and [6] the question whether the set of inner functions has the boundedness property was posed. In [12] it was shown that also the set of Blaschke-products has the w^*-boundedness property in $H^\infty(D)$. It was there also shown that the linear span of the Blaschke products is a first category set in $H^\infty(D)$.

Let us now give a more precise definition of the surjectivity property in normed spaces. We say that a set $A \subset X$ has the *surjectivity property* if, for every Banach space Y, every $T : Y \to X$ onto A is onto X. If the conclusion holds for a subset $\mathcal{A} \subset \mathcal{L}(X,Y)$, we say that A has the *$\mathcal{A}$-restricted surjectivity property* . A special case of this is the case where $\mathcal{A}$ is the space of adjoint operators between two duals. In this case we will say that A has the *w^*-surjectivity property* .

We have already mentioned Seever's theorem in a particular case. Let us state it in full generality. $B(\Sigma)$ denotes as usual the Banach space of bounded measurable functions on a σ-algebra Σ.

Theorem 3.1.3 (Seever's theorem). *Let U be a Banach space and let A be the subset of $B(\Sigma)$ consisting of the characteristic functions on Σ. If $T : U \to B(\Sigma)$ is such that $TU \supset A$, then T is onto.*

The following list is meant to sum up what we have discussed above:

(a) Second category sets have the boundedness property and the surjectivity property.

(b) The set of characteristic functions in $B(\Sigma)$ has the boundedness property and the surjectivity property, but is nowhere dense and its span is first category.

(c) The set of Blaschke products in $H^\infty(D)$ has the weak-star boundedness property and its span is first category.

(d) In Banach spaces, bounded sets have the surjectivity property if and only if they are not a countable increasing union of non-norming sets.

(e) In this paper it is shown that in Banach spaces the surjectivity property and the boundedness property are equivalent. The same is true in the weak-star case, and boundedness of A is not needed.

The Nikodým-Grothendieck theorem has interesting consequences and one might expect nice consequences of analogous theorems in other Banach spaces as well. In the following sections we will show different techniques to obtain such theorems.

Some open questions are posed at the end of the paper. One question which is not asked, but which seems relevant, is the following: In how general situations can theorems analogue to what we have found here in the normed case and in the weak-star case be proved? More specifically: What about operator spaces with pointwise topologies, or with the topology so important for the approximation property; the topology of uniform convergence on compact sets?

3.2 Some more preliminaries

One objective in this paper is to prove that in Banach spaces the (w^*-) surjectivity property and the (w^*-) boundedness property are equivalent to a common property, called (w^*-) thickness. Here are some concepts we will need:

Definition 3.2.1. *Let X be a normed space.*

(i) *A set $A \subset X$ such that*

$$\inf_{f \in S_{X^*}} \sup_{x \in A} |f(x)| \geq \delta,$$

for some $\delta > 0$, is called norming (for X^*).

(ii) *A set $A \subset X$ such that for all $\epsilon > 0$, there exists $0 < t < \infty$ such that $tA + \epsilon B_X \supset B_X$ is called* almost absorbing.

(iii) *A set $A \subset X$ such that there exist $0 < \lambda < 1$ and $0 < t < \infty$ such that $tA + \lambda B_X \supset B_X$ is called* λ-almost absorbing.

By the Hahn-Banach separation theorem the following lemma is easy to prove (see Remark 3.2.1 for the complex case).

Lemma 3.2.2. *The following statements are equivalent for a set A in a normed space X.*

(a) *A is norming for X^*.*

(b) *$\overline{co}(\pm A)$ is norming for X^*.*

(c) *There exists a $\delta > 0$ such that $\overline{co}(\pm A) \supseteq \delta B_X$.*

More specifically, we can speak about δ-norming sets, where the δ refers to the δ in (c).

Suppose a set $B \subset X^*$ is such that $\inf_{x \in S_X} \sup_{f \in B} |f(x)| \geq \delta$ for some $\delta > 0$. In this case we will call the set *norming for X* or *w^*-norming.* Of course we have a similar lemma for sets which are norming for X.

Lemma 3.2.3. *The following statements are equivalent for a set B in the dual X^* of a normed space X.*

(a) *B is norming for X.*

(b) *$\overline{co}(\pm B)$ is norming for X.*

(c) *$\overline{co}^{w^*}(\pm B)$ is norming for X.*

(d) *There exists a $\delta > 0$ such that $\overline{co}^{w^*}(\pm B) \supseteq \delta B_{X^*}$.*

Proposition 3.2.4. *Let X be a normed space. Consider the following statements:*

(a) *A is almost absorbing.*

(b) *A is λ-almost absorbing.*

(c) *A is norming.*

(d) *A is fundamental (i.e. $\overline{\mathrm{span}}\, A = X$.)*

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d). For convex sets (a) $\Leftrightarrow$ (b). (d) does not imply (c) even if A is closed, convex and symmetric.

Proof. (a) $\Rightarrow$ (b) is trivial.

(b) $\Rightarrow$ (c): If A is λ-absorbing, there exists λ, t such that $B_X \subset tA + \lambda B_X$. Then, of course $B_X \subset t \cdot \overline{co}(\pm A) + \lambda B_X$. Thus

$$B_X \subset t \cdot \overline{co}(\pm A) + \lambda B_X \subset t \cdot \overline{co}(\pm A) + \lambda(t \cdot \overline{co}(\pm A) + \lambda B_X) \subset \cdots$$

$$\subset t \cdot \overline{co}(\pm A) + \lambda t \cdot \overline{co}(\pm A) + \cdots + \lambda^n t \cdot \overline{co}(\pm A) + \lambda^{n+1} B_X.$$

Since $\overline{co}(\pm A)$ is convex we obtain

$$B_X \subset t \cdot \overline{co}(\pm A)(1 + \lambda + \cdots \lambda^n) + \lambda^{n+1} B_X.$$

But this is true for every n. Hence we can take limits:

$$B_X \subset t \cdot \overline{co}(\pm A) \frac{1}{1-\lambda},$$

which gives

$$\overline{co}(\pm A) \supset \frac{1-\lambda}{t} B_X,$$

and by Lemma 3.2.2, A is norming.

