MA-224 - WEEK 41
INTRODUCTION TO RECURRENCE RELATIONS

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Contents

1. Introduction

We study sequences of numbers where each number is defined by a recursive rule in terms of its k predecessors. We can say that such sequences satisfy a *recurring relation*, i.e. a relation that holds for any k consecutive numbers in the sequence. We will define what it means to be such a relation more precisely, and look at some classes of relations where we get an explicit description of the sequence of numbers they define.

These notes span the topics in [1, §10.1-4]. In addition to the material presented in the present text, we will also discuss examples from the text book during the lectures.

2. Definition

We encountered an example of a recurrence relation already in the very first week of this course, when we studied the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 \ldots . \tag{1}$$

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The recurrence relation governing this sequence is the one we used to define it, namely

$$F_n = F_{n-1} + F_{n-2}\,. \tag{2}$$

We call (2) a *recurrence (or difference) relation of order 2* since it relates each number in the sequence to its two closest predecessors. Notice that equation (2) is in itself insufficient for producing the sequence (1), since it will only tell us what the n'th number is relative to the previous neighbors, but not how and where to start the sequence. Information like that is called *an initial condition*, which for the Fibonacci sequence is given by declaring what the initial values should be: $F_0 = F_1 = 1$.

Definition 2.1. *Fix a natural number k. A recurrence relation of order k corresponding to a sequence of numbers $(a_0, a_1, \ldots)$ is a set of equations of the form*

$$a_n = \phi(n, a_{n-1}, a_{n-2}, \ldots, a_{n-k}) \ ; n \geq k$$

where ϕ is a function of $k+1$ input variables.

The first k numbers $a_0, a_1, \ldots, n_{k-1}$ of the sequence is referred to as initial values.

Remark 2.2 We do not specify what kind of numbers we allow in our sequences. Most of the time though, they will be integers.

A set of initial values and the recurrence relations together determine the entire sequence. We often think of a recurrence relation as a method of building the number sequence it governs.

Exercise 2.3 Show by induction that solutions to recurrence relations with given initial value are unique. In other words, if $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ are two sequences which both solve a given recurrence relation (and its initial condition), show that $a_n = b_n$ for all $n \geq 0$.

Example 2.4 The sequence $1, 2, 6, 24, 120, 720, 5040, 40320, 362880, \ldots$ where the n'th term is given by the factorial function $a_n = n!$, satisfies the recurrence relation $a_n = n \cdot a_{n-1}$ or order 1, and is a solution to the initial value problem specified by $a_0 = 1$.

Example 2.5 The Fibonacci sequence (1) satisfies the recurrence relation $F_n = F_{n-1} + F_{n-2}$, and is therefore of order 2.

Example 2.6 The numbers in the sequence $2, 6, 18, 54, 162, \ldots$ follow the pattern that the ratio between consecutive numbers is constant $a_n/a_{n-1} = 3$. Therefore the sequence satisfies the recurrence relation $a_n = 3 \cdot a_{n-1}$ and is of order 1.

Any sequence satisfying the recurrence relation $a_n = r \cdot a_{n-1}$ for some constant r, is called a *geometric progression* with *common ratio* r.

3. Solving recurrence relations

A recurrence relation together with an initial condition is often called an *initial value problem*. An explicit non-recursive formula for the n'th term is called a *particular solution* for the initial value problem. If we manage to write down a formula for that solves all the initial value problems for a given recurrence relation, we call this a *general solution* for the relation.

We will spend some time solving various recurrence relations.

3.1. **Geometric progressions.**

Consider the geometric progression with common ratio r and initial value a_0. By an easy inductive proof, the n'th term can be written explicitly as $a_n = a_0 \cdot r^n$. This is the general solution for all initial value problems for geometric progressions with common ratio r.

For the particular sequence in Example 2.6, we have $r = 3$ and initial condition $a_0 = 2$. Hence the solution to that particular initial value problem is $a_n = 2 \cdot 3^n$.

