Sequential Properties of the Weak Topology in a Banach Space.

Olav Nygaard, Agder University College

Abstract. We will prove the Eberlein-Smulian theorem. Along with this we will speak about weak convergence of bounded sequences. Finally we will introduce the concept of "(I)-generating set", "Rainwater set" and "James boundary" and tell some story about these kind of sets, in particular we indicate a proof of James' characterization of reflexive spaces for weakly compactly generated spaces.

1. Four types of compactness and some notation

Let $A$ be a subset of a topological space $(X, \tau)$. Then

- $A$ is compact (C) if every open cover has a finite subcover.
- $A$ is countably compact (CC) if every countable cover has a finite subcover.
- $A$ is sequentially limit point compact (SLPC) if every sequence $(a_n)$ has a limit point $x$, that is, every neighborhood of $x$ contains some $a_k$.
- $A$ is sequentially compact (SC) if every sequence in $A$ has a convergent subsequence.

Then, in general (C) $\Rightarrow$ (CC) $\Rightarrow$ (SLPC) and (SC) $\Rightarrow$ (SLPC).

Recall that (C) is equivalent to the property that every net in $A$ has a convergent subnet. The (CC) and (SLPC) are linked together much the same way, but now only that sequences have convergent subnets. The reason why (SC) is stronger than (SLPC) is that now the convergent subnet can be taken as a subsequence.

For first countable spaces (SC) and (SLPC) are equivalent. There are a lot of different types of compactness and a big and powerful theory exists. A beautiful book is [Dug].

In this notes, let $X$ be a Banach space and let $X^*$ be its dual. The weak topology on $X$ is the weakest topology on $X$ for which all members of $X^*$ are still continuous. From general theory we now that it is enough to describe a neighborhood base at the origin and that such a base is given by sets of the type $W(0; x_1^*, x_2^*, \ldots, x_n^*; \epsilon) = \{x \in X : |x_1^*(x)|, |x_2^*(x)|, \ldots, |x_n^*(x)| < \epsilon\}$.

General theory of weak topologies can be found in [Ru, Ch.3], and more specific descriptions and properties of the neighborhoods for this particular topology can be found in [D, Ch.II].

We will benefit from the following point of view: Let $\Gamma$ be some subset of $X^*$ that separates points on $X$. Then a locally convex Hausdorff topology can be defined on $X$ like above using the $x_i^*$'s only from $\Gamma$. From general theory this topology is nothing but the topology of pointwise convergence on $\Gamma$. Considering $\Gamma = X \subset X^{**}$ gives the so called weak-star topology on $X^*$; the weak and the weak-star topology are the most famous ones on Banach spaces.
But there are clearly more: We could for example let the extreme points of $B_X$ be $\Gamma$. That case is interesting and well studied. In the theory of Banach algebras one often works with $\Gamma$ as the set of multiplicative functionals.

Fundamental theorems that we will need are that $B_X$ is compact in the weak-star topology (Alaoglu’s theorem) and that $B_X$ is weak-star dense in $B_{X^{**}}$ (Goldstine’s theorem). Note further that by definition the formal embedding of $X$ into $X^{**}$ is weak to weak-star continuous. Finally, recall Mazur’s theorem, that for convex sets the norm-closure and the weak closure coincide.

2. Eberlein’s theorem

We are going to prove that if $A$ is a subset of $X$ that is relatively sequentially compact in the weak topology, then this set is relatively compact in that topology. For this proof we make some preliminary observations.

Observation 2.1. If $A \subset X$ is relatively sequentially compact, then $A$ is bounded.

Proof. Suppose $A$ was unbounded. Then we can find a sequence $(x_n) \subset A$ such that $\|x_n\| > n$. Thus any subsequence must be unbounded. But according to the Banach-Steinhaus theorem weakly convergent sequences are bounded and hence $(x_n)$ can’t have any convergent subsequence. □

Observation 2.2. A bounded set $A \subset X$ is relatively weakly compact if and only if the weak-star closure of $A$ up in $X^{**}$ is in $X$.

Proof. Suppose $A$ is relatively weakly compact and look at the weak-star closure $\tilde{A}$ of $A$ in $X^{**}$. This must be the image of the natural embedding since this embedding is weak to weak-star continuous.