That (c) $\Rightarrow$ (d) is trivial.

We now prove that (b) implies (a) for convex sets. Let $\epsilon > 0$. Since A is λ-almost absorbing and convex

$$B_X \subset tA + \lambda B_X \subset tA + \lambda(tA + \lambda B_X) = t(1+\lambda)A + \lambda^2 B_X$$

$$\subset \cdots \subset (1 + \lambda + \lambda^2 + \cdots + \lambda^{n-1})tA + \lambda^n B_X,$$

for every natural number n. Since $\lambda < 1$, λ^n is eventually less than ϵ.

To see that (d) does not imply (c), let

$$A = \overline{co}(\pm \frac{e_n}{n}) \subset l_1.$$

It is easy to check that A has the desired properties. $\square$

Later we will need the now well-known construction of Davis, Figiel, Johnson and Pełzcynski. See [1] or [2, p. 227-228], or [15] for some recent results. We call the space constructed by this procedure from a bounded, absolutely convex set K, the DFJP-space constructed on K.

Proposition 3.2.5. *Let X be a Banach space and let $K \subset X$ be bounded, convex and symmetric. Let X_K be the DFJP-space constructed on K. Then the natural embedding J_K of X_K into X is an isomorphism if and only if K is norming for X^*.*

Proof. If K is norming for X, $\overline{J_K(K)} \supset \lambda \cdot B_X$ for some $\lambda > 0$. Thus $\overline{J_K(B_{X_K})} \supset \lambda \cdot B_X$ and J_K is invertible.

If X_K and X are isomorphic by J_K then, for some $\delta > 0$,

$$\delta B_X \subset C \stackrel{\text{def}}{=} J_K(B_{X_K}).$$

But, by the construction of X_K (see [15]),

$$C \subset a^n K + a^{-n} B_X$$

for all n. We use that $\delta B_X \subset C$ inductively for a constant n:

$$a^n K + a^{-n} B_X \subset a^n K + \frac{a^{-n}}{\delta}(a^n K + a^{-n} B_X)$$

$$\subset (a^n + \frac{1}{\delta})K + \frac{a^{-2n}}{\delta} B_X \subset (a^n + \frac{1}{\delta})K + \frac{a^{-2n}}{\delta^2}(a^n K + a^{-n} B_X)$$

$$\subset (a^n + \frac{1}{\delta} + \frac{a^{-n}}{\delta^2})K + \frac{a^{-3n}}{\delta^2} B_X$$

Continuing this way gives after r steps

$$a^n K + a^{-n} B_X \subset a^n K + \left\{ \frac{1}{\delta} \sum_{k=0}^{r-1} \frac{1}{(\delta \cdot a^n)^k} \right\} K + (\frac{1}{\delta a^n})^r \cdot a^{-n} B_X.$$

This is true for any n. Now choose N so big that $\delta a^N > 2$. Then,

$$B_X \subset \frac{2(1 - (\frac{1}{2})^r) 2^N}{\delta} K + \frac{1}{2^r} B_X.$$

Now, by letting $r \to \infty$, we obtain

$$\frac{\delta}{2a^N} B_X \subset \overline{K}.$$

This proves that K is norming. $\square$

Remark 3.2.1. When the space under consideration is complex, a norming set A is a set such that *for some* $\delta > 0$, $\overline{co}(\cup_{|r|=1} rA) \supset \delta B_X$. It is easy to verify that all the results so far are true with complex scalars instead of real scalars.

Remark 3.2.2. Note that a set A has the surjectivity property (boundedness property) if and only if $co(\cup_{|r|=1} rA)$ has the surjectivity property (boundedness property).

Fundamental sets are useful when testing for weak-star and weak convergence of nets. The following Proposition is well-known and classic.

Proposition 3.2.6. *Let X be a normed space.*

(a) *Suppose a* bounded *net in X^* converges pointwise on a fundamental set $A \subset X$. Then it converges weak-star.*

(b) *Suppose a* bounded *net in X converges pointwise on a fundamental set $A \subset X^*$. Then it converges weakly.*

Remark 3.2.3. The Rainwater-Simons theorem [24, e.g.] "is trivial" (and works also for bounded nets) whenever the James boundary under consideration is fundamental in X^* (e.g. when X^* has the RNP). If we have a sequence, an interesting question is to give conditions on the fundamental set such that boundedness automatically follows from the pointwise convergence of the sequence.

Example 3.2.1. A famous theorem of Marshall says: *$H^\infty(T)$ is the closed, linear span of the Blaschke products.* In other words, the Blaschke products form a fundamental subset of H^∞. Later it has been shown that the Blaschke products form a subset of H^∞ which is 1-norming for the dual (see [11, p. 195-197]). The most recent theorem in this direction is, as far as I know, the result from [26] saying that the interpolating Blaschke products form a set which is 10^{-7}-norming for the dual.

We now define the terms "thick" and "thin":

Definition 3.2.7. *A set is called (w^*)-* thin *if it can be written as a countable increasing union of (w^*)-non-norming sets. A set which is not (w^*-) thin is called* (w^*-) thick.

This classification of sets is not standard and the terms thick and thin sets are often used to describe properties of sets. Maybe it would be better to call thick sets *Fonf sets* since I think he is the one who first and best demonstrated the relevance and importance of the thick sets. Fonf however never uses the word thick, but in his works he always operates with "thin" and "not thin" sets. See e.g. [14], [8] and [7] for examples of earlier use and applications of these concepts.