3.2. **Inspection.**

Sometimes we are lucky and a pattern in the number sequence will appear to us if we only write down the first few terms of it. If so, we can often guess the solution and use induction to prove that it holds for every number in the sequence.

Exercise 3.1 Solve the recurrence relation $a_n = \frac{n-2}{n} \cdot a_{n-1} + \frac{2}{n}$ with initial condition $a_1 = 1$.

Solution: We know $a_1 = 1$, so $a_2 = \frac{2-2}{2} \cdot 1 + \frac{2}{2} = 0 + 1 = 1$ as well. If we go on, we get the sequence

$$a_1 = 1, a_2 = 1, a_3 = 1, \ldots$$

for as many terms as we care to compute by hand. So the guess is that $a_n = 1$ for all n, and we try to prove this by induction on n: We have checked that the induction hypothesis is satisfied, namely that $a_1 = 1$. So for the induction step, assume that $a_k = 1$ for all $k < n$. In particular $a_{n-1} = 1$, and so the recurrence relation implies that

$$a_n = \frac{n-2}{n} \cdot a_{n-1} + \frac{2}{n} = \frac{n-2+2}{n} = \frac{n}{n} = 1\,.$$

3.3. **Generating functions.** A number sequence $s = (a_0, a_1, a_2, \ldots)$ gives rise to a formal sum

$$f(x) = \sum_{n=0}^{\infty} a_n \; x^n \, .$$

We call $f(x)$ the generating function for the sequence s. ((Comment on how to manipulate such infinite sums, and when we can consider them as functions.))

We now assume that the a_n's satisfy a given recurrence relation, and will try to find a non-recursive formula for the coefficients a_n. To see that the formalism of generating functions is useful to this end, we focus on a particular example, namely the Fibonacci sequence (2.5). By basic algebraic manipulation, we get that

$$\begin{aligned} f(x) - x \cdot f(x) - x^2 \cdot f(x) = \; & a_0 + \\ & (a_1 - a_0) \; x + \\ & (a_2 - a_1 - a_0) \; x^2 + \\ & (a_3 - a_2 - a_1) \; x^3 + \\ & \ldots \\ & (a_n - a_{n-1} - a_{n-2}) \; x^n + \ldots \, . \end{aligned}$$

The recurrence relation governing the coefficients implies that the terms on the right hand side vanish for $n \geq 2$, and we get

$$f(x) \cdot (1 - x - x^2) = a_0 + (a_1 - a_0)x = 1 + (1 - 1)x = 1$$

which we solve for $f(x)$ to get

$$f(x) = \frac{1}{1 - x - x^2} \, .$$

That we managed to write the generating function on a very compact form is in itself interesting, but we still do not have a more convenient way of expressing each Fibonacci number. To accomplish this, we need one more step in our analysis of $f(x)$.

The polynomial $1 - x - x^2 = -(x^2 + x - 1)$ has two distinct real roots

$$r_1 = \frac{-1 + \sqrt{5}}{2} \qquad \text{and} \qquad r_2 = \frac{-1 - \sqrt{5}}{2}$$

which means that

$$-(x^2 + x - 1) = -(x - r_1)(x - r_2)$$

and by the method of partial fractions,

$$f(x) = \frac{1}{-(x - r_1)(x - r_2)} = \frac{c_1}{x - r_1} + \frac{c_2}{x - r_2} \tag{3}$$

for some constants c_1 and c_2. By straight forward calculation, it follows that $c_2 = 1/\sqrt{5}$ and $c_1 = -c_2 = -1/\sqrt{5}$. Lastly, we use the identity

$$\frac{1}{1-a} = \sum_{n=0}^{\infty} a^n$$

to rewrite (3) as

$$\begin{aligned} f(x) &= -\frac{c_1}{r_1} \cdot \frac{1}{(1 - x/r_1)} - \frac{c_2}{r_2} \cdot \frac{1}{(1 - x/r_2)} \\ &= \frac{1}{\sqrt{5}r_1} \cdot \sum_{n=0} (x/r_1)^n - \frac{1}{\sqrt{5}r_2} \cdot \sum_{n=0} (x/r_2)^n \\ &= \frac{1}{\sqrt{5}} \sum_{n=0} \left(\frac{1}{r_1^{n+1}} - \frac{1}{r_2^{n+1}} \right) x^n \end{aligned}$$