Now suppose $\tilde{A} \subset X$. By Alaoglu’s theorem, $\tilde{A}$ is weak-star compact. But on $\tilde{A}$ the weak topology of $X$ and the weak-star topology of $X^{**}$ coincide by definition. □

Observation 2.3. If $E$ is a finite-dimensional linear subspace of $X^*$, then there exists a finite set in $E' \subset S_X$ such that for every $x^* \in E$

$$\frac{\|x^*\|}{2} \leq \max_{x \in E'} |x^*(x)|$$

Proof. Since $E$ is finite-dimensional $S_E$ is norm-compact. Thus we can find a 1/4-net $E = \{x_1^*, x_2^*, ..., x_n^*\}$ for $E$. And we can choose $x_1, x_2, ..., x_n$ in $S_X$ such that $\|x_i^*(x_i)\| > 3/4$. Now, let $x^*$ be arbitrary in $E$. Then

$$x^*(x_k) = x_k^*(x_k) + (x^* - x_k^*)(x_k) > \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$$

for some appropriate $k$. Thus, we may simply take $E' = \{x_1, x_2, ..., x_n\}$. □

In the last observation here, let us say that "$E'$ finds $E$". The point is that at least half the norm of $E$’s members is attained on $E'$ when $E'$ finds $E$. Clearly, for every $\delta > 0$, we can take $1 - \delta$ instead of 2, but that doesn’t help us here.

We are ready for Whitley’s proof of Eberlein’s theorem, first published in Mathematische Annalen in 1967. Eberlein’s original proof was published in 1947. A few years later, Grothendieck showed that Eberlein’s theorem is true in a broader class of locally convex spaces, namely those who are quasi-complete in its Mackey topology. This was done in 1953. Brace gave a proof in 1955 which is the one found
in the book of Dunford and Schwartz. Pelczynski gave a completely different proof using basic sequences in 1964. We will follow Whitley’s proof more or less as given in [D].

**Theorem 2.4.** (Eberlein 1947) Let a subset $A$ of a Banach space $X$ be relatively weakly sequentially compact. Then it is relatively weakly compact.

**Proof.** According to Observation 1, $A$ is bounded. By Observation 2, what we then have to do, is to show that every member of the weak-star closure $\tilde{A}$ of $A$ up in $X^{**}$ are in $X$. This will be done the following way: For $x^{**} \in \tilde{A}$ we will construct a clever sequence in $A$. By the relative weak sequential compactness of $A$, this sequence has a subsequence that converges to some $x \in X$. The cleverness of the sequence is that this subsequence also converges weak-star to that $x^{**}$.

Let $x^{**} \in \tilde{A}$.

**Step 1: Constructing the sequence.**

Take some $x_1^* \subset S_{X^*}$. Look at

$$W_1 = \{y^{**} \in X^{**} : |(y^{**} - x^{**})(x_1^*)| < 1\}.$$

Then clearly $W_1$ is a weak-star neighborhood of $x^{**}$ and thus has to contain some $a_1$ from $A$. Of course

$$|(x^{**} - a_1)(x_1^*)| < 1.$$

That was the first element in the sequence. For the next, let

$$E_1 = [x^{**}, x^{**} - a_1] \text{ (norm-closed linear span of } x^{**} \text{ and } x^{**} - a_1)$$

and choose a finite

$$E_1' = \{x_2^*, x_3^*, ..., x_{n(2)}^*\} \subset S_{X^*}$$

such that $E_1'$ finds $E_1$. Put

$$W_2 = \{y^{**} \in X^{**} : |(y^{**} - x^{**})(x_i^*)| < \frac{1}{2}, \quad 1 \leq i \leq n(2)\}.$$

Inside $W_2$ somewhere there has to sit some $a_2 \subset A$, and of course

$$|(x^{**} - a_2)(x_i^*)| < \frac{1}{2}, \quad 1 \leq i \leq n(2).$$

Let

$$E_2 = [x^{**}, x^{**} - a_1, x^{**} - a_2]$$

and pick

$$E_2' = \{x_{n(2)+1}^*, x_{n(2)+2}^*, ..., x_{n(3)}^*\} \subset S_{X^*}$$

such that $E_2'$ finds $E_2$. Put

$$W_3 = \{y^{**} \in X^{**} : |(y^{**} - x^{**})(x_i^*)| < \frac{1}{3}, \quad 1 \leq i \leq n(3)\}.$$