Example 3.2.2. To get an idea of these properties, one can think of the extreme points A of the unit ball in l_1. This is a countable, hence thin (and thus w^*-thin) set. B_{l_1} is the norm-closure of the convex hull of its extreme points $A = ext\, B_{l_1} = \{\pm e_i\}_{i=1}^\infty$ so it is 1-norming. Let $f_n = ne_n$. Then $\{f_n\} \subset c_0$ is pointwise bounded on A, but obviously not bounded as a subset of c_0. So this A does not have the w^*-boundedness property. Now define an operator $T \in L(l_\infty, l_1)$ by $T(x_1, x_2, ...) = (2^{-n}x_n)_{n=1}^\infty$. Then T is onto A, but since it is injective, it can't be onto l_1. Thus A does not have the surjectivity property. Since T is the adjoint of $S : c_0 \to l_1$ defined by $S(x_1, x_2, ...) = (2^{-n}x_n)_{n=1}^\infty$, A does not have the w^*-surjectivity property either.

We want to show that the simple Example 3.2.2 is just a special case of a very general principle in Banach spaces.

3.3 The boundedness property in normed spaces

We recall the definition of the boundedness property.

Definition 3.3.1. *A subset A of a normed linear space X is said to have the boundedness property if for every normed space Y, every family $(T_\alpha) \subset \mathcal{L}(X, Y)$, which is pointwise bounded on A, is bounded.*

A special variant of the following theorem was first published in [18]. Also parts of it are implicit in [7].

Theorem 3.3.2. *Suppose A is a subset of a normed space X. The following statements are equivalent.*

(a) *A has the boundedness property.*

(b) *Every sequence $(f_n) \subset X^*$ which is pointwise bounded on A is a bounded sequence in X^*.*

(c) *A is thick.*

Proof. (a) clearly implies (b). To prove that (b) implies (c) suppose A is thin. Then we can pick a countable, increasing covering, $\cup A_n$, of A, consisting of sets which are non-norming for X^*. Thus, we can find a sequence $(f_n) \subset X^*$ such that $f_n \in nS_{X^*}$ but $\sup_{A_n} |f_n(x)| < 1$. Let x be an arbitrary element of A. Then there is a natural number m such that $x \in A_m$. Thus, since (A_n) is increasing,

$$|f_k(x)| \leq ||f_k||\,||x|| < m||x|| \qquad \text{if } k < m,$$

while

$$|f_k(x)| < 1 \qquad \text{if } k \geq m.$$

This proves that (b) implies (c).

To show that (c) implies (a), suppose A is thick and (T_α) is pointwise bounded on A, i.e.

$$\sup_\alpha \|T_\alpha x\| < \infty \qquad \text{for every } x \in A.$$

Put $A_m = \{x \in A \ : \sup_\alpha \|T_\alpha x\| \leq m\}$. Then (A_m) is an increasing family of sets which covers A. Since A is thick, some A_q is norming. Then, using Lemma 3.2.2, there exists a $\delta > 0$ such that

$$\overline{co}(\pm A_q) \supseteq \delta B_X.$$

But then, for arbitrary α,

$$\delta\|T_\alpha\| = \sup_{x\in\delta S_X} \|T_\alpha x\| \leq \sup_{x\in\overline{co}(\pm A_q)} \|T_\alpha x\| \leq q.$$

Thus $\sup_\alpha \|T_\alpha\| \leq q/\delta < \infty$, and the theorem is proved. □

Remark 3.3.1. Note how simple and general Theorem 3.3.2 is. It gives an interesting perspective on the classical Banach-Steinhaus theorem for Banach spaces, as soon as we have shown that sets of second category are thick. This is done in the following easy Lemma.

Lemma 3.3.3. *Let X be a normed space and suppose A is a second category set in X. Then A is thick.*

Proof. Suppose A is covered by an increasing family (A_i). Since A is of second category, some $\overline{A_m}$ contains a ball. Then $\overline{co}(\pm A_m)$ contains a ball centered at the origin, and hence A_m is norming. Since (A_i) is arbitrary, A must be thick. □

We also have a "uniform boundedness theorem" characterizing w^*-thickness.

Theorem 3.3.4. *Let X be a normed space and suppose B is a subset of X^*. Then the following statements are equivalent.*

(a) *B has the w^*-boundedness property.*
(b) *Every sequence $(x_n) \subset X$ which is pointwise bounded on B is a bounded sequence in X.*
(c) *B is w^*-thick.*

Proof. (a) of course implies (b). The proof that (b) implies (c) is completely analogous to the corresponding part of the proof of Theorem 3.3.2. The proof that (c) implies (a) is also similar to the corresponding part of the proof of Theorem 3.3.2. Just put

$$B_m = \{x^* \in B \ : \sup_{\alpha} \|T_\alpha^* x^*\| \leq m\},$$

and use the w^*-continuity of T_α^*. □

Remark 3.3.2. The set of extreme points of the unit ball of l_1 shows that a set can be norming for the dual without being w^*-thick. It is also possible for a set to be w^*-thick without being norming for the dual (although it is of course norming for the pre dual). In fact, the unit ball of any non-reflexive space, considered as a subset of the bidual, give examples of such situations (they are not even fundamental).

Corollary 3.3.5. *The Blaschke products in $H^\infty(D)$ is w^*-thick and 1-norming for the dual.*

Proof. By the main result in [12], the set of Blaschke products satisfies (b) of Theorem 3.3.4. The set is 1-norming by [11, Corollary 2.6 p. 196]. □

3.4 The surjectivity property in Banach spaces

In this section we study the surjectivity property in Banach spaces. Let us start with the formal definition:

Definition 3.4.1. *In a normed linear space X a set A is said to have the surjectivity property if for every normed linear space Y, every $T \in \mathcal{L}(Y, X)$, such that $TY \supset A$, is onto X. If A in a normed space has the surjectivity property for all T's coming from Banach spaces, we say that A has the surjectivity property for Banach spaces.*

Recall that an operator $T : Y \to X$ is called Tauberian if $(T^{**})^{-1}(X) \subset Y$. As an intuition, it is often helpful to think of these operators as opposite to weakly compact operators. A nice reference for the theory of Tauberian operators is [10].