We can now read off the Fibonacci sequence from the coefficients in the power series:

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1}{r_1^{n+1}} - \frac{1}{r_2^{n+1}} \right)$$

or, by substituting for r_1 and r_2:

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{2}{-1+\sqrt{5}} \right)^{n+1} - \left(\frac{2}{-1-\sqrt{5}} \right)^{n+1} \right) . \tag{4}$$

The above formula is an explicit, non-recursive definition of the numbers in the Fibonacci sequence.

3.4. **Linear homogeneous recurrence relations of order 2.** The recurrence relation that gave us the Fibonacci sequence is an example of a recurrence relation of the form

$$A \cdot a_n + B \cdot a_{n-1} + C \cdot a_{n-2} = 0 \tag{5}$$

of order 2 where A, B, and C are real constants. Note that for this to be a recurrence relation, the constant A cannot be zero.

We call these relations *homogeneous* since the constant term on the right hand side is zero, and *linear* since the left hand side defines a linear transformation of the input vector (a_n, a_{n-1}, a_{n-2}).

Linearity implies that if $\{a_n\}$ and $\{a'_n\}$ are two solutions[1] of the same recurrence relation, then any linear combination of the two sequences $\{c \cdot a_n + d \cdot a'_n\}$ (here c, d are constants) is also a solution.

Exercise 3.2 Assume that a_n and b_n are solutions to the recurrence relation

$$A \cdot a_n + B \cdot a_{n-1} + C \cdot a_{n-2} = 0\,. \tag{6}$$

[1] Remember that before we fix initial values, there are infinitely many sequences that satisfy the recurrence relation.

Show that for any pair of constants k, l, the linear combination $ka_n + lb_n$ is also a solution.

Remember that the geometric progression with common ratio r was given by the recurrence relation $a_n = r \cdot a_{n-1}$ and that the general solution was $a_n = a_0 \cdot r^n$. We can think of this relation as a linear homogeneous relation of order two by letting the constant $C = 0$:

$$a_n - ra_{n-1} + 0 \cdot a_{n-2} = 0\,.$$

Motivated by this, we are tempted to see if the solution $a_n = c \cdot r^n$ will also solve the general linear relation (6). We first try with the simplest possibility, by letting $c = 1$. Then $a_n = r^n$ and substituting this into the recurrence relation produces the following equation:

$$A \cdot r^n + B \cdot r^{n-1} + C \cdot r^{n-2} = 0$$

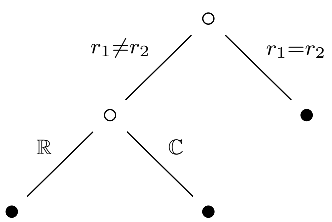
which we divide by $r^{n-2} \neq 0$ to get

$$A \cdot r^2 + B \cdot r + C = 0\,. \tag{7}$$

We call this *the characteristic equation* associated to our recurrence relation, and it tells us that $a_n = r^n$ is a solution if and only if r is a root of (7). Therefore,

$$r_{1,2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}\,.$$

Depending on the sign of the discriminant $B^2 - 4AC$, these roots will be complex or real numbers, and this distinction will lead to different cases in our analysis of the general solution. We break the situation down according to the following decision tree which depends on the roots r_1 and r_2:



3.4.1. *Case of $r_1 \neq r_2$:* We first assume that the roots are distinct. Note that they may be complex. Since the recurrence relation is linear we know that all linear combinations of the solutions we have found are also solutions:

$$a_n = c \cdot r_1^n + d \cdot r_2^n\,. \tag{8}$$

We will now show that all solutions are on this form. Given initial values a_0, a_2, we find the special solution by solving the system of equations

$$a_0 = c \cdot r_1^0 + d \cdot r_2^0 = c + d$$
$$a_1 = c \cdot r_1^n + d \cdot r_2^n = c \cdot r_1 + d \cdot r_2$$

which in vector form looks like

$$\begin{bmatrix} 1 & 1 \\ r_1 & r_2 \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \tag{9}$$