Inside $W_3$ again somewhere there has to be some $a_3 \subset A$, and

$$|(x^{**} - a_3)(x_i^*)| < \frac{1}{3}, \quad 1 \leq i \leq n(3).$$

Once more, quickly: We put

$$E_3 = [x^{**}, x^{**} - a_1, x^{**} - a_2, x^{**} - a_3]$$
\[ E_3 = \{ x_{n(3)+1}^*, x_{n(3)+2}^*, \ldots x_{n(4)}^* \} \]

\[ W_4 = \left\{ y^{**} \in X^{**} : |(y^{**} - x^{**})(x_i^*)| < \frac{1}{3}, \quad 1 \leq i \leq n(4) \right\} \]

\[ a_4 \in W_4 \cap A \]

and go on with \( E_4 \) to produce \( W_5 \) and thus \( a_5 \) and so on.

**Step 2: Some consequences of the construction.**

Ok, we have the sequence \((a_i)\). What properties does it have? We know, since \( A \) is relatively weakly sequentially compact that it has a subsequence \((a_{n_k})\) with a weak limit \( x \) and, by Mazur’s theorem \( x \in [a_i] \).

The way \( Y \) is built gives us a possibility to control norms there, we have for any \( y^{**} \in Y \) that

\[ \frac{\|y^{**}\|}{2} \leq \sup_m |y^{**}(x_m^*)|. \]

So more than half the norm is found along the sequence \( x_m^* \). But this is also the case if we weak-star close \( Y \), and so

\[ \|x^{**} - x\| \leq 2 \sup_m |(x^{**} - x)(x_m^*)|. \]

**Step 3: \( x^{**} = x \).**

This follows as soon as we can show that for any \( \epsilon > 0 \) and any \( m \), \( |(x^{**} - x)(x_m^*)| < \epsilon/2 \). So, let \( m \) be arbitrary. We know that \( x \) is the weak limit of \((a_{n_k})\), so \( x_m^*(a_{n_k} - x) \to 0 \). Thus, we can find a natural number \( p(m) \) such that \( n_k > p \) assures \( |x_m^*(a_{n_k} - x)| < \epsilon/4 \). Now, if necessary, increase \( p \) such that \( n_k > p \) implies that also \( |x^{**} - a_{n_k}| < \epsilon/4 \). By the triangle inequality

\[ |(x^{**} - x)(x_m^*)| \leq |(x^{**} - a_{n_k})(x_m^*)| + |x_m^*(x^{**} - x)|, \]

and we are done. \( \square \)

A remark is that we have only used very elementary means in a clever way. We needed only the Banach-Steinhaus theorem, Mazur’s theorem and Alaoglu’s theorem from functional analysis, together with basics from topology. Moreover, Observation 1 was used just to avoid stating that \( A \) is bounded. Thus, Eberlein’s theorem goes through for bounded sets in any normed space \( X \).

If one thought of proving Eberlein’s theorem with pointwise topologies on \( \Gamma \) along the same lines, one quickly comes into many problems. But if \( \Gamma \) has the property that bounded \( \Gamma \)-convergent sequences are weakly convergent, then Eberlein’s theorem holds for bounded sets.

**Definition 2.5.** A set \( \Gamma \subset X^* \) is called a Rainwater set if every bounded \( \Gamma \)-convergent sequence is weakly convergent.

We obtain the following theorem:

**Theorem 2.6.** (Generalized Eberlein’s theorem) Let \( \Gamma \) be a Rainwater set in \( X^* \) and assume that \( A \) is a bounded set in \( X \). Then, if \( A \) is relatively sequentially \( \Gamma \)-compact, it is relatively weakly compact.
We will meet Rainwater sets later on, we just mention now that the extreme points of \( B_X \), provides a such set.

3. Smulian’s theorem

Suppose \( A \subset X \) is relatively weakly compact. Then it is weakly bounded, hence bounded. Every sequence in \( A \) must now have a weak limit point. We will show that this limit point in fact is the weak limit of a sequence in \( A \), and this is the essence of Smulian’s theorem. We follow the very wise leadership of Dunford and Schwartz. Recall that a set \( \Gamma \subset X^* \) is called total if its span is weak-star dense in \( X^* \).