The next theorem shows the connection between thickness and the surjectivity property. The theorem is a generalization of Theorem 3.1.1, discovered by Kadets and Fonf [14].

Theorem 3.4.2. *Suppose A is a subset of a Banach space X. The following statements are equivalent.*

(a) *A has the surjectivity property for Banach spaces.*

(b) *For every Banach space Y, every injection $T : Y \to X$, which is onto A, is an isomorphism.*

(c) *For every Banach space Y, every Tauberian injection $T : Y \to X$, which is onto A, is an isomorphism.*

(d) *A is thick*

Proof. Of course (a) implies (b) and (b) implies (c).

To show that (c) implies (d) suppose (d) is not true, i.e. A is thin. We will construct a Tauberian injection which is onto A but not onto all of X. Let (A_i) be an increasing family of non-norming subsets of A such that $A = \cup_{i=1}^{\infty} A_i$. Since $\cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty} A_i \cap i \cdot B_X$ and $A_i \cap i \cdot B_X$ is non-norming, we may assume each A_i to be contained in $i \cdot B_X$. Put $C_1 = A_1$ and $C_i = A_i \setminus A_{i-1}$. Define

$$C = \overline{co}(\pm \bigcup_{i=1}^{\infty} \frac{C_i}{i^2}).$$

Then C is closed, bounded, convex and symmetric. We now show that C is non-norming for X^*. To do this, let $\epsilon > 0$ and take j such that $1/j < \epsilon$. Since A_j is not a norming set, there is a functional $f \in S_{X^*}$ such that $\sup_{x \in A_j} |f(x)| < \epsilon$. By the definition of C,

$$\sup_{x \in C} |f(x)| = \sup_i \left\{ \frac{1}{i^2} \sup_{x \in C_i} |f(x)| \right\}.$$

Take an arbitrary $y \in C$. Then either $y \in A_i$, $i \leq j$ or $y \in A_i$, $i > j$. In the first case

$$|f(y)| \leq \sup_{x \in A_j} |f(x)| < \epsilon.$$

In the second case, since $C_i \subset i \cdot B_X$,

$$|f(y)| \leq \sup_{i>j} \left\{ \frac{1}{i^2} \sup_{x \in C_i} |f(x)| \right\} \leq \sup_{i>j} \left\{ \frac{1}{i^2} \cdot i \right\} \leq \frac{1}{j} < \epsilon.$$

Thus

$$\sup_{x \in C} |f(x)| < \epsilon,$$

and C is not norming for X^*.

Hence, by Proposition 3.2.5, the Davis-Figiel-Johnson-Pelzcynski construction on C will produce a Banach space Y and an operator $J : Y \to X$ with the desired properties, i.e. it is injective, Tauberian, onto A but not onto all of X.

It remains to show that (d) implies (a). To do this, let T be any bounded, linear operator, from a Banach space Y, into X and onto A. Put $A_i = T(i \cdot B_Y) \cap A$, where Y is the domain space of T. Since T is onto A, $(A_i)_{i=1}^{\infty}$, is an increasing covering of A. Since A is thick some A_j is norming for X^*. By Lemma 3.2.2, there exists a $\delta > 0$ such that

$$j \cdot \overline{co}(\pm T B_Y) = j \cdot \overline{T B_Y} \supset \delta B_X.$$

Hence $\overline{T B_Y} \supset (\delta/j) \cdot B_X$ and, by e.g. [19, Thm 4.13], T is onto. □

We now combine Theorem 3.3.2 and Theorem 3.4.2 to obtain our main result.

Corollary 3.4.3. *In Banach spaces, the surjectivity property for Banach spaces and the boundedness property are both equivalent to thickness.*

Remark 3.4.1. By the Nikodým-Grothendieck boundedness theorem the characteristic functions is a thick set in $B(\Sigma)$. Thus Seever's theorem (see Theorem 3.1.3) follows as a special variant of the very general Theorem 3.4.2.

Remark 3.4.2. The DFJP-embedding J is "a little more" than Tauberian because J^{**} is Tauberian. In [15] an easy argument is given to show that, in fact, J is a norm-norm homeomorphism when restricted to the set on which the DFJP-space is constructed.

It is interesting to compare Theorem 3.4.2 with the following observation of R. Neidinger [17, p.119]. Its proof is given by a close inspection of a standard proof of the open mapping theorem, see e.g. [19, p.48]. An interesting application can be found in [16].

Lemma 3.4.4 (R. Neidinger). *Let X and Y be Banach spaces and let $T \in \mathcal{L}(X, Y)$. Then the following statements are equivalent.*

(a) *TB_X almost λ-absorbs B_Y.*

(b) *TB_X almost absorbs B_Y.*

(c) *TB_X absorbs B_Y.*

(d) *T is onto.*

An analogue to Lemma 3.4.4, using the terms thick set and norming set is the following:

Lemma 3.4.5. *Let X and Y be Banach spaces and let $T \in \mathcal{L}(X, Y)$. Then the following statements are equivalent.*

(a) *TB_X is norming for Y^*.*

(b) *TB_X is thick in Y.*

(c) *T is onto.*

Note that, by Proposition 3.2.4, (a) and (b) of Lemma 3.4.4 is equivalent to (a) of Lemma 3.4.5.