By our assumption that $r_1 \neq r_2$, we see that the coefficient matrix of this system is invertible with inverse

$$\frac{1}{r_2 - r_1} \cdot \begin{bmatrix} r_2 & -1 \\ -r_1 & 1 \end{bmatrix}.$$

This means that there exists coefficients c, d such that (8) satisfies the initial value problem. Thus, every initial value problem of the recurrence relation has a solution within the set of solutions given by (8). Since solutions are unique, this also implies that there are no other types of solutions to any given initial value problem of this type of recurrence relation.

Sub case: Real roots. When r_1 and r_2 are real numbers, equation (8) describes all solutions to any real-valued initial condition, using only real constants. We cannot make this expression any simpler, so we are done.

Sub case: Complex roots. Given a set of real initial values a_0, a_1, the solution to our recurrence relation is a sequence of real numbers. But can we find a formula which describes all the possible solutions using only real constants, as in the previous case? The solutions are described by (8), but that formula involves complex numbers r_1 and r_2, and also complex free variables c, d. Thus, a priori, it is not clear that the answer to this question is yes.

However, it turns out that it is indeed possible, when the initial values of the sequence are reals. We proceed to show why this is so.

The first observation is that since the characteristic equation (7) has real coefficients A, B, C, then the roots must be complex conjugates: $r_1 = \bar{r}_2$. With the previous case taken care of, we can also assume that $-\operatorname{im}(r_1) = \operatorname{im}(r_2) \neq 0$. Therefore, the polar form of the roots are $r_1 = ke^{i\theta}$ and $r_2 = ke^{-i\theta}$ for some real number k and angle $\theta \in (0, \pi)$. In particular, $\sin(\theta) \neq 0$.

Remark 3.3 As an alternate route to the one below, we can form two linear combinations $a_n = r_1^n + r_2^n$ and $b_n = -ir_1^n + ir_2^n$ which both turn out to be real solutions. Then we can show that these two span the entire space of solutions by the usual method, leading in this case to the determinant

$$\det \begin{bmatrix} 1 & 0 \\ k\cos(\theta) & k\sin(\theta) \end{bmatrix} = k\sin(\theta) \neq 0.$$

Thus, equation (8) turns into

$$a_n = c \cdot k^n e^{i \cdot n\theta} + d \cdot k^n e^{-i \cdot n\theta}\,.$$

By dividing by k^n and using the identity $e^{i \cdot n\theta} = \cos(n\theta) + i\sin(n\theta)$, we can write this as

$$a_n/k^n = c \cdot (\cos(n\theta) + i\sin(n\theta)) + d \cdot (\cos(-n\theta) + i\sin(-n\theta))$$

or, more compactly,

$$a_n/k^n = (c+d)\cos(n\theta) + i(c-d)\sin(n\theta)\,. \tag{10}$$

Given real initial values $a_0, a_1 \in \mathbb{R}$ we see that

$$\begin{aligned} a_0/1 &= (c+d)\cos(0) + i(c-d)\sin(0) = c+d \\ a_1/k &= (c+d)\cos(\theta) + i(c-d)\sin(\theta)\,. \end{aligned}$$

From the first equation we deduce that $(c+d)$ is real which means that $\operatorname{im}(c) = -\operatorname{im}(d)$. Then from the second equation we deduce that $i(c-d)$ is also real (this uses that $\sin(\theta) \neq 0$). By basic complex arithmetic, this implies that $\operatorname{re}(c) = \operatorname{re}(d)$. All in all we conclude that $c = \bar{d}$.

Using this, we write the general solution (10) as

$$a_n = k^n(x\cos(n\theta) + y\sin(n\theta)) \tag{11}$$

for real constants $x = c+d = 2\operatorname{re}(c)$ and $y = i(c-d) = -2\operatorname{im}(c)$. Note that the complex number c is freely chosen, therefore any pair of real constants x and y can be obtained this way.