**Theorem 3.1.** (Smulian 1940) Suppose a subset \( A \) of a Banach space \( X \) is relatively weakly compact. Then it is relatively weakly sequentially compact.

**Proof.** First we show that we may assume \( X \) is separable. To this end, suppose the theorem has been proved in the separable case and let \( A \subset X \) be relatively weakly compact. Let \( (a_n) \) be a sequence in \( A \). Let \( Y = [a_n] \). Then \( Y \) is a separable, closed subspace. \( A \cap Y \) is relatively weakly compact in \( Y \). Now pick a subsequence \( (a_{n_k}) \) that converges weakly to some \( y \in Y \). Then, since every \( x^* \in X^* \) is nothing but an extension of some \( y^* \in Y^* \), \( (a_{n_k}) \) converges weakly to \( y \) in \( X \) as well.

So, the situation is reduced to studying a relatively weakly compact compact set \( A \) in a separable Banach space \( X \). We take a sequence \( (a_n) \) in \( A \), which we know has a limit point \( x \in X \) and we try to find a subsequence of \( (a_n) \) that could have the possibility to converge weakly to \( x \). So, first of all we need a subsequence that has a tendency of converging weakly to something. For this, suppose \( H \) is a countable subset of \( S_X \), say \( (x_n^*) \). Look at the collection

\[
\begin{align*}
x_1^*a_1 & \quad x_1^*a_2 & \quad x_1^*a_3 & \quad x_1^*a_4 & \cdots \\
x_2^*a_1 & \quad x_2^*a_2 & \quad x_2^*a_3 & \quad x_2^*a_4 & \cdots \\
x_3^*a_1 & \quad x_3^*a_2 & \quad x_3^*a_3 & \quad x_3^*a_4 & \cdots \\
\vdots & \quad \vdots & \quad \vdots & \quad \vdots & \quad \vdots
\end{align*}
\]

By the Bolzano-Weierstrass theorem, we can pick subsequences such that every horizontal line consist of convergent scalar sequences. Assume so has been done. Then take the diagonal sequence, and denote the corresponding subsequence of \( (a_n) \) by \( (y_n) \). By construction \( \lim_n x^*y_n \) exists for every \( x^* \in H \), so some kind of weak convergence seems to emerge.

Now we will argue that there is a \( y_0 \in X \) such that \( \lim_n x^*y_n = x^*y_0 \) for every \( x^* \in H \). We know \( (y_n) \) has a weak limit point \( y_0 \). This means that every weak neighborhood at \( y_0 \) contains at least one \( (y_n) \). In particular all the sets

\[
\{x \in X : |x^*(x - y_0)| < \epsilon \}
\]

contains some \( y_m \). Since \( x^*y_m \) converges for every \( x^* \in H \), the limit has to be \( x^*y_0 \).

This can be done for every countable \( H \). Let us now make \( H \) "big" to get closer to weak convergence of \( (y_n) \). Let \( (x_n) \) be dense in \( S_X \) and pick \( x_n^* \subset S_{X^*} \) such that \( x_n^*x_n = 1 \). Let \( H = (x_n^*) \). Then \( H \) is total in \( X^* \). To see this, let \( x \in S_X \) be such that \( x_n^*x = 0 \) for every \( n \) and let \( x^* \in S_{X^*} \). Let \( \epsilon > 0 \) and find some \( x_n \) such that \( \|x - x_n\| < \epsilon \). Then, since \( \|x^*\| = 1 \), \( |x^*x| < \epsilon \). Since \( \epsilon \) was arbitrary, it follows from the Hahn-Banach separation theorem that \( H \) is total.
To end the proof we show that the subsequence \((y_m)\) we obtain with a total \(H\) does the trick, that is \(x^*y_m \to x^*y_0\) for every \(x^* \in X^*\). Suppose not. Then there must be some \(x_0^* \in X^*\), some \(\delta > 0\) and some subsequence \((y_{m_k})\) of \((y_m)\) such that
\[
|x_0^*(y_{m_k} - y_0)| > \delta, \quad k = 1, 2, ...
\]
and, by again invoking the Bolzano-Weierstrass theorem, we may assume \(\lim_{k} x_0^*y_{m_k}\) exists. But \((y_{m_k})\) sits in \(A\) and has a weak limit point, say \(y_0\). Just as above, we obtain \(x^*y_{m_k} \to x^*y_0\) for every \(x^* \in H\) and for \(x_0^*\). But then
\[
x_0^*y_0 = x_0^*y_0 \quad \text{for every } x^* \in H,
\]
and since \(H\) is total, \(y_0 = y_0\).
So \(|x_0^*(y_{m_k} - y_0)| > \delta, \quad k = 1, 2, ...\) and still \(x_0^*(y_{m_k}) \to x_0^*y_0\), a hopeless situation. The theorem is proved.