Proof. Only the implication (a) $\Rightarrow$ (b) needs proof. TB_X is norming, so there is a $\delta_1 > 0$ such that $\overline{TB_X} \supset \delta_1 B_Y$. Suppose TB_X has been written as an increasing countable union, $TB_X = \cup_{i=1}^{\infty} A_i$. Then $B_X = \cup_{i=1}^{\infty}(T^{-1}(A_i) \cap B_X) = \cup_{i=1}^{\infty} B_i$, an increasing union. Since B_X is thick, there exists a number m and a $\delta_2 > 0$ such that $\overline{co}(\pm B_m) \supset \delta_2 B_X$. Thus

$$\overline{co}(\pm A_m) = \overline{co}(\pm TB_m) \supset T\overline{co}(\pm B_m) \supset \delta_2 \overline{TB_X} \supset \delta_1 \delta_2 B_Y.$$

□

Let us now consider w^*-thick sets. Here is a characterization of such sets in terms of surjectivity properties:

Theorem 3.4.6. *Suppose B is a subset of a dual space X^*. Then the following statements are equivalent.*

(a) *B has the surjectivity property for all dual operators into X^*, i.e. B has the w^*- surjectivity property.*

(b) *B has the surjectivity property for all dual injections into X^*.*

(c) *B is w^*-thick.*

Proof. That (a) implies (b) is trivial. To show that (b) implies (c) we make necessary adjustments in the corresponding proof of Theorem 3.4.2. First substitute A's with B's. Then define C by w^*-closure. Note that C is now a non-w^*-norming set. Define Y to be the w^*-closure of *span* C.

If $Y \neq X^*$ let T be the embedding of Y into X^*. Then, since Y is w^*-closed, $Y = (X/M)^*$, where M is the annihilator of Y in X. Moreover, T is the adjoint of the quotient map $q : X \to X/M$.

If $Y = X^*$ the set C can be used to define a new norm $\|\cdot\|_C$ on X by the formula

$$\|x\|_C = \sup_{c \in C} |c(x)|.$$

Then, by the definition of C, $\|\cdot\|_C$ is strictly weaker than the original norm. Let E be the completion of X in this weaker norm, let j be the embedding of X into E. The adjoint T of j is then continuous and injective (since j is continuous and dense). Moreover, T is by definition onto C and hence onto B. Thus (b) implies (c).

To show that (c) implies (a), mimicking the corresponding proof of Theorem 3.4.2 gives the existence of a natural number j such that

$$\overline{(TB_Y)}^{w^*} \supset \frac{\delta}{j} B_{X^*}.$$

Now we use that T is an adjoint operator. This gives us that the set TB_Y is w^*-compact, and hence

$$TB_Y \supset \frac{\delta}{j} B_{X^*},$$

which concludes the proof. □

The following corollary is known from [7].

Corollary 3.4.7. *Suppose B is a bounded subset of a dual space X^*. Then the following statements are equivalent.*

(a) *There exists a Banach space Y and an injection $T : X \to Y$ such that T^* is injective and $T^*Y^* \supset B$, but T is not invertible.*

(b) *B is w^*-thin.*

Corollary 3.4.8. *In the Banach space setting, the w^*- surjectivity property and the w^*- boundedness property are both equivalent to w^*- thickness.*

By the Fernandez-Hui-Shapiro theorems (see Theorem 3.1.2 and the comments after it), the set of Blaschke-products is a w^*- thick, 1-norming subset of H^∞. Thus, if T is an adjoint operator from a dual Banach space into H^∞ which is onto the set of Blaschke products, then T is onto X. We state this in an alternative form:

Theorem 3.4.9. *Let X be an arbitrary Banach space. Let B denote the set of Blaschke-products in H^∞. Suppose $S \in \mathcal{L}(L_1/H_0^1, X)$ is such that $S^*(X^*) \supset B$. Then X contains L_1/H_0^1 as a closed subspace.*

Proof. Since S^* is dense, S is 1-1. Since B is weak-star thick S^* is onto. Thus S has closed range. □

3.5 The Seever property and the Nikodým property

In [3, Example 5 p.18] an example is given to show that the Nikodým-Grothendieck boundedness theorem may fail when the measures are not defined on a σ- algebra, but just on an algebra.

In [21] five properties for algebras $\mathcal{A}$ of sets are discussed. They are as follows:

(i) $\mathcal{A}$ has the Vitali-Hahn-Saks property (VHS) if the Vitali-Hahn-Saks theorem holds on $\mathcal{A}$.

(ii) $\mathcal{A}$ has the Nikodým property (N) if the Nikodým-Grothendieck boundedness theorem holds on $\mathcal{A}$.

(iii) $\mathcal{A}$ has the Orlicz-Pettis property (OP) if, for every Banach space X, weak countably additivite X-valued measures are countably additive.

(iv) $\mathcal{A}$ has the Grothendieck property (G) if $B(\mathcal{A})$ is a Grothendieck space, i.e. if every weak-star convergent sequence in the dual is weakly convergent.

(v) $\mathcal{A}$ has the Rosenthal property (R) if $B(\mathcal{A})$ is a Rosenthal space, i.e. if every continuous, non-weakly compact, linear operator into a Banach space X fixes a copy of ℓ_∞.

A σ-algebra has all these properties. It is shown in different papers (see [21] and [4] for references) that (VHS) $\Leftrightarrow$ (N) and (G), that (G) $\Rightarrow$ (OP), that (R) $\Rightarrow$ (G) and that no other implications hold. For some time it was open whether (G) alone might imply (N). A counterexample was given by M. Talagrand in [25].

Let us say that an algebra has the Seever property (S) if Seever's theorem works on $\mathcal{A}$.

Theorem 3.5.1. *The following statements about an algebra $\mathcal{A}$ are equivalent:*

(a) *$\mathcal{A}$ has the Nikodým property (N)*

(b) *$\mathcal{A}$ has the Seever property (S)*

(c) *The set $\{\chi_A : A \in \mathcal{A}\}$ is thick*

3.6 Some results on thickness in $\mathcal{L}(X, Y)^*$

Let X and Y be Banach spaces. A very useful set in $\mathcal{L}(X, Y)^*$ is the tensor product $X^{**} \otimes Y^*$. Recall that the action of a functional $x^{**} \otimes y^*$ on an operator $T \in \mathcal{L}(X, Y)$ is defined by $(x^{**} \otimes y^*)(T) = x^{**}(T^* y^*)$. In [13, Lemma 1.7b p. 268] it is shown that $extB_{X^{**}} \otimes extB_{Y^*}$ is 1-norming for $L(X, Y)$. It need not be w^*- thick. But often it is.

