FAMILIES OF VECTOR MEASURES OF UNIFORMLY BOUNDED VARIATION

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Abstract. We characterize families of vector measures of uniformly bounded variation and semivariation in terms of additivity properties. A classical theorem of Rickart and a simplified proof of Nikodým's boundedness theorem follow.

Throughout this note, X will be a Banach space with dual space X^* and unit ball B_X, Ω a non-empty set, $\mathcal{F} \subset \mathcal{P}(\Omega)$ an algebra, and $\{F_\tau\} = \{F_\tau \colon \tau \in T\}$ a family of X-valued vector measures (i.e., finitely additive set functions) defined on $\mathcal{F}$. Recall [1, page 2] that the *variation* of a vector measure $F \colon \mathcal{F} \to X$ is the (finitely additive) set function $|F| : \mathcal{F} \to X$ whose value on a set $A \in \mathcal{F}$ is defined by

$$|F|(A) = \sup \left\{ \sum_{j=1}^{n} \|F(A_i)\| : n \in \mathbb{N},\, A_1, \dots, A_n \in \mathcal{F} \text{ are pairwise disjoint} \right\}.$$

If $|F|(\Omega) < \infty$, then F is said to be of *bounded variation.* We shall say that the family $\{F_\tau\}$ is *of uniformly bounded variation*, if $\sup_{\tau \in T} |F_\tau|(\Omega) < \infty$.

Theorem 1. *The following assertions are equivalent.*

(i) *The family $\{F_\tau\}$ is of uniformly bounded variation.*

(ii) *Whenever $A_j \in \mathcal{F}$, $j \in \mathbb{N}$, are pairwise disjoint sets, then*

$$\sup_{\tau \in T} \sum_{j=1}^{\infty} \|F_\tau(A_j)\| < \infty.$$

Proof. Only (ii)⇒(i) has to be proven. Denote $\Phi(A) := \sup_{\tau \in T} |F_\tau|(A)$, $A \in \mathcal{F}$. Let (ii) hold and suppose for contradiction that (i) fails. It suffices to find indices $\tau_k \in T$ and pairwise disjoint sets $E_k \in \mathcal{F}$ such that $|F_{\tau_k}|(E_k) \geq k$, $k \in \mathbb{N}$. To this end in turn, it suffices to show that, given $k \in \mathbb{N}$ and a set $A \in \mathcal{F}$ with $\Phi(A) = \infty$, there are disjoint measurable sets $D, E \subset A$ and an index $\tau_k \in T$ such that $\Phi(D) = \infty$ and $|F_{\tau_k}|(E) \geq k$.

So, let $k \in \mathbb{N}$ and let $A \in \mathcal{F}$ satisfy $\Phi(A) = \infty$. There are two alternatives:

1) There are an index $\tau_k \in T$ and a measurable set $B \subset A$ such that

$$\|F_{\tau_k}(B)\| \geq \max \left\{ 2k, 2 \sup_{\tau \in T} \|F_\tau(A)\| \right\} =: \delta.$$

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2) For all $\tau \in T$ and all measurable $B \subset A$, one has $\|F_\tau(B)\| < \delta$.

1) In this case, denote $C = A \setminus B$. One has

$$\|F_{\tau_k}(C)\| \geq \|F_{\tau_k}(B)\| - \|F_{\tau_k}(A)\| \geq \|F_{\tau_k}(B)\| - \frac{\|F_{\tau_k}(B)\|}{2} = \frac{\|F_{\tau_k}(B)\|}{2} \geq k.$$

Now $A = B \cup C$, $B \cap C = \emptyset$ and $\min\{|F_{\tau_k}|(B), |F_{\tau_k}|(C)\} \geq k$. Since $\Phi(A) = \infty$, at least one of the quantities $\Phi(B)$ and $\Phi(C)$ has to be ∞. If $\Phi(B) = \infty$, put $D = B$ and $E = C$. If $\Phi(B) < \infty$, then $\Phi(C) = \infty$; in this case put $D = C$ and $E = B$.

2) Since $\Phi(A) = \infty$, there are an index $\tau_k \in T$ and pairwise disjoint measurable sets $B_1, \ldots, B_n \subset A$ ($n \in \mathbb{N}$) such that $\sum_{j=1}^n \|F_{\tau_k}(B_j)\| \geq 3\delta$. Clearly, $n \geq 4$. Let m be the least integer such that $\sum_{j=1}^m \|F_{\tau_k}(B_j)\| \geq \delta$. Since $\|F_{\tau_k}(B_j)\| < \delta$ for all $1 \leq j \leq n$, we have $\sum_{j=1}^m \|F_{\tau_k}(B_j)\| < 2\delta$ and hence $\sum_{j=m+1}^n \|F_{\tau_k}(B_j)\| > \delta$. Denote $B = \bigcup_{j=1}^m B_j$ and $C = A \setminus B$. Then $\min\{|F_{\tau_k}|(B), |F_{\tau_k}|(C)\} \geq \delta \geq k$. At least one of the quantities $\Phi(B)$ and $\Phi(C)$ has to be ∞. So, if $\Phi(B) = \infty$, put $D = B$ and $E = C$. If $\Phi(B) < \infty$, then $\Phi(C) = \infty$; in this case put $D = C$ and $E = B$.