4. Weak convergence of bounded sequences and a new type of closed convex hull

In 1963 (see [R] or [D, p. 155]) the following theorem was published by people from Seattle under the pseudonym J. Rainwater: For a bounded sequence in a Banach space \(X\) to converge weakly it is enough that it converges pointwise on the extreme points of the unit ball in the dual, \(B_{X^*}\). It is of course this result that motivates our term Rainwater set.

Later on S. Simons (see [S1] or [S2]) gave a completely different argument to show that Rainwater’s theorem is true with any boundary of the dual unit ball. So what is a boundary?

**Definition 4.1.** Let \(K\) be a weak-star compact convex subset of \(X^*\). A subset \(B \subset K\) is called a boundary (or a James boundary) for \(K\) if, for every \(x^* \in X^*\) (which we do know attain max on \(K\)), the max over \(K\) is attained somewhere on \(B\).

**Observation 4.2.** The extreme points of \(B_{X^*}\) is a boundary for \(B_{X^*}\).

**Proof.** Let \(x \in X\). Let \(m\) be the max over \(B_{X^*}\) and put \(J = \{x^* \in B_{X^*} : x^*(x) = m\}\). Then \(J\) is a weak-star closed face (extreme set), and hence it contains an extreme point by the proof of the Krein-Milman theorem.

We formulate Simon’s generalization of Rainwaters theorem:

**Theorem 4.3.** (Simons, 1972) Every boundary for \(B_{X^*}\) is a Rainwater set.

**Definition 4.4.** Let \(K\) be a weak-star compact convex subset of \(X^*\). A subset \(B \subset K\) is said to
\begin{itemize}
  \item[(a)] \((S)\)-generate \(K\) if \(K\) is the norm-closed convex hull of \(B\) and \(K\) is then said to be the \((S)\)-hull of \(B\).
  \item[(b)] \((W)\)-generate \(K\) if \(K\) is the weak-star-closed convex hull of \(B\) and \(K\) is then the \((W)\)-hull of \(B\).
  \item[(c)] \((I)\)-generate \(K\) if whenever \(B\) is built as a countable union, \(B = \bigcup_n B_n\), then \(K\) is the \((I)\)-hull of \(B\), that is:
  \[
  K = \| \cdot \|\text{-clco}\left( \bigcup_i \text{w}^*\text{-clco}B_n \right).
  \]
\end{itemize}
In other words, \( K \) is the (S)-hull of the union of the (W)-hulls of the \( B_i \)'s. In this case \( K \) is called the (I)-hull of \( B \).

It is not difficult to see that any (S)-generating set is (I)-generating and that any (I)-generating set is (W)-generating. Moreover, an example can be given to show that the (S)-hull in general lies strictly in between the (S)-hull and the (W)-hull. It can also easily be seen that to produce the (I)-hull of \( B \) we can take increasing unions. In this case the union is itself convex and we end up with only taking norm-closure of the union of the (W)-hulls.

Here is an illustration of how the fact that \( B \) (I)-generates \( B_X^\ast \) can be used in proving theorems (compare to Rainwater-Simons' theorems):

**Theorem 4.5.** Let \( X \) be a normed space and suppose \( B \) (I)-generates \( B_X^\ast \). Then \( B \) is a Rainwater set.