Lemma 3.6.1. *Suppose A and B are w^*- thick subsets of X^{**} and Y^* respectively. Then $A \otimes B$ is a w^*- thick subset of $\mathcal{L}(X, Y)^*$.*

Proof. We will use Theorem 3.3.4. Let (T_n) be a sequence in $L(X,Y)$ such that

$$\sup_n |x^{**} \otimes y^*(T_n)| = \sup_n |x^{**}(T_n^* y^*)| < \infty$$

for all $x^{**} \in A$ and all $y^* \in B$. Since A is w^*- thick we conclude that

$$\sup_n \|(T_n^* y^*)\| < \infty$$

for all $y^* \in B$. Since B is w^*- thick,

$$\sup_n \|T_n^*\| < \infty$$

and the result follows. □

Since, by definition, no countable set can be w^*-thick, the extreme points of B_{l_1} is a w^*-thin set. This is in fact a special case of a rather difficult theorem discovered by V.P. Fonf. Recall that a James boundary J for X is a subset of X^* such that every $x \in X$ attains its norm on J. As an example, the set of extreme points of the dual unit ball is always a James boundary for X.

Theorem 3.6.2. *If a Banach space X admits a w^*-thin James boundary J, then X contains a copy of c_0.*

We only present a list to show how the theorem can be proved with help of different papers. The list points to the simplest proof known to the author.

Proof.
(a) Note that the restriction of a James boundary to a subspace Y is a James boundary for Y.
(b) Put $J = \cup_n A_n$. By Simons' generalization of the Rainwater lemma [24], there is a sequence (x_n) on S_X which converges weakly to 0. By the Bessaga-Pełzcynski selection principle (see e.g. [2, p.42]) (x_n) can be assumed to be a basic sequence. Let $Y = [x_n]$. We look for c_0 inside Y.
(c) Let T be the natural embedding of Y into X. Put $B_n = T^*(A_n)$. Then show that $J' = \cup B_n$ is a James boundary for Y.
(d) Show that each B_n is relatively norm-compact as done on page 489 in [8]. Thus Y has a σ-compact James boundary J'.
(e) Use Lemma 27 in [9] to renorm Y equivalently to have a countable James boundary J''.
(f) Follow the proof of [9, Theorem 23] to construct a copy of c_0 inside a once more equivalently renormed version of Y. This copy is also a copy in X.

□

An interesting result follows from Lemma 3.6.1 and Theorem 3.6.2:

Corollary 3.6.3. *Suppose X^* and Y does not contain a copy of c_0. Then the set $E = ext\, B_{X^{**}} \otimes ext\, B_{Y^*}$ is w^*-thick in $L(X,Y)^*$.*

Proof. Since neither X^* nor Y contains a copy of c_0, the sets $ext\, B_{X^{**}}$ and $ext\, B_{Y^*}$ are both w^*-thick. Hence, by Lemma 3.6.1, E is w^*-thick. $\square$

Remark 3.6.1. Note that the set E is not necessarily a James boundary for $L(X,Y)$. But being "identical" to the set $ext B_{K(X,Y)^*}$ it is a James boundary for $K(X,Y)$.

By combining the main result from [7] with the knowledge of the exposed points of the dual unit ball of $K(X,Y)$ (see e.g. [20, Theorem 5.1]), we obtain the following theorem on w^*-thickness of $exp\, B_{K(X,Y)^*}$.

Theorem 3.6.4. *Suppose X^* and Y are separable and Y does not contain a copy of c_0. Then $exp\, B_{K(X,Y)^*}$ is w^*-thick.*

Proof. Since X^* is a separable dual it has the RNP and thus does not contain a copy of c_0. By the main result from [7], the sets $A = exp\, B_{X^{**}}$ and $B = exp\, B_{Y^*}$ are both w^*-thick. Hence, by Lemma 3.6.1 $A \otimes B$ is w^*-thick. But, by [20], $A \otimes B$ is exactly the set of exposed points of $B_{K(X,Y)^*}$. $\square$

Remark 3.6.2. When X^* and Y both are separable we obtain that $K(X,Y)$ is separable. However, $K(X,Y)$ may very well contain c_0 even though X^* and Y doesn't. For example, the space $K(l_2)$ contains a copy of c_0.

Corollary 3.6.5. *Suppose X and Y are separable, reflexive spaces. Then $w^* - exp\, B_{K(X,Y)^*}$ is w^*-thick.*

Proof. The result follows since $w^* - exp\, B_{K(X,Y)^*} = w^* - exp\, B_X \otimes w^* - exp\, B_{Y^*}$. $\square$

Corollary 3.6.6. *Suppose X and Y are separable, reflexive spaces. Then every James boundary of $K(X,Y)$ is w^*-thick.*

Proof. Every James boundary must contain the w^*-exposed points of B_{X^*}. $\square$

3.7 Some questions and remarks

Suppose a Banach space contains a thin, fundamental set. Then, by the definition of such a set, there exists a w^*- null sequence (x_n^*) on S_{X^*}. Thus, the Josefsson-Nissenzweig theorem is just a triviality in such spaces. Of course, every separable Banach space contains a thin, even norming, set (take any dense countable subset of B_X). Also, it is immediate that any Banach space containing a complemented separable subspace contains a thin, norming set. Thus, WCG spaces contain thin, norming sets.