□

Corollary 2. *A vector measure $F\colon \mathcal{F} \to X$ is of bounded variation if and only if whenever $A_j \in \mathcal{F}$, $j \in \mathbb{N}$, are pairwise disjoint sets, then*

$$\sum_{j=1}^\infty \|F(A_j)\| < \infty.$$

We now turn to uniformly bounded families of vector measures. Recall [1, page 2] that the *semivariation* of a vector measure $F\colon \mathcal{F} \to X$ is the (finitely subadditive) set function $\|F\|\colon \mathcal{F} \to X$ whose value on a set $A \in \mathcal{F}$ is defined by $\|F\|(A) = \sup_{x^* \in B_{X^*}} |x^*F|$ where $|x^*F|$ is the variation of the real-valued vector measure x^*F. By [1, page 4, Proposition 11], one has, for all $A \in \mathcal{F}$,

$$\sup_{A \supset B \in \mathcal{F}} \|F(B)\| \leq \|F\|(A) \leq 4 \sup_{A \supset B \in \mathcal{F}} \|F(B)\|,$$

thus F is bounded (i.e., it has bounded range) if and only if $\|F\|(\Omega) < \infty$, and the family $\{F_\tau\}$ is uniformly bounded if and only if $\sup_{\tau \in T} \|F_\tau\|(\Omega) < \infty$.

Theorem 3. *The following assertions are equivalent.*

(i) *The family $\{F_\tau\}$ is uniformly bounded.*

(ii) *Whenever $A_j \in \mathcal{F}$, $j \in \mathbb{N}$, are pairwise disjoint sets and $x^* \in X^*$, then*

$$\sup_{\tau \in T} \sum_{j=1}^\infty |x^* F_\tau(A_j)| < \infty.$$

Proof. Only (ii)⇒(i) has to be proven. Let (ii) hold. Then, whenever $x^* \in X^*$, Theorem 1 says that the family of scalar-valued vector measures $\{x^*F_\tau\}_{\tau \in T}$ is of uniformly bounded variation. Denote $C_j = \{x^* \in X^*\colon \sup_{\tau \in T} |x^*F_\tau|(\Omega) \leq j\}$, $j \in \mathbb{N}$. Notice that $X^* = \bigcup_{j=1}^\infty C_j$ with each C_j being norm (and even weak*) closed. Baire's theorem says that there is some C_m containing a ball, and since C_m is absolutely convex, a ball centered at the origin, i.e., $\delta B_{X^*} \subset C_m$ for some

$\delta > 0$. This means that, for all $x^* \in B_{X^*}$, one has $\sup_{\tau\in T} |x^* F_\tau|(\Omega) \le \frac{m}{\delta}$, thus $\sup_{\tau\in T} \|F_\tau\|(\Omega) \le \frac{m}{\delta}$. ☐

Corollary 4. *A vector measure $F\colon \mathcal{F} \to X$ is bounded if and only if whenever $A_j \in \mathcal{F}$, $j \in \mathbb{N}$, are pairwise disjoint sets and $x^* \in X^*$, then*

$$\sum_{j=1}^{\infty} |x^* F(A_j)| < \infty.$$

Recall [1, page 7] that a vector measure $F\colon \mathcal{F} \to X$ is said to be *strongly additive*, if whenever $A_j \in \mathcal{F}$, $j \in \mathbb{N}$, are pairwise disjoint sets, then the series $\sum_{j=1}^{\infty} F(A_j)$ converges in norm. The following classical result of Rickart [1, page 9, Corollary 19] is immediate from Corollary 4.

Corollary 5. *If a vector measure $F\colon \mathcal{F} \to X$ is strongly additive, then it is bounded.*

The converse to Corollary 5 fails in general: By the Diestel-Faires theorem [1, page 20, Theorem 2], *X does not contain any isomorphic copies of c_0 if and only if every bounded X-valued vector measure is strongly additive.* It is well known that the classical Bessaga-Pełczyński test (see e.g. [1, page 22]) for "c_0-freeness" — *X does not contain any isomorphic copies of c_0 if and only if every weakly unconditionally Cauchy series in X is (unconditionally) convergent* — can be derived from the Diestel-Faires theorem. Notice that the Diestel-Faires result above, in turn, follows immediately from the Bessaga-Pełczyński test via Corollary 4.

Theorem 3 enables one to simplify the standard Darst proof of the Nikodým boundedness theorem [1, page 14, Theorem 1].