**Proof.** Let \((x_i)\) be a bounded sequence in \( X \) that converges pointwise on \( B \). We must show that \((x_n)\) is weakly convergent. Let \( M \) be such that \(||x_i||, ||x|| \leq M\) for all \( i \). Pick an arbitrary \( x^\ast \in B_X^\ast \) and let \( \epsilon > 0 \). Define

\[
B_i = \{ y^\ast \in B : \forall j \geq i, |y^\ast(x_j - x)| < \epsilon \}.
\]

Then, since \( y^\ast(x_i) \to y^\ast(x) \) for every \( y^\ast \in B \), \( (B_i) \) is an increasing covering of \( B \).

Since \( B \) has property (I), there is a \( y^\ast \) in some \( w^\ast\text{-clco}B_N \) such that \(||x^\ast - y^\ast|| < \epsilon \). Note that for every \( y^\ast \in w^\ast\text{-clco}B_N, j \geq N \) implies that \(|y^\ast(x_j - x)| \leq \epsilon \). Now, the triangle inequality show that for \( j \geq N \)

\[
|y^\ast(x_j - x)| \leq |x^\ast(x_j) - y^\ast(x_j)| + |y^\ast(x_j) - y^\ast(x)| + |y^\ast(x) - x^\ast(x)|
\]

\[
\leq (1 + 2M)\epsilon,
\]

and hence \((x_i)\) converges weakly to \( x \). \( \square \)

The property that \( B \) (I)-generates \( K \) was introduced by Fonf and Lindenstrauss in a recent paper, see [FL]. Here the following theorem is the main observation:

**Theorem 4.6.** (Fonf-Lindenstrauss, 2003) Suppose \( B \) is a boundary for \( K \). Then \( B \) (I)-generates \( K \).

Note that Simons’ theorem follows from the two theorems above. Another corollary is the following:

**Corollary 4.7.** Suppose every \( x^\ast \in B_X^\ast \) attain its norm on \( B_X \). Then \( X \) is a Grothendieck space, that is, every weak-star convergent sequence in \( X^\ast \) is weakly convergent.

**Proof.** That every \( x^\ast \in B_X \) attain its norm on \( B_X \) tells us exactly that \( B_X \) is a boundary for \( B_X^{\ast\ast} \). By the Fonf-Lindenstrauss theorem \( B_X \) (I)-generates \( B_X^{\ast\ast} \). Now, let \((x_n^\ast)\) be a weak-star convergent sequence in \( X^\ast \). Then, by the Banach-Steinhaus theorem, it is bounded. By definition it converges pointwise on \( B_X \). Hence, by Theorem 4.5 it is weakly convergent. \( \square \)

A much stronger theorem than the above corollary is true. This is a difficult theorem of R. C. James and reads as follows: Let \( X \) be a Banach space. Suppose every \( x^\ast \in B_X^\ast \) attain its norm on \( B_X \). Then \( X \) is reflexive. James’ theorem for a big class of spaces follows, however from the above corollary and Eberlein’s theorem:
Corollary 4.8. Let $X$ be a Banach space such that $B_{X^*}$ is sequentially weak-star compact. If every $x^* \in B_{X^*}$ attain its norm on $B_X$, then $X$ is reflexive.

Proof. We need only observe that any Grothendieck space with a weak-star sequentially compact dual unit ball is reflexive. To prove this we will show that then $B_{X^*}$ is weakly compact, which is equivalent to $X$ being reflexive. By Eberlein’s theorem we need only show it is weakly sequentially compact and so it clearly is, by the Grothendieck property. □

Which spaces have weak-star sequentially compact dual unit balls? Well, separable spaces for sure, since then $B_{X^*}$ in its weak-star topology is metrizable. It is a theorem (not very difficult, one only needs the so called Davis-Figiel-Johnson-Pelczynski construction) of Amir and Lindenstrauss that every subspace of a weakly compactly generated space (WCG-space) is also of this type. A WCG-space is a space where one can find a weakly compact set such that the span of that set is dense.

Definition 4.9. A Banach space such that $B_X$ (I)-generates $B_{X^*}$ is called an (I)-space.

Note that by Goldstine’s theorem, every Banach space is a (W)-space (the definition should be obvious) and that the (S)-spaces are exactly the reflexive spaces.

Question 4.10. Who are the (I)-spaces?

References