Question 3.7.1. *Does every Banach space contain a fundamental, thin set? Does every Banach space contain a norming, thin set?*

Fernandez, Hui and Shapiro have asked (in our notation) whether the Blaschke-products is a thick subset in H^∞ (not only w^*- thick). We formulate an extended question:

Question 3.7.2. *Is the set of inner functions (Blaschke-products) a thick subset in H^∞? Is the set of interpolating Blaschke-products thick or w^*- thick?*

By a theorem of Mooney (see [11, p. 206-207]), the pre-dual of H^∞ is weak sequentially complete. Thus, it doesn't contain a copy of c_0. Hence, any James boundary in H^∞ is w^*-thick. In light of V.P Fonf's theorem (Theorem 3.6.2) it is natural to ask:

Question 3.7.3. *Is the set of inner functions (Blaschke-products, interpolating Blaschke-products) a James boundary in H^∞?*

If so, Fernandez' and Shapiro's results would follow as special cases of Theorem 3.6.2.

We end this paper by giving a list of sets for which results on thickness are known:

Theorem 3.7.4. *The following results on thickness are valid:*

(a) *Any James-boundary of a Banach space not containing c_0 is w^*-thick (Fonf [8]).*
(b) *If X is a separable Banach space not containing c_0, then $exp\, B_{X^*}$ is w^*-thick (Fonf [7]).*
(c) *The set of characteristic functions in $B(\mathcal{A})$ when $\mathcal{A}$ has the Nikodým (Seever) property, is thick (Nikodým-Grothendieck [3]). Thus, $extB_{l_\infty}$ is thick.*
(d) *The set of inner functions and Blaschke products in H^∞ are w^*-thick and norming for the dual ([5], [12]).*
(e) *The tensor product of two w^*-thick sets in X^{**} and Y^* is a w^*-thick subset in $L(X,Y)^*$ (this paper).*
(f) *Suppose X and Y are separable, reflexive spaces. Then every James boundary of $K(X,Y)$ is w^*-thick (this paper).*

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Chapter 4

Slices in the unit ball of a uniform algebra

4.1 Introduction

It is an important task in Banach space theory to determine the extreme point structure of the unit ball for various examples of Banach spaces. The most common way to describe "corners" of convex sets is by looking for *extreme points, exposed points, denting points* and *strongly exposed points*.

Let C be a closed, bounded subset of a Banach space Y. A *slice* of C is a set of the form

$$S(y^*, \varepsilon, C) = \{y \in C\colon \operatorname{Re} y^*(y) \geq \sup \operatorname{Re} y^*(C) - \varepsilon\},$$

where $y^* \in Y^*$.

Let $B_\varepsilon(y)$ denote the ball with radius ε centred at y. A *denting point* of C is a point y_0 in C such that $y_0 \notin \overline{\text{co}}(C \setminus B_\varepsilon(y_0))$ for every $\varepsilon > 0$. Thus, by the Hahn-Banach theorem, if $y_0 \in C$ is a denting point, then there are slices of C of arbitrarily small diameter containing y_0. When C has slices of arbitrarily small diameter it is called *dentable.*

For the definitions of an extreme point, an exposed point and a strongly exposed point, we refer to [2]. More or less directly from the definitions it follows that every strongly exposed point is both denting and exposed, and every denting or exposed point is extreme.

In this note $\mathcal{A}$ denotes an infinite-dimensional uniform algebra, i.e., an infinite-dimensional closed subalgebra of some $C(K)$-space which separates the points of K and contains the constant functions. In [1] Beneker and Wiegerinck demonstrated the non-existence of strongly exposed points in $B_{\mathcal{A}}$, the closed unit ball of $\mathcal{A}$. Here we shall prove a stronger result by more elementary means. A corollary of our result is that the set of denting points is, in fact, also empty.

The set of extreme points and the set of exposed points of the unit ball have been characterised in many uniform algebras. For example, there are lots of such points in the unit ball of H^∞, the algebra of bounded analytic functions on the unit disk, since every inner function is an exposed point (see [3, p. 221]).

Let $\mathcal{A}$ be a function algebra on a compact space K. A point $x \in K$ is called a *strong boundary point* if for every neighbourhood V of x and every $\delta > 0$, there is some $f \in \mathcal{A}$ such that $f(x) = \|f\| = 1$ and $|f| \leq \delta$ off V. The closure of the set of strong boundary points is the *Silov boundary* of $\mathcal{A}$. It is a fundamental result in the theory of uniform algebras that one can identify $\mathcal{A}$ as a uniform algebra on its Silov boundary; cf. [8, p. 49 and p. 78]. In the sequel we shall therefore assume that the set of strong boundary points is dense in K.

4.2 The slices have diameter 2

We now turn to our first result, which gives a quantitative statement of the non-dentability of $B_{\mathcal{A}}$.

Theorem 4.2.1. *Every slice of the unit ball of an infinite-dimensional uniform algebra $\mathcal{A}$ has diameter* 2.

Proof. Take an arbitrary slice $S = \{a \in B_{\mathcal{A}}\colon \operatorname{Re} \ell(a) \geq 1 - \varepsilon\}$, where $\|\ell\| = 1$. We will produce two functions in S having distance nearly 2. Let $0 < \delta \leq \varepsilon/11$. We first pick some $f \in B_{\mathcal{A}}$ such that

$$\operatorname{Re} \ell(f) \geq 1 - \delta.$$

The functional ℓ can be represented by a regular Borel measure μ on K with $\|\mu\| = 1$, i.e., $\ell(a) = \int_K a\, d\mu$ for all $a \in \mathcal{A}$.