Theorem 6 (Nikodým boundedness theorem)**.** *Suppose that $\mathcal{F}$ is a σ-algebra. Let each F_τ be bounded and let the family $\{F_\tau\}$ be setwise bounded (i.e., $\sup_{\tau\in T} \|F_\tau(A)\| < \infty$ for all $A \in \mathcal{F}$). Then the family $\{F_\tau\}$ is uniformly bounded.*

Proof. Suppose for contradiction that the family $\{F_\tau\}$ is not uniformly bounded. Then, by Theorem 3, there are $x^* \in B_{X^*}$, pairwise disjoint sets $A_j \in \mathcal{F}$, $j \in \mathbb{N}$, and a sequence $(\tau_k)_{k=1}^{\infty} \subset T$ such that, with $\mu_k = x^* F_{\tau_k}$, one has

$$\lim_{k\to\infty} \sum_{j=1}^{\infty} |\mu_k(A_j)| = \infty.$$

Step 1. We may assume that $\lim_{k\to\infty} |\mu_k(A_k)| = \infty$.

Indeed, we can choose increasing sequences of indices $(n_k)_{k=1}^{\infty}$ and $(\nu_k)_{k=1}^{\infty}$ such that $\sum_{j=\nu_{k-1}+1}^{\nu_k} |\mu_{n_k}(A_j)| \ge 4k$ for all $k \in \mathbb{N}$ (here $\nu_0 = 0$). Given $k \in \mathbb{N}$ and defining $I_k = \{\nu_{k-1}+1, \dots, \nu_k\}$, at least one of the sets

$$\{j \in I_k\colon \operatorname{Re} \mu_{n_k}(A_j) \ge 0\}, \quad \{j \in I_k\colon \operatorname{Re} \mu_{n_k}(A_j) < 0\},$$
$$\{j \in I_k\colon \operatorname{Im} \mu_{n_k}(A_j) \ge 0\}, \quad \{j \in I_k\colon \operatorname{Im} \mu_{n_k}(A_j) < 0\},$$

which we denote by J_k, must satisfy $\left|\mu_{n_k}(\bigcup_{j\in J_k} A_j)\right| \geq k$. It remains to relabel μ_{n_k} by μ_k and $\bigcup_{j\in J_k} A_j$ by A_k.

Step 2. By passing to subsequences, we may clearly assume that, for all $k \in \mathbb{N}$, $k \geq 2$, one has

$$|\mu_k(A_k)| - \sum_{j=1}^{k-1} |\mu_k(A_j)| \geq k+1.$$

The rest of the proof follows the last part of the proof presented in [1, pages 15–16]. We include it for completeness.

Step 3. We may assume that, for all $k \in \mathbb{N}$, one has $|\mu_k|\left(\bigcup_{j=k+1}^{\infty} A_j\right) < 1$.

Indeed, put $n_1 = 1$ and partition $\mathbb{N} \setminus \{n_1\}$ into pairwise disjoint infinite sets N_i, $i \in \mathbb{N}$. Since

$$\sum_{i=1}^{\infty} |\mu_{n_1}|\Big(\bigcup_{j\in N_i} A_j\Big) \leq |\mu_{n_1}|\Big(\bigcup_{i=1}^{\infty}\bigcup_{j\in N_i} A_j\Big) \leq |\mu_{n_1}|(\Omega) < \infty,$$

it follows that some N_i, which we denote by $\tilde{N}_1$, satisfies $|\mu_{n_1}|(\bigcup_{j\in\tilde{N}_1} A_j) < 1$. Put $n_2 = \min \tilde{N}_1$ and repeat the preceding argument with $\mathbb{N} \setminus \{n_1\}$ replaced by $\tilde{N}_1 \setminus \{n_2\}$ and μ_{n_1} by μ_{n_2} to obtain an infinite set $\tilde{N}_2 \subset \tilde{N}_1 \setminus \{n_2\}$ satisfying $|\mu_{n_2}|(\bigcup_{j\in\tilde{N}_2} A_j) < 1$. Put $n_3 = \min \tilde{N}_2$. Continuing in an obvious manner, we obtain indices $n_1 < n_2 < \dots$. It remains to relabel A_{n_j} by A_j and μ_{n_j} by μ_j, $j \in \mathbb{N}$.

Step 4. To arrive at a contradiction, define $A = \bigcup_{j=1}^{\infty} A_j$ and observe that, for all $k \in \mathbb{N}$, $k \geq 2$,

$$\begin{aligned}|\mu_k(A)| &\geq |\mu_k(A_k)| - \sum_{j=1}^{k-1} |\mu_k(A_j)| - \left|\mu_k\Big(\bigcup_{j=k+1}^{\infty} A_j\Big)\right| \\ &\geq k+1 - |\mu_k|\Big(\bigcup_{j=k+1}^{\infty} A_j\Big) \geq k+1-1 = k,\end{aligned}$$

and thus

$$\sup_{\tau\in T} \|F_\tau(A)\| \geq \sup_{k\in\mathbb{N}} |x^* F_{\tau_k}(A)| = \sup_{k\in\mathbb{N}} |\mu_k(A)| = \infty.$$

□

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References

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