3.4.2. *Case of* $r_1 = r_2$. If both roots are equal to zero, this means that the recurrence relation collapses to relation $A \cdot a_n = 0$ of order 0, which only has the trivial solution $a_n = 0$ since $A \neq 0$. Therefore, assume that $r_1 = r_2 = r \neq 0$ is the common root.

We claim that the general solution is given by

$$a_n = c \cdot r^n + d \cdot n \cdot r^n\,. \tag{12}$$

Exercise 3.4 Check that sequences on the form of (12) satisfy the recurrence relation (6).

To solve a given initial value problem, we are, just as in the previous two cases, led to a vector equation

$$\begin{bmatrix}1 & 0\\ r & r\end{bmatrix}\cdot\begin{bmatrix}c\\ d\end{bmatrix}=\begin{bmatrix}a_0\\ a_1\end{bmatrix}$$

which consistent since the determinant of the coefficient matrix is equal to $r \neq 0$.

The argument from the very first case can now be repeated to show that given an initial condition, the solution will be on the form described by equation (12). Hence, we have found that all solutions can be written on the given form.

3.5. **Non-homogeneous linear relations of order 2.** Assume that g_n is a general solution to the homogeneous recurrence relation

$$A \cdot a_n + B \cdot a_{n-1} + C \cdot a_{n-2} = 0$$

of order k and that p_n is a particular solution to the non-homogeneous relation

$$A \cdot a_n + B \cdot a_{n-1} + C \cdot a_{n-2} = q(n)$$

with a given initial condition a_0, a_1. Then the general solution to the non-homogeneous relation is given by the sum sequence $a_n = g_n + p_n$.

Exercise 3.5 Assume that g_n is a solution to the homogeneous recurrence relation

$$A \cdot a_n + B \cdot a_{n-1} + C \cdot a_{n-2} = 0\,, \tag{13}$$

and that p_n is a solution to the non-homogeneous relation

$$A \cdot a_n + B \cdot a_{n-1} + C \cdot a_{n-2} = q(n)\,. \tag{14}$$

Show that the sum

$$a_n = g_n + p_n$$

is a solution to the non-homogeneous relation (14).

Example 3.6 We know from exercise **??** that the solution to the initial value problem

$$\begin{cases} a_n = 2a_{n-1} + 1 \\ a_0 = 1 \end{cases} \tag{15}$$

is given by $a_n = 2^{n+1} - 1$. This is in other words a particular solution to the general problem. The homogeneous recurrence relation associated to (15) looks like

$$b_n = 2b_{n-1}$$

which is easily seen to have general solution given by $b_n = 2^n b_0$.

Therefore, the general solution to (15) is the sum of these:

$$a_n = (2^{n+1} - 1) + (2^n a_0) = 2^n \cdot (2 + a_0) - 1\,.$$

3.5.1. *Linear non-homogeneous recurrence relations of order* 1. Given a linear recurrence relation of the form

$$a_n = Ba_{n-1} + f(n)\,, \tag{16}$$

where B is a real constant and $f(n)$ is a function of n. Then we can deduce the following formula

$$a_n = B^k a_{n-k} + \sum_{i=0}^{k-1} B^i f(n-i)\,, \tag{17}$$

for $k \leq n$. In particular, when $k = n$, we get the general solution right away:

$$\begin{aligned} a_n &= B^n a_0 + \sum_{i=0}^{n-1} B^i f(n-i) \\ &= B^n a_0 + B^{n-1} f(1) + \ldots + B^2 f(n-2) + Bf(n-1) + f(n)\,. \end{aligned}$$

Sometimes we are even lucky enough to find a nice formula for the sum on the right hand side of the equation.

Exercise 3.7 Show by induction that formula (17) holds.

Exercise 3.8 Find the solution to the initial value problem given by $a_n - a_{n-1} = 2n$, and $a_0 = 2$.

References

[1] Ralph P. Grimaldi, *Discrete and combinatorial mathematics: an applied introduction.*, Pearson Education, 2013.

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