Let $\emptyset \neq V_0 \subset K$ be an open set with $|\mu|(V_0) \leq \delta$; such a set exists since K is infinite. Fix a strong boundary point $x_0 \in V_0$. Using the definition of a strong boundary point, inductively construct functions $g_1, g_2, \ldots \in \mathcal{A}$ and nonvoid open subsets $V_0 \supset V_1 \supset V_2 \supset \ldots$ such that

$$g_n(x_0) = \|g_n\| = 1, \quad |g_n| \leq \delta \text{ on } K \setminus V_{n-1}$$

and

$$V_n = \{x \in V_{n-1}\colon |g_n(x) - 1| < \delta\}.$$

Let $N > 1/\delta$ and define

$$g = \frac{1}{N} \sum_{k=1}^{N} g_k, \quad h = f(1 - g) \in \mathcal{A}.$$

By construction, $|h| \leq \delta$ on V_N and $|h| \leq 1+\delta$ on $K \setminus V_0$. We claim that $\|h\| \leq 1+3\delta$. In fact, if $x \in V_{r-1} \setminus V_r$, then $|1-g_k(x)| \leq \delta$ if $1 \leq k < r$, $|g_r(x)| \leq 1$ and $|g_k(x)| \leq \delta$ if $r < k \leq N$, and therefore

$$|h(x)| \leq \frac{1}{N}\sum_{k=1}^{N}|1-g_k(x)| \leq \frac{(N-1)(1+\delta)+2}{N} \leq 1+3\delta.$$

We now estimate $|\ell(f)-\ell(h)|$:

$$\begin{aligned}|\ell(f)-\ell(h)| &\leq \int_{K\setminus V_0}|f-h|\,d|\mu| + \int_{V_0}|f-h|\,d|\mu| \\ &\leq \int_{K\setminus V_0}|g|\,d|\mu| + \int_{V_0}(|f|+|h|)\,d|\mu| \\ &\leq \delta + (2+3\delta)|\mu|(V_0) \leq 4\delta.\end{aligned}$$

Next, we produce a function $\varphi \in \mathcal{A}$ such that

$$\varphi(x_0) = \|\varphi\| = 1, \quad |\varphi| \leq \delta \text{ on } K \setminus V_N.$$

We then have $\|h \pm \varphi\| \leq 1+4\delta$, and the functions $\psi_\pm = (h \pm \varphi)/(1+4\delta)$ are in the unit ball of $\mathcal{A}$. We have $|\ell(\varphi)| \leq 2\delta$ and thus

$$|\ell(\psi_\pm)-\ell(h)| \leq |\ell(h)|\frac{4\delta}{1+4\delta} + \frac{2\delta}{1+4\delta} \leq 6\delta.$$

Consequently,

$$\operatorname{Re}\ell(\psi_\pm) \geq \operatorname{Re}\ell(f) - 10\delta \geq 1-11\delta \geq 1-\varepsilon$$

so that $\psi_\pm \in S$; but $\|\psi_+ - \psi_-\| = 2/(1+4\delta) \to 2$ as $\delta \to 0$. Hence $\operatorname{diam} S = 2$. ☐

The point of working with g rather than g_1 in the proof is to control $\|1-g\|$. Another way to achieve this is to construct a suitable conformal map ϕ from the unit disk to a neighbourhood of $[0,1]$ in $\mathbb{C}$ and to consider $\phi \circ g_1$.

We now extend Theorem 4.2.1 to relatively weakly open subsets.

Theorem 4.2.2. *Every nonvoid relatively weakly open subset W of the unit ball of an infinite-dimensional uniform algebra $\mathcal{A}$ has diameter* 2.

Proof. Every nonvoid relatively weakly open subset of the unit ball of a Banach space contains a convex combination of slices, see [4, Lemma II.1] or [12]. Thus, if $W \subset B_{\mathcal{A}}$ is given as above, there are slices $S^{(1)}, \dots, S^{(n)}$ and $0 \leq \lambda_j \leq 1$, $\sum_{j=1}^n \lambda_j = 1$, such that $\sum_{j=1}^n \lambda_j S^{(j)} \subset W$.

Let $S^{(j)} = \{a \in B_{\mathcal{A}} \colon \operatorname{Re}\ell_j(a) \geq 1-\varepsilon_j\}$ with $\|\ell_j\| = 1$ and representing measures μ_j. We now perform the construction of the proof of Theorem 4.2.1 with $\varepsilon = \min \varepsilon_j$, $0 < \delta \leq \varepsilon/11$ as before and a nonvoid open set $V_0 \subset K$ such that $|\mu_j|(V_0) \leq \delta$ for all j. We obtain functions $h^{(j)}$ and φ (independently of j) such that $(h^{(j)} \pm \varphi)/(1+4\delta) \in S^{(j)}$ and $\|\varphi\| = 1$. Therefore $\sum_{j=1}^n \lambda_j h^{(j)} \pm \varphi \in (1+4\delta)W$, and $\operatorname{diam} W = 2$. ☐

4.3 Some remarks

In case K does not have isolated points, Theorem 4.2.1 is a formal consequence of the Daugavet property of $\mathcal{A}$, proved in [14] or [13], and [7, Lemma 2.1]. Likewise Theorem 4.2.2 follows from [12] in this case. We are grateful to K. Jarosz for pointing out to us that Theorem 4.2.2 can also be deduced from results of his in [6].

It is clear from the description in terms of slices that a denting point of the unit ball of a Banach space Y is a point of continuity for the identity mapping from (B_Y, weak) to (B_Y, norm). Conversely, it is known that a point of continuity which is an extreme point of B_Y is a denting point [9], [10].

The following corollary is immediate from Theorem 4.2.2.

Corollary 4.3.1. *The unit ball of an infinite-dimensional uniform algebra does not contain any denting points or merely points of continuity for the identity mapping with respect to the weak and the norm topology.*

A related result we would like to mention is the theorem due to Hu and Smith stating that the unit ball in the space of continuous functions from a compact Hausdorff space into a Banach space equipped with its weak topology has no denting points [5]. This result was also obtained by T.S.S.R.K. Rao [11], who has informed us that he has also given a proof of Corollary 4.3.1 based on techniques from that paper.

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